Information geometry and the renormalization group

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Information theoretic geometry near critical points in classical and quantum systems is well understood for exactly solvable systems. Here, we show that renormalization group flow equations can be used to construct the information metric and its associated quantities near criticality for both classical and quantum systems in a universal manner. We study this metric in various cases and establish its scaling properties in several generic examples. Scaling relations on the parameter manifold involving scalar quantities are studied, and scaling exponents are identified. The meaning of the scalar curvature and the invariant geodesic distance in information geometry is established and substantiated from a renormalization group perspective.

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I. INTRODUCTION

Information geometry provides a unique arena where geometric notions are applied to physical systems, often leading to new and alternative insights into the physics of classical and quantum phase transitions. A Riemannian metric defined on the parameter space for classical systems or the space of coupling constants for quantum systems define a distance on the parameter manifold (PM) [1,2]. Geometric properties of this distance translate into useful physical quantities to understand phase transitions. Although this method is primarily used to study second-order continuous transitions, first-order phase transitions can also be treated in the geometric framework [3,4].

Geometric methods have often been applied to statistical systems that are solvable. Namely, one calculates the metric and its associated quantities analytically using an equation of state or a solvable Hamiltonian and studies their limiting behavior as one approaches criticality. For example, in classical liquid-gas or magnetic systems, one can use the van der Waals equation of state to analytically compute the metric or use experimental data (based on multiparameter fits to equations of state) to compute the same [3,4]. In the context of quantum systems, one normally alludes to a complex quantum geometric tensor whose real (symmetric) part is the Riemannian metric on the parameter manifold and whose imaginary part is the Berry curvature. In exactly solvable quantum systems, knowledge of the ground state leads to the metric [5].

Scaling analyses of metrics in quantum phase transitions were first performed in the important work of [6]. There, the authors provided an integral representation of the quantum geometric tensor in terms of imaginary time correlation functions and hence were able to extract information regarding the scaling properties of the metric tensor (see also [7] for related work regarding the scaling behavior of the Gaussian curvature in the context of the XY spin chain model). The broad issue that we address in this work is whether there is a generic way of understanding the geometry of phase transitions, both classical and quantum, particularly in situations where an exact solution to the model (or an equation of state) may not be available.

To this end, we first note that there are indications that as far as geometry is concerned, the descriptions of classical and quantum phase transitions might be very similar. Let us briefly elaborate on this by focusing on two-dimensional PMs which will mainly be of our interest here. Important in the study of any geometric setup are scalar invariants in these. These invariants, which are coordinate independent, provide an invariant characterization of a curved manifold. As is well known, in two dimensions, the scalar curvature (or the Ricci scalar) completely characterizes the curvature. Associated to this is the scalar expansion parameter (which will be elaborated on towards the end of this section), which measures how geodesics (which are analogs of straight lines in curved spaces) converge (or diverge) towards a point in the PM [8]. Further, the line element, identified with an affine parameter that measures infinitesimal distances along geodesics, provides a third scalar quantity. As pointed out in [9], relations between these scalar quantities reveal universal behavior in classical and quantum phase transitions. Namely, the scaling behavior of the Ricci scalar and the expansion parameter with the affine parameter near criticality are universal for these in any two-dimensional PM under the assumption that the scalar curvature diverges at criticality as a power law. This hint of universality naturally leads one to suspect that there might be a generic way to compute metrics on the PM, at least near criticality, and we indicate how this can be achieved by using ideas from scaling symmetries near critical points.

As far as classical phase transitions are concerned, the usefulness of this method is that we are able to compute the information metric in various scenarios. For example, as we elaborate upon later, the metric for Ising-type models close to and at four dimensions computed from our method shows interesting and nontrivial behavior of scalar invariants in information geometry, consistent with the physics near their fixed points, and we will show that the scaling behavior of the Ricci scalar acquires logarithmic corrections in this example.

Although most of this work deals with classical phase transitions, we apply our method to one example in the context of zero-temperature quantum phase transitions, compute the metric using scaling arguments, and show that we get consistent results. Our method here should be contrasted with the one developed in [6]. As we have mentioned, the latter

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used the imaginary time correlation function to derive the scaling relations of the metric in the context of quantum phase transitions. Here, we will directly use the renormalization group (RG) equations to construct the metric and show that this gives sensible results. In a sense this complements the approach of [6] while retaining universal applicability.

It is useful to point out here the conventional definitions of the information metric in classical and quantum systems. In the former case, Ruppeiner's definition of the metric [1] reduces to derivatives of a thermodynamic potential. For example, if we consider the entropy density s, then the line element and the metric on the PM are defined via

$$d\lambda^2 = g_{ab}dx^a dx^b, \quad g_{ab} = -\left(\frac{\partial^2 s}{\partial x^a \partial x^b}\right), \tag{1}$$

where $x^a, a = 1,2$, denotes the internal energy and the particle number (both per unit volume) and Boltzmann's constant is set to unity. These are the coordinates on the parameter manifold in the "entropy representation." The metric can also be computed in various other representations, and a full list of such metrics is available in Table II of [1]. We mention here that an alternative definition of the metric had been advocated earlier in the seminal works of Weinhold [10]. That the two definitions are related by a conformal transformation is well known.

On the other hand, the information metric in quantum systems is defined by considering two infinitesimally separated quantum states (in the parameter space) and computing

$$|\psi(\vec{x}+d\vec{x}) - \psi(\vec{x})|^2 = \langle \partial_a \psi | \partial_b \psi \rangle dx^a dx^b = \alpha_{ab} dx^a dx^b, \tag{2}$$

where x^a [collectively denoted as $\vec{x}$ on the left-hand side of Eq. (2)] denotes the parameters on which the wave function ψ depends and ∂_a is a derivative with respect to x^a. The approach of [2] is to construct, from α_{ab} (which are not gauge invariant), a gauge-invariant metric tensor given by

$$g_{ab} = \alpha_{ab} - \beta_a \beta_b, \quad \beta_a = -i\langle \psi(\vec{x}) | \partial_a \psi(\vec{x}) \rangle. \tag{3}$$

Here, g_{ab} is the metric induced on the PM from the natural structure of the Hilbert space of quantum states. Equations (1) and (3) are the standard definitions of the classical and quantum mechanical Riemannian metrics on the parameter manifold. Our approach here is to compute these metrics in the critical regime, without using Eqs. (1) and (3) directly.

For this purpose, we use an existing notion in the literature, namely, the geometric equivalent of scale invariance near a fixed point. To the best of our knowledge, such a proposal first appeared in [11]. In this work, the scaling relations for classical liquid-gas phase transitions are recovered from a geometric perspective. Related work has appeared in the literature in the context of quantum field theory [12–17] and statistical mechanics [18]. The main idea in these works is that the renormalization group flow equations determine the so-called homothetic vector fields, which are mathematically related to scale invariance.

Specifically, if K^a is a homothetic vector field on a manifold with metric g_{ab} (we will mostly consider two-dimensional manifolds so that $a,b = 1,2$), it satisfies the condition $\mathcal{L}_K g_{ab} = D g_{ab}$, where $\mathcal{L}_K g_{ab}$ is the Lie derivative (see, e.g., Sec. 1.4 of [19]) of the metric along a curve whose tangent is K^a. Also D is a constant that we will identify with the spatial dimension of the system (note that this is different from the dimensionality of the parameter manifold, which will mostly be two in this paper). This equation reduces by standard manipulations to the condition $K_{a;b} + K_{b;a} = D g_{ab}$, where a semicolon denotes a covariant derivative (defined in the next section) on the parameter manifold (again, see Sec. 1.4 of [19]). This last equation can be alternatively written in a simpler form as

$$g_{ac}K^c_{,b} + g_{bc}K^c_{,a} + g_{ab,c}K^c = D g_{ab}, \tag{4}$$

where the comma indicates an ordinary derivative with respect to the coordinate label that follows it and repeated indices imply a summation (which will always be the case in this paper). To fix ideas, let us consider a textbook example, the two-dimensional Euclidean space. This is flat space, with coordinates (x,y) and metric $g_{ab} = \mathrm{diag}(1,1)$. Considering the vector field $K^a = (x,y)$, which represents the tangent at any point on the flat manifold, it is seen that $K_{a;b} + K_{b;a} = \mathrm{diag}(2,2)$, thus confirming Eq. (4). This is expected since two-dimensional Euclidean space is flat and looks the same at any length scale. Curved manifolds, which will be of interest to us here, are more challenging to deal with. Indeed, properties of homothetic vectors (when they exist) are of great importance in general relativity and cosmology. Here, we apply this notion to information geometry of phase transitions.

In this paper, we will use the notion of scale invariance of the parameter manifold near a critical point. Our starting point is a metric on the parameter manifold of a system, i.e., the space of the coupling constants in the theory. This metric is assumed to be *a priori* unknown. Near criticality, following [11,12], we demand that the β functions of the theory are the components of a tangent vector field which is homothetic. From Eq. (4), we then get a set of coupled partial differential equations for the components of the metric. These equations, if solvable, will lead to solutions of the metric on the PM, without detailed knowledge of the full solution of the system.

For classical systems where, in the conventional approach of Ruppeiner [1], the metric components are defined via derivatives of the free energy (or entropy) and are related to response functions, this has been demonstrated in [11]. Here, up to linear order in RG, it was found that Eq. (4) implied that the metric components are generalized homogeneous functions. Euler's theorem was then invoked to read off the scaling behavior of the metric components. In this paper, we will consider situations where this may not be possible and solve for the metric components directly from Eq. (4). Using scale invariance of the parameter manifold, our method should be viewed as a tool for obtaining the geometry of any system sufficiently near to criticality. This nontrivially generalizes the analysis of [11] by providing a universal approach to computing such metrics, and as we show in the following, we obtain unique properties of the information metric for a wide variety of systems, consistent with the physics near the fixed points of these systems. It should be kept in mind that if, in general, the coordinates on the coupling constant space are (x,y), while linearizing about a nontrivial fixed point (x^*,y^*), it is more natural to use as coordinates $(\delta x, \delta y) = (x - x^*, y - y^*)$. This should be understood by the context.

Viewed in this perspective, this method bypasses the standard requirement of knowledge of the equation of state (for classical systems) or the many-body ground state (for quantum systems). Of course, this assumes that the system of equations generated from Eq. (4) are solvable, but we will show in the following that this is true in a variety of examples. An objection could be that even if Eq. (4) does yield a solution up to some level in the RG, this solution might not hold when higher-order terms are introduced. This is certainly a drawback, which we will address towards the end of this paper. For most of this paper, we will concentrate on cases where Eq. (4) admits an analytic solution. We will see in the following that we are able to capture a large class of models within this simple approach.

As mentioned earlier, important ingredients in any geometric setup are scalars, which are invariant under coordinate transformations and indicate global properties of a curved manifold. In contrast, tensor components such as metric components will change under a coordinate transformation, and their scaling relations will not in general be coordinate invariant. Ruppeiner conjectured [1] that near criticality, the scalar curvature $R \sim \xi^D$ [20]. The arguments of Ruppeiner are based on the notion of relative flatness in a curved space [see Eq. (4.76) of [1] and the arguments preceding this equation]. From an RG perspective, we will prove that this result is exact up to linear order but that there are important subtleties when one includes a class of higher-order terms. Further, the infinitesimal geodesic distance $d\lambda$ along a curve, defined [see Eq. (1)] as $d\lambda = \sqrt{g_{ab}dx^a dx^b}$, is an interesting quantity and is known to be related to the concept of a statistical distance. We study this object and show that the geodesic distance is related to the length scale of the problem.

It is well known that that geodesics converge (or diverge) at singularities of a given manifold [21]. From a celebrated equation due to Raychaudhuri (see, e.g., [22]), the convergence of geodesics can be quantified in this case by a scalar expansion parameter (denoted Θ in the following). In two-dimensional Euclidean cases which interest us here, the expansion parameter is also a universal indicator of phase transitions, as is the Ricci scalar. There are algebraic relations between the three scalar quantities mentioned above, defined on a two-dimensional manifold. These give rise to the so-called geometric exponents [9]. While in that paper, these exponents were calculated in solvable systems, we show here that they emerge from the perspective of the RG, even beyond linear order.

This paper is organized as follows. In Sec. II, we elaborate on linearized RG flow equations, generalizing the work of [11] and providing a number of results. In Sec. III, we study nonlinear RG flow equations and their geometric significance, including logarithmic corrections. Section IV ends this paper with our conclusions and directions for future research.

Before we embark on our analysis, a word about the notations and conventions used in the paper is in order. We consider a variety of examples, and using different symbols for the variables will unnecessarily clutter the notation. We will proceed with the understanding that the notations used in a particular subsection of this paper find usage only in that subsection and are not to be related to the other subsections of the paper. Also, the examples used in this paper are standard and can be found in textbooks [23,24]. We will refrain from a detailed discussion of the models themselves, which would make the paper unnecessarily lengthy, and instead refer the reader to these excellent texts for more details. Another important issue should be kept in mind. In order to have a valid notion of geometry, the line element $d\lambda^2 = g_{ab}dx^a dx^b$ should be positive definite. This means that along with the diagonal element, the determinant of the metric tensor should be positive [see, e.g., the discussion around Eq. (3.20) of [1]]. In all the examples considered in this paper, we have checked that this condition is satisfied. We will not mention this in the following.

II. LINEARIZED RG AND INFORMATION GEOMETRY

In this section, we demonstrate the construction of the information theoretic metric near criticality for linearized RG flows. The flows may arise in any statistical system when one linearizes the RG equations near a critical point. Here, we will be concerned with two-parameter examples, and comments on the generalization to higher-dimensional manifolds will be given towards the end of this section.

A. Case I

In this section, we first recast the results of [11] in a form that will be useful for us to make some general statements regarding two-parameter information theoretic models in a linear RG and then go on to study geodesics for these models. We start with a theory with two coupling constants x and y and assume that near a generic fixed point (x^*,y^*), the linearized RG equations can be written in terms of the eigenvalues a and b (these should not be confused with the coordinate labels) as

$$\dot{x} = ax, \quad \dot{y} = by, \tag{5}$$

where the overdot indicates a derivative with respect to a logarithmic length scale, $l = \ln(L)$. As pointed out in the Introduction, appropriate coordinates on the PM in this case are $(\delta x,\delta y) = (x - x^*, y - y^*)$, which we will still call (x,y) by a slight abuse of notation. Let us denote the metric components on the PM by g_{xx}, g_{yy}, and g_{xy} [25].

As mentioned in the Introduction, for this example the homothetic vector field has components $K^a = (ax,by)$ [26]. We take this field as an input and insert it in Eq. (4) (or, alternatively, we use $K_{a;b} + K_{b;a} = Dg_{ab}$ after lowering the indices of K^a). Writing out the components of Eq. (4) then leads to differential equations for the metric tensor. For the chosen off-diagonal form of the metric, Eq. (4) gives rise to the following three equations (as before, a comma denotes an ordinary derivative with respect to the variable that follows it):

$$\begin{aligned} ax g_{xx,x} + by g_{xx,y} + (2a - D)g_{xx} &= 0, \\ ax g_{yy,x} + by g_{yy,y} + (2b - D)g_{yy} &= 0, \\ ax g_{xy,x} + by g_{xy,y} + (a + b - D)g_{xy} &= 0. \end{aligned} \tag{6}$$

These equations reveal that the metric components are generalized homogeneous functions near criticality [11] and immediately reproduce the well-known static scaling relations. This is true for any linearized set of RG equations. Say the variable x drives the phase transition. Then we can write the general solution for Eq. (6) as

$$g_{xx} = x^{\frac{D}{a}-2}\mathcal{G}_1\left(yx^{-\frac{b}{a}}\right), \quad g_{yy} = x^{\frac{D-2b}{a}}\mathcal{G}_2\left(yx^{-\frac{b}{a}}\right),$$
$$g_{xy} = x^{-\frac{a+b-D}{a}}\mathcal{G}_3\left(yx^{-\frac{b}{a}}\right). \tag{7}$$

Here, $\mathcal{G}_i, i = 1,2,3$, are functions of a single variable $yx^{-\frac{b}{a}}$ and reminiscent of Widom scaling of the free energy near criticality [23,24]. However, the functions $\mathcal{G}_i$ are not the same as scaling functions that appear in the free energy since the metric is more naturally interpreted as the second derivative of the free energy in the conventional picture of information geometry. It will be assumed that the functions $\mathcal{G}_i$ are analytic and equal a constant value near criticality, $y = 0$. We will advocate two arguments to justify this and go on to check these with the known example of the one-dimensional Ising model.

First, note that in the classical notion of information geometry, the metric components are related to the classical response functions. For example, in a magnetic system, if x is identified with the reduced temperature $t = (T - T_c)/T_c$, with T being the temperature and T_c being its critical value, and y is identified with the reduced magnetic field H/T_c, then the metric components g_{xx}, g_{yy}, and g_{xy} are related to the specific heat, the magnetic susceptibility, and the derivative of the magnetization, respectively. Equation (7) then indicates that these have the correct critical exponents if the functions $\mathcal{G}_i$ are analytic at $y = 0$ and equal a constant of order unity.

This can also be seen by noting that Eq. (6) translates into the fact that the metric components are generalized homogeneous functions up to linear order so that standard scaling arguments can be applied [see Eq. (5.5) of [11]]. This is indicative of the fact that the functions $\mathcal{G}_i$ can be taken to be analytic and constants of order unity close to criticality. There is a small subtlety here. If we take all functions $\mathcal{G}_i$ to equal the same constant near criticality, the metric of Eq. (7) become singular. Hence, this should be avoided, and $\mathcal{G}_i, i = 1,2,3$, have to be taken to equal different constant numbers of order one. These multiplicative constants can, at most, affect our results for the metric, the scalar curvature, and the expansion parameter by some overall constants and will not affect our scaling analysis. Keeping these explicitly in the computations will clutter the notation, and without loss of generality, we will take two of these to equal unity and set the other one to 2. This is just a particular choice, and any other choice would affect the results only by an overall numerical constant. With this choice, we are also able to consistently satisfy the positivity constraint on the line element, as can be checked.

Second, assuming that the scaling function is analytic and equals a constant of order unity near criticality, we obtain the scalar curvature in terms of the driving parameter x up to an overall constant as

$$R = \frac{2b}{a^2}(D - 2b)x^{-\frac{D}{a}}. \tag{8}$$

This shows that the scalar curvature blows up if x is relevant and goes to zero if x is irrelevant. If we assume that $R \sim \xi^D$, where ξ is the correlation length, then Eq. (8) implies that $\xi \sim x^{-\frac{1}{a}}$, i.e., correctly reproduces the correlation length exponent for classical phase transitions where x is identified with the reduced temperature. For quantum phase transitions, we note that if x is a relevant variable, then a perturbation in this direction produces a gap in the spectrum that in turn indicates that the correlation length scales as $x^{-\frac{1}{a}}$. This is again consistent with $R \sim \xi^D$ (and also justifies the assumption of $R \sim \xi^D$). We thus see that assuming that the functions $\mathcal{G}_i$ are constants of order unity near criticality produces a consistent geometric picture up to linear order in RG. This will be assumed in what follows. We will not explicitly indicate these functions in the following.

It is instructive to validate our analysis thus far by comparing it to a known example. We choose the standard example of the classical one-dimensional Ising model in a magnetic field, with the Hamiltonian given by

$$H = -J\sum_{j=1}^{N} S_j S_{j+1} - h\sum_{j=1}^{N} S_j. \tag{9}$$

Information geometry for this model was worked out in [27] in the limit of large N, and we quote their result for the metric. First, we define the variables $x = J/T$ and $y = h/T$ (where we set Boltzmann's constant to unity). Further, writing $t = e^{-4x}$ and near the critical point substituting $t = \epsilon$ and $h = \delta$, the metric components [Eq. (4.17) of [27]] read, after some algebra,

$$g_{xx} = \frac{\epsilon^{-\frac{3}{2}}}{4}\frac{1 + 2\left(\delta\epsilon^{-\frac{1}{2}}\right)^2}{\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}}, \quad g_{xy} = \frac{\epsilon^{-1}}{2}\frac{\delta}{\epsilon^{\frac{1}{2}}}\frac{1}{\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}},$$
$$g_{yy} = \epsilon^{-\frac{1}{2}}\frac{1}{\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}}. \tag{10}$$

Thus, the metric is similar to the one in Eq. (7) (with $x \equiv \epsilon$ and $y \equiv \delta$) upon identifying $a = 2$, $b = 1$, with $D = 1$. The functions defined in that equation read

$$\mathcal{G}_1\left(\delta\epsilon^{-\frac{1}{2}}\right) = \frac{1}{4}\frac{1 + 2\left(\delta\epsilon^{-\frac{1}{2}}\right)^2}{\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}},$$
$$\mathcal{G}_2\left(\delta\epsilon^{-\frac{1}{2}}\right) = \frac{\delta\epsilon^{-\frac{1}{2}}}{2\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}}, \tag{11}$$
$$\mathcal{G}_3\left(\delta\epsilon^{-\frac{1}{2}}\right) = \frac{1}{\left[1 + \left(\delta\epsilon^{-\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}},$$

which are analytic near criticality, as expected, if we assume ϵ and δ to be of the same order. Note that these are different functions which equate to different numerical values near criticality, as alluded to before. Also, from our discussion it follows that the RG flow equations here are governed by $\dot{t} = 2t$, $\dot{h} = h$, in agreement with Eqs. (32) and (33) of [18]. The analysis of the scalar curvature and geodesics for the one-dimensional Ising model has been done in [9], to which we refer the reader for more details.

The metric of Eq. (7) is to be used when x is the driving parameter in the phase transition. An equivalent form of writing the solutions of Eq. (6) is

$$g_{xx} = y^{\frac{D-2a}{b}}\mathcal{F}_1\left(xy^{-\frac{a}{b}}\right), \quad g_{yy} = y^{\frac{D-2b}{b}}\mathcal{F}_2\left(xy^{-\frac{a}{b}}\right),$$
$$g_{xy} = y^{-\frac{a+b-D}{b}}\mathcal{F}_3\left(xy^{-\frac{a}{b}}\right). \tag{12}$$

Here, $\mathcal{F}_i, i = 1,2,3$, are arbitrary functions of the variable $xy^{-\frac{a}{b}}$ and are assumed to approach a constant value at $x = 0$. The metric of Eq. (12) should be used when y drives the phase transition. This metric has a scalar curvature given by

$$R = \frac{a}{b^2}(D - 2a)y^{-\frac{D}{b}}. \tag{13}$$

Comparing Eqs. (8) and (13), we see that the divergence of the scalar curvature is controlled by the coefficient of the driving parameter in the RG, which is expected. As before, the functional forms of $\mathcal{F}_i(xy^{-\frac{a}{b}})$ may be different for $i = 1,2,3$. However, the only assumptions here are that all these are of order unity and that the leading behavior of the metric near criticality is controlled by the exponents of y.

Equation (8) is applicable to any model of linearized RG and shows that if the parameter x is relevant, the divergence of the scalar curvature is controlled by the relevant eigenvalue. If we identify the scalar curvature with the correlation volume (up to a possible arbitrary constant), it is seen that the correct correlation length exponent is recovered.

An interesting quantity that we will now focus our attention on is a set of geodesics (called a geodesic congruence) on the PM. Geodesics are analogs of straight lines in curved spaces, and these are paths that minimize distances between points on a curved manifold. For the curved information theoretic manifolds that we describe here, geodesics provide a further characterization of classical and quantum phase transitions, as shown in [28]. Namely, one considers a geodesic congruence on the PM, and it can be shown that near a critical point, the congruence converges (or diverges).

Let us make this statement more precise. If our PM is defined by the coordinates (i.e., coupling constants) x^a, then geodesic paths on the manifold satisfy the equation $(x^a)'' + \Gamma^a_{bc}(x^b)'(x^c)' = 0$. Here, $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})$ are the Christoffel connections defined from the metric, and the prime denotes a derivative with respect to an affine parameter λ along the geodesic, which is conventionally taken to be the square root of the line element, i.e., $d\lambda^2 = g_{ab}dx^a dx^b$. For such an affinely parametrized geodesic, the geodesic equations can be obtained from a variational principle from the Lagrangian $\mathcal{L} = \frac{1}{2}[g_{ab}(x^a)'(x^b)']$. This will be useful for us later.

If we denote as the normalized tangent vectors $u^a = (x^a)'$, curvature effects on geodesics (near criticality) are measured by the tensor $B^a{}_b = \nabla_b u^a$. Recall that the covariant derivative on a generic vector V^a is defined by the action $\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac}V^c$. Then it can be shown that $\Theta = B^a{}_a$, called the expansion scalar, gives an effective measure of the convergence or divergence of a geodesic congruence. At critical points on the PM, i.e., at phase transitions, Θ diverges. Because Θ is a scalar quantity, this is a coordinate-independent characterization of phase transitions. To compute Θ, we require a solution for the vectors u^a. In general this might be difficult to obtain, but when the metric is independent of one of the coordinates (which will always be the case here), such solutions can be found analytically if u^a is normalized, i.e., $u^a u_a = 1$. These analytic solutions constitute our geodesic congruence. We refer the reader to [9] for more details.

To illustrate the procedure, we first recall the two-dimensional metric of Eq. (7), where we will include a multiplicative factor of k_1 in the g_{yy} component. Now we denote the tangent vectors to geodesic trajectories by the vector $u^a = (x'(\lambda), y'(\lambda))$, where a prime denotes a derivative with respect to an affine parameter λ. Normalization of u^a imposes the condition

$$x(\lambda)^{\frac{-2a-2b+D}{a}}\left[x'(\lambda)^2 x(\lambda)^{\frac{2b}{a}} + 2x'(\lambda)y'(\lambda)x(\lambda)^{\frac{a+b}{a}} + k_1 x(\lambda)^2 y'(\lambda)^2\right] = 1. \tag{14}$$

The left-hand side of Eq. (14) is, in fact, proportional to the Lagrangian alluded to before. Noting that this is independent of the coordinate y (as is the metric), the Euler-Lagrange equation for $y(\lambda)$ imposes a further constraint $\partial\mathcal{L}/\partial y' = k_2$, where k_2 is an arbitrary constant. Then, if this constraint is solved in conjunction with the normalization condition, we obtain

$$x'(\lambda) = \sqrt{x(\lambda)^{2-\frac{2D}{a}}\left[2x(\lambda)^{\frac{D}{a}} - k_2^2 x(\lambda)^{\frac{2B}{a}}\right]},$$
$$y'(\lambda) = \frac{1}{2}\Big(k_2 x(\lambda)^{\frac{2B-D}{a}} - x(\lambda)^{\frac{B}{a}-1}\sqrt{x(\lambda)^{2-\frac{2D}{a}}\left[2x(\lambda)^{\frac{D}{a}} - k_2^2 x(\lambda)^{\frac{2B}{a}}\right]}\Big). \tag{15}$$

The first of these equations can be solved to obtain an expression for the geodesic distance in terms of the Gauss hypergeometric function,

$$\lambda = \frac{\sqrt{2}}{D}ax_2^{\frac{D}{2a}}F_1\left(\frac{1}{2},\frac{D}{4b-2D};\frac{D}{4b-2D}+1;\frac{1}{2}k_2^2 x^{\frac{2b-D}{a}}\right) - k_3, \tag{16}$$

where k_3 is another arbitrary constant. In order to obtain a real value of λ which is physically reasonable, we require $b > D/2$ and a further constraint on the constant k_2. For small values of x, this can be seen to restrict the hypergeometric function to values close to unity. This last fact indicates that it is reasonable to set $k_2 = 0$, without loss of generality. Indeed, Eq. (15) simplifies in this limit, and we obtain as a solution $\lambda \sim x^{\frac{D}{2a}} - k_3$. Now note that we are interested in geodesics that reach very close to the critical point. It is natural to measure λ from the critical point, so that we will require $\lambda \to 0$ as $x \to 0$. This indicates that the constant k_3 can be set to zero as well [29].

From Eq. (15), we now obtain the following solutions for x and y as a function of λ:

$$x(\lambda) = 2^{-\frac{a}{D}}\left(\frac{D\lambda}{a}\right)^{\frac{2a}{D}}, \quad y(\lambda) = -\frac{2^{-\frac{b+D}{D}}a}{b}\left(\frac{D\lambda}{a}\right)^{\frac{2b}{D}}, \tag{17}$$

where we have imposed the condition $x(\lambda = 0) = 0$; that is, the affine parameter is measured from criticality. These equations can now be inverted to obtain an analytic expression for the affine parameter, namely, $\lambda \sim x^{\frac{D}{2a}}$, apart from constant factors. Also, using Eq. (15) and the metric of Eq. (7), we obtain, by some simple manipulations,

$$\Theta = \frac{(D - 2b)x^{-\frac{D}{2a}}}{\sqrt{2}a}. \tag{18}$$

Using the solution for the affine parameter, we obtain as $x \to 0$ from Eqs. (13) and (18)

$$R \sim \lambda^{-2}, \quad \Theta \sim \lambda^{-1}. \tag{19}$$

The same conclusion can be reached for the metric of Eq. (12), as can be easily checked. Equation (19) can thus be understood as a universal indicator of phase transitions for any system with linearized RG flow. The exponents appearing in this equation were dubbed geometric critical exponents in [9]. In that paper, the analysis was conducted by exploiting the behavior of the information metric close to criticality for exactly solvable systems. Here, we have given proof that the relations hold for any arbitrary two-parameter system, at least up to linear order in RG.

We record a couple of observations before we move on. From Eqs. (13) and (18), note that the scalar curvature and the expansion parameter diverge if the operator x (or y) is relevant. If it is irrelevant, i.e., a or b is negative so that x or y is a stable direction, then these quantities go to zero in the limit that the coupling constants go to zero. In that case, λ calculated from Eq. (16) (after setting k_2 and k_3 to zero) approaches infinity. This is a typical feature of information geometry that we will also come across later [30]. Note that the relations of Eq. (19) remain valid irrespective of whether the operators are relevant or irrelevant.

Also note that the geodesic distance λ can be related to the length scale of the problem as follows. Using the fact that under an RG transformation, $\xi = \xi_0 e^l$, we find that $R \sim \xi^D$ translates into $l = -(2/D)\ln\lambda$. This is a generic feature for all linearized cases.

B. Case II

Our next example is that of an RG flow of the form

$$\dot{x} = a_1 x + a_2 y, \quad \dot{y} = b_1 y. \tag{20}$$

This form of the RG equations occurs in perturbation theory for a linearized one-loop approximation in Landau-Ginzburg models. In this case, one can obtain a metric using the original variables, as we illustrate in a moment. The important point is that geometric methods can be applied to a set of redefined coordinates, consistent with the eigendirections of the RG flow equations. For example, in this case, if we define a new variable $z = x + a_2 y/(a_1 - b_1)$, the RG flow equations reduce to $\dot{z} = a_1 z$, $\dot{y} = b_1 y$. The results of our previous analysis can now be readily applied in this new set of coordinates. In particular, for $y = 0$, we obtain the components of the information metric as

$$g_{zz} = z^{\frac{D}{a_1}-2}, \quad g_{yy} = z^{\frac{D-2b_1}{a_1}}, \quad g_{yz} = z^{-\frac{a_1+b_1-D}{a_1}}, \tag{21}$$

and an entirely similar analysis holds for $z = 0$. In both cases it can be seen that our previous results $R \sim \lambda^{-2}$ and $\Theta \sim \lambda^{-1}$ hold. As before, the scalar curvature can be computed, and the expressions are similar to the ones in Eqs. (8) and (13), and the identification $R \sim \xi^D$ shows that while a_1 is the critical exponent in the direction $y = 0$, b_1 is the one in the direction $z = 0$. We have thus constructed the information metric for the Landau-Ginzburg model at one loop, Eq. (21), solely by using the RG flow equations.

We should mention here that using the original set of equations [Eq. (20)], it is also possible to compute the metric tensor. However, this has a complicated structure and does not reveal any meaningful physics. Our scalar relations are, however, expected to hold here as well. It should thus be kept in mind that to interpret the various quantities associated with information geometry, one needs to correctly choose coordinates. Once this is done, analysis of the metric becomes meaningful, and the correlation length exponent comes out correctly with the identification $R \sim \xi^D$, with D being the spatial dimension of the system. To summarize, for any set of linearized RG flow equations, the information metric can be written down simply from the scaling dimension of the operators. An appropriate choice of coordinates then predicts the correct exponents of the system.

C. Case III

Before we close this section, we will comment on a situation in which a system has a critical line, for example, a gapless line in the parameter space for quantum phase transitions. This is exemplified by the one-dimensional anisotropic Heisenberg spin-1/2 chain. This model was considered in [31], where the gapless line was interpreted as a spin flip transition. The model Hamiltonian is given by

$$H = \sum_n \big[(1+\gamma)S_n^x S_{n+1}^x + (1-\gamma)S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z - h S_n^x\big], \tag{22}$$

where $S^i, i = 1,2,3$, are spin operators, γ is an anisotropy parameter, Δ is the coupling in the z direction, and h is a magnetic field along the x direction. As shown in [31], bosonization techniques yield the following perturbative RG equations in terms of h, γ, and b (the latter being the coefficient of an operator which arises in an operator product expansion):

$$\begin{aligned} \dot{h} &= a_1 h - a_2 \gamma h - a_3 b h, \quad \dot{\gamma} = b_1 \gamma - b_2 h^2, \\ \dot{b} &= c_1 b + c_2 h^2, \end{aligned} \tag{23}$$

where the coefficients are determined by the scaling dimension of the corresponding operators and read

$$\begin{aligned} a_1 &= 2 - K - 1/(4K), \quad a_2 = 2 - 1/K, \quad a_3 = 2 - 4K, \\ b_1 &= (2 - 1/K), \quad b_2 = [2K - 1/(2K)] = c_2, \\ c_1 &= (2 - 4K). \end{aligned} \tag{24}$$

Here, K is related to Δ [Eq. (3) of [31]] and takes values $1/2 \leqslant K < \infty$ (for details, see [31]). Now suppose we are at a fixed point $h = h^*$ and look at information geometry in the γ-b plane, where the RG equations are linear. We will not go into the details here and simply state the result that the information metric at a fixed point of h can be obtained as

$$g_{\gamma\gamma} = z^{-2-\frac{K}{1-2K}}, \quad g_{\gamma b} = z^{\frac{1-K(4K-3)}{1-2K}}, \quad g_{bb} = z^{\frac{K(3-8K)}{1-2K}}, \tag{25}$$

where $z = [2\gamma(1 - \frac{1}{2K}) - 2h^{*2}(K - \frac{1}{4K})]$ and a coordinate defined by $y = b(2 - 4K) + h^{*2}[(2K - 1)/2K]$ is set to zero. The scalar curvature and the expansion parameter diverge as

$$R \sim z^{-\frac{2K}{2K-1}}, \quad \Theta \sim z^{-\frac{K}{2K-1}} \tag{26}$$

and it can be checked that $R \sim \lambda^{-2}$ and $\Theta \sim \lambda^{-1}$, with λ being the geodesic length, as expected. For different values of K that are within its specified range, the scalar curvature and the expansion parameter diverge at $\gamma^* = (2K + 1)h^{*2}/2$. An entirely similar analysis holds for $z = 0$ when the information metric is determined by y. This results in $b^* = \gamma^*/2K$. These values of γ^* and b^* determine a fixed line in the γ-b plane and also determine the value of h^*, which is entirely consistent with [31], proving the validity of Eq. (25).

As an aside, we point out that an RG flow equation similar to Eq. (23) was obtained in [32] for a model of two weakly coupled Luttinger chains, where the second and third terms of the first expression of Eq. (23) were absent. In this case, a simple transformation of variables $z_1 = \gamma - b_2h^2/(2a_1 - b_1)$ and $z_2 = b - c_2h^2/(2a_1 - c_1)$ renders the RG equations linear, i.e., $\dot{h} = a_1h$, $\dot{z}_1 = b_1z_1$, $\dot{z}_2 = c_1z_2$. These equations can be used to define the homothetic vector field in three dimensions, where an analysis entirely similar to our two-dimensional examples so far can be done. We will, however, not present the details here; higher-dimensional examples will be treated elsewhere.

Finally, we point out that setting $\Delta = 0$ in Eq. (22) reduces it to the transverse XY model considered in the context of information geometry in [5]. This model is exactly solvable, so as a curiosity we check what our RG method predicts as the information metric. A bosonization procedure near $\gamma = 0$, $h = 0$ here yields the RG equations $\dot{\gamma} = \gamma$, $\dot{h} = h$. Along with Eq. (7) and the fact that for linearized RG flows, the information metric can be taken to be diagonal [25], we recover $g_{\gamma\gamma} \sim \gamma^{-1}$, $g_{hh} \sim \gamma^{-1}$, in agreement with the metric derived in Eq. (7) of [5], near $\gamma = 0$.

III. NONLINEAR EXAMPLES AND LOGARITHMIC CORRECTIONS

In this section, we will focus on cases in which the RG flow equations include nonlinear higher-order terms. As in the previous section, an appropriate combination of variables will be seen to render the solutions tractable.

A. Case IV

Our first example is given by the flow equations

$$\dot{x} = a_1x + a_2x^2, \quad \dot{y} = b_1y. \tag{27}$$

These flows arise, for example, in the critical dynamics of a time-independent random field interaction introduced in an Ising spin or quantum rotor model [33] in the ϵ expansion, above the lower critical dimension $d_c = 2$. The Hamiltonian for the model is

$$H = -J\sum_{\langle ij\rangle} S_i^zS_i^z - \Gamma\sum S_i^x - \sum_i h_iS_i^z, \tag{28}$$

where Γ is the strength of a transverse field and h_i are site-dependent magnetic fields, with the random fields correlated in one direction. RG flows in this model were studied in [33], and the equations are of the form of Eq. (27), with the identification $x = h/J$ and $y = T/J_0$, where T is the temperature, h is a measure of the randomness of the random field h_i defined via its distribution [Eq. (2) of [33]], and J_0 is the interaction in the direction in which the fields are correlated. Also, for this model, $a_1 = -\epsilon/2$ and $b_1 = -(1+\epsilon)$, with $d = 2+\epsilon$. In general, obtaining the information metric for such a system is not possible exactly but can be done using the symmetry of the RG flow equations, as we show below.

To keep the discussion general, we proceed with arbitrary a_1, a_2, and b_1. First, note that a transformation of variables $z = x/(a_1 + a_2x)$ renders the equations linear in the variables z and y, i.e., $\dot{z} = a_1z$ and $\dot{y} = b_1y$. Now we can construct the information metric using the methods described in the previous section. Since both x and y are irrelevant up to linear order, i.e., are stable directions, following our previous discussion, the scalar curvature does not blow up here. This is the trivial fixed point. In order to obtain the information metric near the nontrivial fixed point, it is more convenient to work in terms of the original variables, and we illustrate the results below. The differential equations for the metric components are obtained in the variables x and y as

$$\begin{aligned} 2(a_1+2a_2x)g_{xx} + b_1yg_{xx,y} + x(a_1+a_2x)g_{xx,x} - Dg_{xx} &= 0,\\ (a_1+b_1+2a_2x)g_{xy} + b_1yg_{xy,y} + x(a_1+a_2x)g_{xy,x} - Dg_{xy} &= 0,\\ 2b_1g_{yy} + b_1yg_{yy,y} + x(a_1+a_2x)g_{yy,x} - Dg_{yy} &= 0. \end{aligned} \tag{29}$$

These have the solutions given by

$$\begin{aligned} g_{xx} &= x^{\frac{D}{a_1}-2}(a_2x+a_1)^{-\frac{D}{a_1}-2},\\ g_{xy} &= x^{-\frac{a_1+b_1-D}{a_1}}(a_2x+a_1)^{-\frac{a_1-b_1+D}{a_1}},\\ g_{yy} &= \left(\frac{x}{a_2x+a_1}\right)^{\frac{D-2b_1}{a_1}}. \end{aligned} \tag{30}$$

The geodesic equations can be obtained in the same way as outlined in the previous section, and denoting the tangent vector as $(x'(\lambda), y'(\lambda))$, we obtain as a normalized solution

$$\begin{aligned} x'(\lambda) &= \sqrt{2}x(a_1+a_2x)\left(\frac{x}{a_1+a_2x}\right)^{-\frac{D}{2a_1}},\\ y'(\lambda) &= -\frac{1}{\sqrt{2}}\left(\frac{x}{a_1+a_2x}\right)^{-\frac{D}{2a_1}+\frac{b_1}{a_1}}. \end{aligned} \tag{31}$$

The above equation then yields

$$\lambda = \frac{\sqrt{2}}{D}\left(\frac{x}{a_1+a_2x}\right)^{\frac{D}{2a_1}}. \tag{32}$$

Noting that the scalar curvature and the expansion parameter are given by

$$\begin{aligned} R &= 2b_1(D-2b_1)\left(\frac{x}{a_2x+a_1}\right)^{-\frac{D}{a_1}},\\ \Theta &= \frac{D-2b_1}{\sqrt{2}}\left(\frac{x}{a_2x+a_1}\right)^{-\frac{D}{2a_1}}, \end{aligned} \tag{33}$$

we obtain $R \sim \lambda^{-2}$, $\Theta \sim \lambda^{-1}$, as expected. Note that these are true in $D = 2+\epsilon$ dimensions. Since $a_1 = -\epsilon/2$ is negative, the curvature and the expansion parameters diverge at the nontrivial fixed point $x^* = -a_1/a_2$ and $t^* = 0$. There is no divergence of these quantities at the trivial fixed point $x^* = t^* = 0$.

B. Case V

Next, we come to the case where the RG equations are taken to be

$$\dot{x} = a_1x + a_2x^2, \quad \dot{y} = b_1y + b_2xy. \tag{34}$$

These are the standard textbook form for RG equations of Ising-like models near four dimensions, i.e., in $d = 4 - \epsilon$, if we identify $a_1 = \epsilon$, $a_2 = -72$, $b_1 = 2$, $b_2 = -24$. For $O(N)$ vector models, the coefficients a_2 and b_2 are given by $-8(n + 8)$ and $-8(n + 2)$, respectively [24].

First, note that upon making the coordinate transformations $z_1 = x/(a_1 + a_2 x)$, $z_2 = y(a_1 + a_2 x)^{-\frac{b_2}{a_2}}$, these equations reduce to $\dot{z}_1 = a_1 z_1$, $\dot{z}_2 = b_1 z_1$. This clearly defines the Gaussian fixed point, with the critical exponent along the y direction being $b_1 = 2$. Next, if we linearize the RG equations near the Wilson-Fisher fixed point $x^* = -a_1/a_2 = \epsilon/72$, we get back the thermal exponent $2 - \epsilon/3$, which follows from our discussion of the previous section.

It is interesting to see what the information metric reveals when applied to a nonlinear set of equations, Eq. (34). Here, it is convenient to first define a new variable $z = y(a_2 + a_1/x)^{b_1/a_1}$, so that we have an equivalent set of equations

$$\dot{x} = a_1 x + a_2 x^2, \quad \dot{z} = b_2 x z. \tag{35}$$

Note that since b_1 and ϵ are positive, the coordinate transformation mentioned above becomes ill defined near $x \to 0$, i.e., the Gaussian fixed point. However, this is not the case near the Wilson-Fisher fixed point. We will need this fact later. We take the components of the homothetic vector field, generating the scale transformation near criticality as $(\dot{x},\dot{z})$. Then, the equations determining the metric on the parameter manifold [with parameters (x,z)] are given by

$$\begin{aligned} x[2b_2 g_{zz} + b2z g_{zz,z} + (a_1 + a_2 x) g_{zz,x}] - D g_{zz} &= 0, \\ a_1 + x(2a_2 + b_2) g_{xz} + b_2 z g_{zz} + x[b_2 y g_{xz,z} + (a_1 + a_2 x) g_{xz,x}] - D g_{xz} &= 0, \\ 2(a_1 + 2a_2 x) g_{xx} + 2b_2 z g_{xz} + x[b_2 z g_{xx,z} + (a_1 + a_2 x) g_{xx,x}] - D g_{xx} &= 0. \end{aligned} \tag{36}$$

The metric can be determined from the above equations by first solving for g_{zz} and hence g_{xz} and g_{xx}. The expressions are lengthy but simplify in the limit $z = 0$ (equivalently, $y = 0$) and read

$$\begin{aligned} g_{xx} &= x^{\frac{D}{a_1}-2}(a_2 x + a_1)^{-\frac{D}{a_1}-2}, \\ g_{xz} &= x^{\frac{D}{a_1}-1}(a_2 x + a_1)^{-\frac{b_2}{a_2}-\frac{a_1+D}{a_1}}, \\ g_{zz} &= x^{\frac{D}{a_1}}(a_2 x + a_1)^{-\frac{2b_2}{a_2}-\frac{D}{a_1}}. \end{aligned} \tag{37}$$

Having obtained the metric, we now focus on geodesics on the PM. As usual, we start with the tangent vector $(x'(\lambda), z'(\lambda))$. The fact that the metric is independent of z leads to

$$\begin{aligned} x'(\lambda) &= \sqrt{2} x^{1-\frac{D}{2a_1}}(a_1 + a_2 x)^{1+\frac{D}{2a_1}}, \\ z'(\lambda) &= -\frac{1}{\sqrt{2}} x^{-\frac{D}{2a_1}}(a_1 + a_2 x)^{\frac{D}{2a_1}+\frac{b_2}{a_2}}. \end{aligned} \tag{38}$$

The first of the above equations can then be solved to give

$$\lambda = \frac{\sqrt{2}}{D}\left(\frac{x}{a_1 + a_2 x}\right)^{\frac{D}{2a_1}}, \tag{39}$$

which was also the expression obtained in the model studied in Sec. III A. The scalar curvature and the expansion parameter are given by

$$\begin{aligned} R &= -2b_2 x^{1-\frac{D}{a_1}}(a_2 x + a_1)^{\frac{D}{a_1}}(-2a_2 x - 2a_1 + 2b_2 x - D), \\ \Theta &= \frac{1}{\sqrt{2}}(D - 2b_2 x)\left(\frac{x}{a_1 + a_2 x}\right)^{-\frac{D}{2a_1}}. \end{aligned} \tag{40}$$

Now note that in this case, $a_1 = \epsilon$ is positive. Hence, the scalar curvature and the expansion parameter seem to diverge at the Gaussian fixed point $x \to 0$. However, as mentioned before, the coordinate transformation used to derive Eq. (35) cannot be trusted here. Let us thus focus on the Wilson-Fisher fixed point, where from Eq. (39) we also see that $\lambda \to \infty$. This may look at odds with the linearized result (where the geodesic distance goes to zero near a nontrivial fixed point), but that is not so. To see this, note that in our nonlinear analysis, we have not linearized about the Wilson-Fisher point. The geodesic distance here is still measured from $x = y = 0$, so our analysis simply reflects the fact that the Wilson-Fisher fixed point is an infrared fixed point. By combining Eqs. (39) and (40), we recover the relations $R \sim \lambda^{-2}$ and $\Theta \sim \lambda^{-1}$ at this fixed point.

C. Case VI

Another interesting situation occurs when one of the variables is marginally irrelevant. We will focus on a set of RG flow equations of the form

$$\dot{x} = a_1 x^2, \quad \dot{y} = b_1 y + b_2 x y. \tag{41}$$

These are the RG flow equations for the Ising model in four dimensions (with $a_1 = -72$, $b_1 = 2$, and $b_2 = -24$), which follows from our analysis of the previous section, by setting $\epsilon = 0$. Clearly, the results of that analysis cannot be used here simply by setting the coefficient of the linear term to zero since the metric components of Eq. (37) are ill defined in this limit. To perform this analysis, we start by defining a new variable, $z = y e^{b_1/(a_1 x)}$. Since a_1 is taken to be negative, this transformation is well defined in the limit $x \to 0$. It follows that the RG equations can be written more conveniently as

$$\dot{x} = a_1 x^2, \quad \dot{z} = b_2 x z. \tag{42}$$

In terms of the variables x and z, the equations for the metric components in this case are seen to be

$$\begin{aligned} x(2b_2 g_{zz} + b_2 z g_{zz,z} + a_1 x g_{zz,x}) - D g_{zz} &= 0, \\ (2a_1 + b_2) x g_{xz} + b_2 z g_{zz} + x(b_2 z g_{xz,z} + a_1 x g_{xz,x}) - D g_{xz} &= 0, \\ 4a_1 x g_{xx} + 2b_2 z g_{xz} + x(b_2 z g_{xx,z} + a_1 x g_{xx,x}) + D g_{xx} &= 0. \end{aligned} \tag{43}$$

As before, the first of these equations can be solved to obtain g_{zz}, which can in turn be used to find g_{xz} and hence g_{xx}. The solutions for the metric components read

$$g_{xx} = \frac{1}{a_1^2} x^{-\frac{2b_2}{a_1}-4} e^{-\frac{D}{a_1 x}} \left(a_1 x^{\frac{b_2}{a_1}} - b_2 x z\right)^2,$$
$$g_{xz} = \frac{1}{a_1} x^{-\frac{2(a_1+b_2)}{a_1}} e^{-\frac{D}{a_1 x}} \left(a x^{\frac{b_2}{a_1}} - b_2 x z\right),$$
$$g_{zz} = k_1 x^{-\frac{2b_2}{a_1}} e^{-\frac{D}{a_1 x}}. \quad (44)$$

Here, k_1 is a constant that we will set to 2 following our previous discussions. We first record the expression for the scalar curvature calculated from the metric of Eq. (44) in the limit $z = 0$,

$$R = \frac{2b_2 x e^{\frac{D}{a_1 x}} (2a_1 x - 2b_2 x + D)}{a_1^2}. \quad (45)$$

Clearly, with negative a_1, R rapidly goes to zero for small values of x. Now let us consider geodesics on the manifold defined by the metric of Eq. (44). For the four-dimensional Ising model, $a_1 = -72$ and $b_2 = -24$, and for small x and z, we can ignore the xz pieces in Eq. (44), which renders the said metric independent of z. As before, we consider a tangent vector to a geodesic, which we denote by $(x'(\lambda), z'(\lambda))$. Here, λ is an affine parameter along the geodesic. Normalization of the tangent vector, along with the fact that the metric does not depend on z, implies that

$$x'(\lambda) = \sqrt{2} x(\lambda)^2 e^{\frac{D}{2a_1 x(\lambda)}}, \quad z'(\lambda) = -\frac{1}{\sqrt{2}} x(\lambda)^{\frac{b_2}{a_1}} e^{\frac{D}{2a_1 x(\lambda)}}. \quad (46)$$

Solving the first of these equations, we obtain

$$x(\lambda) = -\frac{D}{2a_1 \ln\left(\frac{-D\lambda}{\sqrt{2}a_1}\right)}. \quad (47)$$

If a_1 is negative, which happens for the four-dimensional Landau-Ginzburg model, we see that at $x = 0$, λ has to go to infinity, although logarithmically [34]. After some algebra we obtain here

$$R = -\frac{2b_2\left[a_1 \ln\left(-\frac{D\lambda}{\sqrt{2}a_1}\right) - a_1 + b_2\right]}{a_1^2 \lambda^2 \ln^2\left(-\frac{D\lambda}{\sqrt{2}a_1}\right)},$$
$$\Theta = \frac{1}{\lambda}\left(1 + \frac{b_2}{a_1 \ln\left(-\frac{D\lambda}{\sqrt{2}a_1}\right)}\right). \quad (48)$$

In the limit $\lambda \to \infty$, we finally obtain

$$R \sim \frac{1}{\lambda^2 \ln \lambda}, \quad \Theta \sim \frac{1}{\lambda}. \quad (49)$$

The first of these equations indicates that the Ricci scalar picks up logarithmic corrections to geometric scaling relations in four dimensions.

IV. CONCLUSIONS

In this paper, we have provided evidence that scale invariance in the vicinity of a critical point can provide valuable information on the metric of the parameter manifold in classical and quantum phase transitions in a unified fashion. In particular, this method can be applied to systems that are not exactly solvable to read off the scaling behavior of the metric (and hence related quantities like the fidelity susceptibility) near criticality. Our method complements the work of [6] to determine scaling patterns for information geometric quantities. While the work of [11] utilizes the RG equations up to first order to read off the scaling of the metric in classical phase transitions, our method explicitly solves for the metric components in a variety of nonlinear examples.

While most of this paper dealt with classical phase transitions, we gave one nontrivial example of a quantum phase transition that can be studied in this framework. This extends the study of information geometry to novel settings. We have seen here that the relations $R \sim \lambda^{-2}$ and $\Theta \sim \lambda^{-1}$ are universal, except for the four-dimensional Ising model, where the former relation picked up logarithmic corrections. This strengthens the claim made in [9] about universal geometric critical exponents.

We should mention here that in two dimensions, the scalar curvature R and the expansion parameter Θ satisfy the Raychaudhuri equation $\Theta^2 + \Theta' + R/2 = 0$, where a prime denotes a derivative with respect to the affine parameter λ [35]. In principle, given R (Θ), this equation can be used to determine Θ (R). However, it is not always possible to solve this equation analytically. In all the examples considered in this paper, as a cross-check on our results, we have verified that the Raychaudhuri equations are indeed satisfied.

In this paper, we have considered a class of examples where the information metric was obtained from the set of RG flow equations. Clearly, one might argue that this may not be the case for more generic examples. Consider, for example, an RG equation of the form $\dot{x} = a_1 x + a_2 y$, $\dot{y} = b_1 y + b_2 x$. In this case, the homothetic vectors do not have an analytic solution, which can be checked. This is a caveat to our analysis. A further criticism might be that terms that are higher order than those considered here are difficult to take care of. We note, however, that in general, such terms might be solved iteratively; that is, one can use perturbation theory to solve the differential equations for the components of the metric tensor. This is substantially more complicated than the analysis presented here, and a full study of the same is left for the future.

It will be interesting to extend the present analysis to cases in which the parameter manifold has dimensionality higher than two. One example was commented upon in this work, but a broader analysis might reveal interesting facts about the geometry of the renormalization group, as higher-dimensional PMs offer more structure and, in particular, more scalar invariants. What these scalars mean in the context of RG will be an interesting issue for future investigations. It might also be interesting to consider the role of time in information theory [36] in the context of the models considered here. This issue is currently under investigation. Finally, it might be useful to investigate information geometry in the context of Kosterlitz-Thouless-type phase transitions in the two-dimensional XY model. Preliminary analysis indicates that here the scalar curvature of the information metric diverges exponentially, in line with the behavior of the correlation length. However, this case requires further understanding.

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[20] For a two-dimensional manifold with coordinates (x^1,x^2), $R = \frac{2}{\sqrt{g}}[\frac{\partial}{\partial x^2}(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{11}) - \frac{\partial}{\partial x^1}(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{12})]$, where g is the determinant of the metric and the Christoffel symbols are defined by $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\zeta}(\frac{\partial g_{\zeta\nu}}{\partial x^{\rho}} + \frac{\partial g_{\zeta\rho}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\zeta}})$. For higher-dimensional manifolds, the formula is standard but more complicated. See, e.g., [19].

[21] The converse is not true: geodesic focusing does not always indicate a singularity. This is seen, for example, by considering a two-sphere, where great circles focus at the poles which are regular points on the sphere.

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[25] In this particular example, we could also have begun with a diagonal metric with g_{xy} set to zero. This choice also gives a consistent set of differential equations for the metric and leads to the same conclusions as those with a generic off-diagonal form of the metric. However, we proceed with a generic of-diagonal metric to keep the discussion general and to connect to a known example later in this section.

[26] In the examples that we study, the components of the homothetic vector field are obtained from the RG equations in the same way. These are then put in Eq. (4), and equations for the metric components are obtained.

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[30] In this case, it is possible to adjust the constant k_3 appearing in Eq. (16) to very large values to keep the value of λ close to zero. But this requires fine-tuning of the constant k_3. In what follows, we will assume that λ becomes large along an irrelevant (stable) direction.

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[34] If, on the other hand, we wanted $\lambda \to 0$, then x becomes negative, which renders the metric components of Eq. (44) imaginary, which we should avoid from physical considerations.

[35] In two dimensions, the components of the Ricci tensor are related to the metric $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$, with R being the Ricci scalar. This is the required ingredient in this derivation.

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