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Gravity-driven soap film dynamics in subcritical regimes

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We undertake the analysis of soap-film dynamics with the classical approach of asymptotic expansions. We focus our analysis in vertical soap film tunnels operating in subcritical regimes with elastic Mach numbers $M_e = O_{(10^{-1})}$. Considering the associated set of nondimensional numbers that characterize this flow, we show that the flow behaves as a two-dimensional (2D) divergence free flow with variable mass density. When the soap film dynamics agrees with that of a 2D and almost constant mass density flow, the regions where the second invariant of the velocity gradient is non-null correspond to regions where the rate of change of film thickness is non-negligible.

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I. INTRODUCTION

Since the mid-1990s a large number of studies concerning the dynamics of films driven to flow under the action of various forces such as gravity, centrifugation, capillarity, thermocapillarity, and intermolecular forces has been a topic of interest for different researchers [1]. The usual modeling of this dynamics relies on exploiting the disparities in the length scales arising naturally in thin-film flows. In such flows the thickness of the film is much smaller than the characteristic length scales of the midplane flow, enabling the use of small parameters for perturbation expansions. This may simplify the dynamics involved in the depth coordinate through a rational asymptotic approach and thus reduce the Navier-Stokes equations to a more tractable, yet highly nonlinear, set of partial differential equations, capable of capturing the dominant physics.

Within this framework the fluid dynamics of soap films has been studied in different works with different objectives such as determining the stability of the film or establishing the possible analogies that can be found between the behaviors of three-dimensional (3D) draining films and 2D general flows. Among the last group of studies is the work of Chomaz [2], who considered horizontal films and performed an asymptotic analysis under the assumption that the bending wave velocity v_b (velocity of the varicose mode that produces antisymmetric deformations of the sheet) is much larger than the mean velocity of the flow. From the leading-order equations obtained, two limiting cases were derived. The first one is related to the behavior as an incompressible 2D flow. In this case the midplane velocity field distribution of the film complies with the incompressible 2D Navier-Stokes equation and the sufficient condition found for this is that the typical flow velocity U be very small compared to the elastic wave velocity v_e (or, equivalently, elastic Mach number $M_e = \frac{U}{v_e} \ll 1$). The other case corresponds to situations at much larger velocities in which an analogy could be expected with the compressible 2D Euler equation of a compressible gas. However it was found that the observed dynamics becomes very specific to soap films and could only correspond to a two-dimensional compressible gas flow with an unusual ratio of specific heat capacities (equal to unity).

The elastic wave velocity v_e is the speed at which the symmetric perturbations of the film thickness travel on the surface of the film. In general, the symmetric and antisymmetric modes propagate at similar velocities ($v_e \sim v_b$) but often in experiments v_b is smaller than v_e [2,3]. The value of v_e is related to the surfactant concentration at the surface that in turn depends on the soap molecule flux between surface and bulk. In consequence, it is strongly dependent on the experimental device used to produce the film and on the soap solution considered. The ratio of the midplane flow velocity with this celerity, expressed by the value of M_e, enables us to classify the flow as subcritical or supercritical. Without any information on its value, a simple identification of supercritical flow behavior on an experimental facility can be done, for instance, by piercing the stream with pins that will produce diamondlike fringes [4]. The supercritical regime $M_e > 1$ has attracted the attention of different authors [5,6]. The elastic Marangoni shocks characteristic of these regimes have been analyzed and the experimental evidence has been contrasted against models.

The soap tunnels may have a horizontal [7,8] or vertical configuration [9–12]. In vertical soap tunnels typical velocities are larger than the horizontal case and in many experimental conditions it is observed that $M_e = O_{(10^{-1})}$. Here we propose analysis if, under these experimental situations, it is possible for the draining film to behave as a 2D flow with divergence-free velocity fields. To derive the leading master equations we will relax the assumption concerning the relative smallness of the mean flow velocity compared to elastic wave velocity and directly assume that this lies in the range of our interest. This work therefore tries to link the soap film dynamics with the one of 2D flows in a "gray zone" not covered by previous analysis. Let us note that the soap films and classical 2D flows share the existence of a supercritical behavior, respectively, when M_e or $M_a = \frac{U}{c}$ (with c = sound speed) are equal or larger than 1 and also that both flows show velocity fields that are divergence free when the values of these nondimensional numbers are quite small compared to unity, but this coincidence does not necessary imply that both flows will show the same behavior for intermediate values.

Our work could compare in some sense with that of Ida and Miksis [13] in which it was considered the $M_b = \frac{U}{v_b} \sim O_{(1)}$. However, in their work the film was not flat. The curvature introduced in their analysis gives other leading-order equations (in which pressure terms play a significant role) and cannot be applied for the case we study.

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One of the most attractive characteristics of soap film is the quality of flows visualization that can be attained. The usual diagnostic of soap film dynamics is performed with the interferometry technique that gives patterns of fringes that are function of the thickness variations of the soap film. Other techniques to measure thickness of the film in single points or larger regions have also been applied to soap film analysis [14–16]. The thickness variations of the film have been correlated by different authors to the dynamics of either passive tracers or pressure and vorticity fields [17–20]. The work of Chomaz has, however, established that when $M_e \ll 1$ any of this dynamics could be observed depending on the experimental conditions and particularly on the kinetics to attain an equilibrium between the bulk and surface concentration of surfactant. In this work we also try to give a suitable interpretation of the observed patterns for the regime under analysis.

A. Soap film equations

Soap films used in hydrodynamics experiments may be considered a three-layer structure: two surfaces and a bulk fluid in between (Fig. 1).

The vertical unperturbed film has a thickness $2H$ that we assume constant in the region of interest. The film is driven by gravity forces and we consider that the dynamics of the bulk film variables (velocity $\mathbf{u}$ and pressure p) can be described by the 3D incompressible Navier-Stokes equation,

$$\nabla \cdot \mathbf{u} = 0,$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}. \quad (1)$$

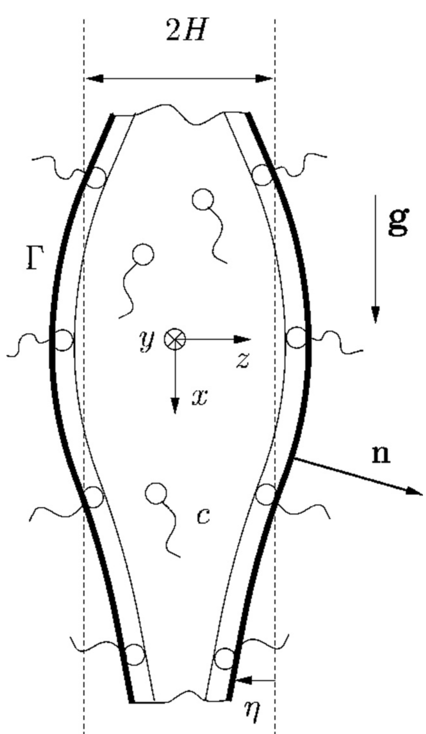

FIG. 1. Soap film internal structure.

The perturbed interfaces are at $z = H + \eta$. Therefore we will consider the following conditions:

$$p - p_a + 2\mathcal{C}\sigma = \mu \hat{\mathbf{n}} \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^t) \cdot \hat{\mathbf{n}},$$
$$0 = \hat{\mathbf{t}}_i \cdot \nabla_s \sigma + \mu \hat{\mathbf{t}}_i \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^t) \cdot \hat{\mathbf{n}}, \quad (2)$$

where ∇_s is the gradient surface operator, $\nabla_s = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \nabla$, with $\mathbf{I}$ the spatial idemfactor. These equations represent the Young-Laplace law projected on the unit normal $\hat{\mathbf{n}}$ and on the tangents $\hat{\mathbf{t}}_\mathbf{i}$ to the interface. In the first expression $\mathcal{C}$ is the mean surface curvature, p_a the ambient pressure, and μ the viscosity coefficient,

$$2\mathcal{C} = -\nabla \cdot \hat{\mathbf{n}}$$
$$= \frac{\frac{\partial^2 \eta}{\partial x^2}\left[1 + \left(\frac{\partial \eta}{\partial y}\right)^2\right] - 2\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 \eta}{\partial x \partial y} + \frac{\partial^2 \eta}{\partial y^2}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]}{\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]^{\frac{3}{2}}}. \quad (3)$$

Note that like in Chomaz and for the sake of simplicity we have not incorporated any drag of the film with the surrounding atmosphere. The following kinematic condition is used to link the surface deformation with w the velocity component along z coordinate,

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta = w. \quad (4)$$

The surface tension σ is related to the surfactant concentration at the surface through the following expression:

$$\sigma = \sigma_a - \sigma_r \Gamma, \quad (5)$$

in which σ_a is the surface tension in the absence of surfactant and the elasticity $\sigma_r = \frac{\partial \sigma}{\partial \Gamma}|_{\Gamma=0}$. The dynamics of the surfactant concentration is quite complex and can be described as follows:

$$\frac{\partial \Gamma}{\partial t} + \Gamma(\nabla_s \cdot \hat{\mathbf{n}})(\mathbf{u} \cdot \hat{\mathbf{n}}) + \nabla_s \cdot (\mathbf{u}_s \Gamma) = D_s {\nabla_s}^2 \Gamma + j, \quad (6)$$

where $\mathbf{u}_\mathbf{s}$ represents the components of the velocity $\mathbf{u}$ along the surface $[\mathbf{u}_\mathbf{s} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{u}]$. In this equation the second term on the left-hand side represents the rate of change of concentration due to the dilatation of the surface, a contribution that is simply related to the product of the mean curvature and the normal velocity [21]. The first term of the right-hand side equation accounts for the process of diffusion of soap film molecules at the surface and is usually described with a constant value for the surface diffusivity coefficient D_s; meanwhile, the flux of soap molecules between the bulk and the surface j is generally described as an adsorption-desorption process for which time constants have to be specified.

B. Nondimensional problem

To obtain the set of equations in terms of nondimensional variables we will consider the same notation and choice of parameters as in the work of Chomaz [2]. As is usually done, we propose an asymptotic expansion of the solution of the problems in terms of a small parameter $\epsilon = \frac{H}{L}$ that is the ratio between the mean half thickness of the film H and the length scale of the in-plane velocity variations L. This parameter is typically quite small $[\epsilon = O_{(10^{-3})}]$. As main differences with

Chomaz work in which it is assumed that $\frac{M_b}{\epsilon} \sim \frac{M_e}{\epsilon} = O_{(1)}$, we will analyze flow regimes in which

$$M_e = O_{(10^{-1})}. \tag{7}$$

This condition agrees well with the situations encountered in vertical soap tunnels for which the free-stream velocities are frequently $U \sim 1-2$ m/s and the elastic wave velocities v_e attain values ~ 4 m/s. For the flows taking place in these tunnels we will take into account as typical values of the mean thickness $2H \sim 10\ \mu$m and length scales of the midplane flow $L \sim 10^{-3}-10^{-2}$ m. The value of the the antisymmetric bending wave velocity v_b depends on fluid solution and on the film thickness. In the facilities we consider it is verified in general that the bending mode speed v_b is of the same order as v_e but customarily v_b is smaller than v_e, which implies that M_b is larger than M_e. The development of the antisymmetric (bending) mode associated with this kind of wave can be excluded as, in general, it is not present in experiments.

1. Scaling

The following nondimensionalization of variables is proposed for the space variables

$$\begin{aligned} &x = x'L; \quad y = y'L; \quad z = z'\epsilon L; \\ &\eta = \eta' H; \quad Z' = \eta' + 1 \quad \text{(interface)}. \end{aligned} \tag{8}$$

Time and velocity variables are scaled with

$$t = t'\frac{L}{U}; \quad u = u'U; \quad v = v'U; \quad w = w'\epsilon U; \tag{9}$$

meanwhile, for surface tension and pressure we take

$$\begin{aligned} \sigma &= \sigma_m + \sigma'\rho H \mathrm{U}^2 = \sigma_m + \sigma'\rho\epsilon L \mathrm{U}^2 \\ p &= p_a + p'\epsilon\sigma_m/L. \end{aligned} \tag{10}$$

Here σ_m is the mean surface tension of the film. It is important to signal here that we keep the same scaling for pressure as did Chomaz. Other authors [4], for instance, have proposed the scaling for pressure with $p = p_a + p'\frac{\mathrm{U}\mu}{L}$. In the derived set of equations with this last scaling the terms where pressure appears become altered by $\epsilon\chi = \epsilon\frac{\sigma_m}{\mathrm{U}\mu}$. Typical values for the experiments we consider give $\chi = O_{(1)}$. Finally, the surfactant concentration at the surface is scaled as

$$\Gamma = \Gamma'\Gamma_m \tag{11}$$

with Γ_m the surface mean concentration.

2. Nondimensional equations

For convenience let us drop the primes of all nondimensionalized quantities; the nondimensional bulk equations then read as

$$\begin{aligned} \nabla\cdot\mathbf{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\mathrm{Fr}^2} - \frac{\epsilon^2}{M_b^2}\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial z^2}\right) \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\epsilon^2}{M_b^2}\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial z^2}\right) \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{M_b^2}\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{\epsilon^2}\frac{\partial^2 w}{\partial z^2}\right), \end{aligned} \tag{12}$$

in which $\mathrm{Re} = \frac{UL}{\nu}$, $\mathrm{Fr}^2 = \frac{U^2}{gL}$, $M_b = \frac{U}{v_b}$, and $v_b = \sqrt{\frac{\sigma_m}{H\rho}}$. Note that this last velocity depends on the experimental condition considered through the amount of surfactant concentration present in the solution considered and through the thickness of the film. At the same time, the thickness of the film depends on the flow rate considered and width of the film. At the interface we have

$$\begin{aligned} &\frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} + V\frac{\partial\eta}{\partial y} = W \\ &P + 2\mathcal{C}\left(1 + M_b^2\sigma\right) = 2M_b^2\frac{1}{\epsilon^2\mathrm{Re}}\left(\frac{\partial w}{\partial z} - \frac{\partial\eta}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial u}{\partial z} + O_{(\epsilon^2)}\right) \\ &2\mathcal{C} = \frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + O_{(\epsilon^2)} \\ &\frac{\partial\sigma}{\partial x} = \frac{1}{\epsilon^2\mathrm{Re}}\left\{\frac{\partial u}{\partial z} + \epsilon^2\left[-2\frac{\partial\eta}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial\eta}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial w}{\partial x} - \left(\frac{\partial\eta'}{\partial x'}\right)^2\frac{\partial u'}{\partial z'} - \frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial v}{\partial z} + 2\frac{\partial\eta}{\partial x}\frac{\partial w}{\partial z}\right] + O_{(\epsilon^4)}\right\} \\ &\frac{\partial\sigma}{\partial y} = \frac{1}{\epsilon^2\mathrm{Re}}\left\{\frac{\partial v}{\partial z} + \epsilon^2\left[-2\frac{\partial\eta}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial\eta}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial w}{\partial y} - \left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial v}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial v}{\partial z} + 2\frac{\partial\eta}{\partial y}\frac{\partial w}{\partial z}\right] + O_{(\epsilon^4)}\right\} \\ &\sigma = \frac{1}{M_e^2}(1 - \Gamma). \end{aligned} \tag{13}$$

C. Asymptotic expansion

The discrepancy of scales allows us to introduce the small parameter $\epsilon = \frac{H}{L}$. Since this parameter is quite small compared to unity we can use this knowledge to identify those terms that vary slowly across and along the fluid layer and thus can be discarded. This procedure then gives a quantifiable and legitimate way of reducing the equations to a simpler form; it also enables us to make clear the limits when the theory will break down. The following expansion is proposed for all variables:

$$f = f_0 + \epsilon^2 f_2 + \epsilon^4 f_4 + \cdots . \tag{14}$$

We define the 2D operators

$$\nabla_\mathbf{0} = \frac{\partial}{\partial x}\hat{\mathbf{r}}_\mathbf{x} + \frac{\partial}{\partial y}\hat{\mathbf{r}}_\mathbf{y} \tag{15}$$

and

$$\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = \nabla_\mathbf{0} \cdot \mathbf{u_0}. \tag{16}$$

1. Leading-order equations

Considering the first-order terms of the expansion we have for the bulk

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0, \tag{17}$$

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial z^2}, \tag{18}$$

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial^2 v_0}{\partial z^2}, \tag{19}$$

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial^2 w_0}{\partial z^2}. \tag{20}$$

At the interface we have

$$\eta = \eta_0 \Longrightarrow Z = 1 + \eta_0, \tag{21}$$

$$2\mathcal{C}_0 = \frac{\partial^2 \eta_0}{\partial x^2} + \frac{\partial^2 \eta_0}{\partial y^2}, \tag{22}$$

$$\frac{\partial \eta_0}{\partial t} + U_0\frac{\partial \eta_0}{\partial x} + V_0\frac{\partial \eta_0}{\partial y} = W_0. \tag{23}$$

As we consider

$$M_b = \frac{M}{\epsilon} = O_{(1)} \tag{24}$$

we have

$$0 = 2M_b^2\frac{1}{\mathrm{Re}}\frac{\partial w_0}{\partial z}|_Z. \tag{25}$$

Note that if it were considered as Chomaz did,

$$M = O_{(1)}, \tag{26}$$

the following equation would result:

$$P_0 + \left(\frac{\partial^2 \eta_0}{\partial x^2} + \frac{\partial^2 \eta_0}{\partial y^2}\right)\left[1 + \left(\frac{M_b}{M_e}\right)^2(1-\Gamma_0)\right] = 2M^2\frac{1}{\mathrm{Re}}\frac{\partial w_0}{\partial z}\ |_Z. \tag{27}$$

For the rest of equations we have

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial u_0}{\partial z_0}\bigg|_Z, \tag{28}$$

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial v_0}{\partial z_0}\bigg|_Z, \tag{29}$$

$$\frac{\partial \Gamma_0}{\partial t_0} + \frac{\partial (u_0\Gamma_0)}{\partial x} + \frac{\partial (v_0\Gamma_0)}{\partial y_0} + \Gamma_0\left(\frac{\partial^2 \eta_0}{\partial x^2} + \frac{\partial^2 \eta_0}{\partial y^2}\right)W_0 = 0, \tag{30}$$

$$\sigma_0 = M_e^{-2}(1-\Gamma_0). \tag{31}$$

As

$$0 = \frac{1}{\mathrm{Re}}\frac{\partial^2 w_0}{\partial z^2}, \tag{32}$$

we can write that $\frac{\partial w_0}{\partial z}$ is only a function of x and y. At the interface

$$\frac{\partial w_0}{\partial z}\ |_Z = 0. \tag{33}$$

Therefore at the bulk it will be verified that

$$\frac{\partial w_0}{\partial z} = 0. \tag{34}$$

Thus the velocity w_0 will be a function only of x and y but, for symmetry reasons, we have

$$w_0\ |_{Z=0} = 0. \tag{35}$$

Hence we conclude that $w_0 = 0$ in any point of the flow. We also have the incompressibility condition defined as

$$\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = \nabla_\mathbf{0} \cdot \mathbf{u_0} = 0 \tag{36}$$

and this is satisfied for the leading-order term. Also as

$$w_0 = \frac{\partial \eta_0}{\partial t} + u_0\frac{\partial \eta_0}{\partial x} + v_0\frac{\partial \eta_0}{\partial y} = 0. \tag{37}$$

The leading-order deformations of the film will not change with time. Therefore when the initial deformations of the membrane are negligible [$\eta(x,y,0) = 0$], the surface at the leading order remains flat, that is,

$$\eta_0(x,y,t) = 0. \tag{38}$$

Note that in this set of equations u_0, v_0, and Γ_0 remain undetermined.

2. Next-order equations

Considering terms of the next order we have

$$\nabla \cdot \mathbf{u_2} = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0,$$
$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = \frac{1}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}\right)$$
$$\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2}\right)$$
$$0 = -\frac{1}{M_b^2}\frac{\partial p_0}{\partial z} + \frac{1}{\mathrm{Re}}\frac{\partial^2 w_2}{\partial z^2}. \quad (39)$$

At the interface $z = Z$ we have

$$W_2 = \frac{\partial \eta_2}{\partial t} + U_0\frac{\partial \eta_2}{\partial x} + V_0\frac{\partial \eta_2}{\partial y}, \quad (40)$$
$$P_0 = 2M_b^2\frac{1}{\mathrm{Re}}\frac{\partial w_2}{\partial z}|_Z. \quad (41)$$

Also we can write

$$\frac{\partial \sigma_0}{\partial x} = \frac{1}{\mathrm{Re}}\frac{\partial u_2}{\partial z}|_Z, \quad (42)$$
$$\frac{\partial \sigma_0}{\partial y} = \frac{1}{\mathrm{Re}}\frac{\partial v_2}{\partial z}|_Z. \quad (43)$$

As the symmetry imposes that (u,v) are even equations at the bulk, this determines that (u_2,v_2) are parabolic in z since all other terms in these equations are leading-order variables that do not depend on z. Taking into account the equations of the interface we can write

$$u_2 = \mathrm{Re}\frac{\partial \sigma_0}{\partial x}\frac{z^2}{2Z} + U_2$$
$$v_2 = \mathrm{Re}\frac{\partial \sigma_0}{\partial y}\frac{z^2}{2Z} + V_2, \quad (44)$$

where (U_2,V_2) are functions of x and y and $Z = 1 + \eta_2\epsilon^2$. Therefore the momentum equation reads as

$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = \frac{1}{\mathrm{Fr}^2} + \frac{1}{Z}\frac{\partial \sigma_0}{\partial x}\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)$$
$$\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} = \frac{1}{Z}\frac{\partial \sigma_0}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right). \quad (45)$$

Introducing the 2D operators

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{u_0}\nabla_\mathbf{0}; \quad \nabla_0^2 = \nabla_\mathbf{0} \cdot \nabla_\mathbf{0} \quad (46)$$

and considering (45) we have

$$\nabla_\mathbf{0} \cdot \mathbf{u_0} = 0, \quad \frac{D_0\mathbf{u_0}}{Dt} = \frac{1}{\mathrm{Fr}^2} - \frac{1}{ZM_e^2}\nabla_\mathbf{0}\Gamma_0 + \frac{1}{\mathrm{Re}}\nabla_0^2\mathbf{u_0}. \quad (47)$$

As we can see under the conditions we analyze, the leading-order equations that govern the soap film dynamics enables us to undertake an analogy with the case of an incompressible two-dimensional fluid flow. The role of pressure is here determined by the surface tension concentration Γ_0 that becomes an equivalent pressure. The role of the equivalent density is determined by ZM_e^2. Also it is of interest here to analyze the laws that determine the variables of the bulk. For a fixed value of x and y the first derivative of the normal velocity has to be parabolic in z because

$$-\frac{\partial w_2}{\partial z} = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}. \quad (48)$$

As u_2 and v_2 are both parabolic in z, then w_2 will be cubic in z [with a null value at the midplane $w_2(0)$ for symmetry reasons] and the second derivative $-\frac{\partial^2 w_2}{\partial z^2}$ has to be linear in z. For this reason we can write

$$-\frac{\partial^2 w_2}{\partial z^2} = \alpha z + \beta, \quad (49)$$

where $\alpha = \alpha(x,y)$ and $\beta = \beta(x,y)$. As pressure is related with the normal velocity through Eqs. (39) we can observe that the derivative of the pressure $\frac{\partial p_0}{\partial z}$ will be linear with z. Because of the symmetry of the deformations of the film at the midplane $\frac{\partial p_0}{\partial z}|_{z=0} = 0$. As a consequence, the pressure p_0 will be parabolic with z across the film thickness and thus β has to be equal to 0.

II. DISCUSSION

It is of interest here to compare our results with the previous results of Chomaz which, as noted, were performed under the assumption that $M_e = O_{(\epsilon^2)}$. With this assumption the nondimensional thickness variations appear as leading-order terms. The leading-order solution changes when modifying the relative values of the nondimensional parameters. Therefore it is not surprising that in our analysis of the leading-order terms the film remains flat and surface deformations appear with higher-order terms. Under both analyses when the flow behaves as an incompressible 2D flow, the equivalent pressure is determined by the leading order of Γ_0. Although Chomaz obtained that the flow would correspond to a fluid with constant mass density we found that it would correspond to a fluid with an equivalent density that will not be uniform in the fluid domain. Typical values of Z lie in the range comprised by ∼0.9−1.1. Therefore it is expected that the equivalent density will show variations of a few percentages. As the flow is divergence free and the density variations from the background density is of a few percentages, an immediate analogy with a Boussinesq flow seems pertinent. The importance of the buoyancy term characteristic of this kind of flow it is not expected to be very considerable as flow velocities are relatively large and therefore the potential energy of typical density perturbation of a parcel of fluid will be, in general, much lower than the associated kinetic energy. Non-negligible density variations may have an influence on vorticity dynamics. We introduce here the operator $\nabla_{\mathbf{0}}\times$ such that for any vector $\mathbf{F} = (F_x, F_y)$

$$\nabla_{\mathbf{0}} \times \mathbf{F} = \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right)\mathbf{r_z}. \tag{50}$$

Applying this operator to the Eq. (47) and in the inviscid limit (Re $\longrightarrow \infty$) we obtain the following equation for the 2D vorticity $\boldsymbol{\varpi} = \varpi \mathbf{r_z}$:

$$\frac{D_0 \boldsymbol{\varpi}}{Dt} = \left(\frac{1}{Z M_e}\right)^2 \nabla_{\mathbf{0}} Z \times \nabla_{\mathbf{0}} \Gamma_0, \tag{51}$$

where we observe that the vorticity of a fluid particle may change with time when the right-hand-side term is non-negligible. The nonalignment of the gradients of thickness (density) and surfactant concentration at the surface (pressure) acts therefore as a source term for vorticity. This effect is obviously absent when the density of the fluid is constant. The contribution of this baroclinic term may be interesting to researchers working in the field of 2D turbulence, as it could be possible to tune surfactant concentration to enhance or reduce such an effect. Previous deviations from 2D hydrodynamics found in experiments may be due to such an effect [9,20].

A simple analysis assuming that the parameter $\Theta = Z M_e^2$ remains small can still be performed. We proceed therefore to a second expansion of any variable in terms of Θ

$$g_0 = g_{0_0} + \Theta g_{0_1} + \Theta^2 g_{0_2} + \cdots . \tag{52}$$

Hence at leading-order terms

$$\nabla_{\mathbf{0}} \Gamma_{00} = 0 \Rightarrow \Gamma_{00} = 1 \tag{53}$$

and to order Θ the system reduces to

$$\nabla_{\mathbf{0}} \cdot \mathbf{u_{00}} = 0, \tag{54}$$

$$\frac{D_0 \mathbf{u_{00}}}{Dt} = \frac{1}{\mathrm{Fr}^2} - \nabla_{\mathbf{0}} \Gamma_{01} + \frac{1}{\mathrm{Re}} \nabla_0^2 \mathbf{u_{00}}, \tag{55}$$

which represent the equations of an incompressible flow with a pseudodensity constant equal to 1 and in which the pseudopressure has to be identified as Γ_{01}. Note that in this case the set of equations agree with the one obtained by Chomaz, as expected.

A. Analysis of film thickness field

In view of the following:

$$\frac{\partial \sigma_0}{\partial x} = \frac{1}{\mathrm{Re}} \frac{\partial u_2}{\partial z} |_Z; \quad \frac{\partial \sigma_0}{\partial y} = \frac{1}{\mathrm{Re}} \frac{\partial v_2}{\partial z} |_Z , \tag{56}$$

taking derivatives of these equations with respect to x and y we get

$$\nabla_0^2 \sigma_0 = \frac{1}{\mathrm{Re}} \frac{\partial}{\partial z} \left(\frac{\partial u_2(x,y)}{\partial x} + \frac{\partial v_2(x,y)}{\partial y}\right) |_Z , \tag{57}$$

$$= -\frac{1}{\mathrm{Re}} \frac{\partial^2 w_2}{\partial z^2} |_Z . \tag{58}$$

As the second derivative of the velocity w_2 is linear with z we can write

$$\frac{\partial^2 w_2}{\partial z^2} |_Z \sim \frac{W_2}{Z^2} \sim \frac{1}{Z^2} \frac{D_0 \eta_2}{Dt} \tag{59}$$

and then

$$\nabla_0^2 \Gamma_0 \sim \frac{M_e^2}{\mathrm{Re} Z^2} W_2. \tag{60}$$

When the expansion in terms of Θ can be performed, the right-hand side of this equation plays the equivalent role of the one played by the source term in the pressure Poisson equation of an incompressible 2D flow:

$$\nabla_0^2 P = -\nabla_{\mathbf{0}} \cdot (\mathbf{u} \nabla_{\mathbf{0}} \mathbf{u}), \tag{61}$$

where $\mathbf{u} = (u,v)$ is the velocity of the analogous 2D flow and P represents the associated pressure field normalized with the mass density of the flow (supposed constant).

The right-hand side of Eq. (61) can also be expressed as

$$-\nabla_{\mathbf{0}} \cdot (\mathbf{u} \nabla_{\mathbf{0}} \mathbf{u}) = \tfrac{1}{2}(\varpi^2 - s^2) = \Lambda \tag{62}$$

and is coincident with the value of the second invariant of the velocity gradient tensor of the 2D flow. The value it attains in different regions has been used as a vortex identification criterion (Q or Okubo-Weiss criterion) [22]. Such a criterion may be established on the basis of dynamical arguments involving a comparison between the squared vorticity (ϖ^2) and the squared rate of strain $[s^2 = (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2]$ or considering local topological aspects of velocity fields [23,24].

For the experiments with soap film flows in which this analogy takes place, as expressed in Eq. (60), the source term is indeed associated with the 3D behavior of the film determined by the field of the normal velocity. Equivalently, from a Lagrangian point of view, this source term can be associated to the rate of change of the thickness of a soap film particle. In both approaches, these magnitudes only appear in equations at the higher orders of the asymptotic expansion.

Hence, the value of the instantaneous thickness of a material element of the film depends not only on the value of the invariant in the region it flows (a measure of the rate of change of thickness) but also on the time of permanence of the particle in the different regions. When a soap film particle flows in regions where the invariant is null (null normal velocity) it keeps its original thickness. On the contrary, if the particle travels in regions where the invariant is non-negligible, then it will change its thickness and the variations can be revealed by the interferometry technique.

Note that regions with $\nabla_0^2 P > 0$ (usually named elliptic domains) can be associated to regions that tend to make humps in the film, while the regions with $\nabla_0^2 P < 0$ (hyperbolic domains) can be associated to those that tend to form depressions in the film.

Even if the sign of the invariant cannot be determined by direct observation of images produced by the interferometry technique one can take advantage of some *a priori* knowledge of the flow and try to associate the topology of the thickness patterns with the hyperbolic or elliptic characteristic of the regions. For instance, in the particular case of 2D turbulent flows without presence of walls, on average, particles are trapped in highly elliptic or hyperbolic regions [25]. The first case corresponds to strong vortex cores, where rotation dominates strain deformation, with quite axisymmetric centers [26]. The regions where deformation highly dominates rotation generally take place in the organized structures surrounding the vortex cores [25]. In consequence, it is expected that regions with the largest thickness variation correspond to regions of the flow with presence of strong centers or saddles (high hyperbolic regions). In the background turbulent regions (the region in between these regions with low values of Λ) the invariant may in general adopt any sign, but as particles are not trapped there, the passage of soap film elements through these regions is expected to produce only slight changes of thickness.

This analysis seems to be in agreement with results concerning the statistics of thickness fluctuations obtained by Ref. [19] for experiments at low flow rates (that can be associated to ranges of low values of Θ) and those of the statistics of centers and saddles obtained by [23,26]. The asymmetric shapes of the curves of probability distribution functions (PDF) of film thickness and those of parameter Λ are quite in concordance, showing in both cases an asymmetric distribution with a bias towards the positive values (associated respectively to the humps of the films and to predominant vorticity). In the experiments of Ref. [19] at the higher values of flow rates (that can be associated to ranges of larger values of Θ) the PDF curves for the thickness are distorted and, therefore, as expected, the analogy no longer holds.

Note that, in general, the identification of the elliptic or hyperbolic flow regions is performed within the framework of a Eulerian approach with snapshots of the velocity field of the flows [23,26]. Although direct visualizations of soap-film dynamics does not enable a direct identification of these regions, it has the advantage of being a complementary tool of analysis that does not rely on velocity field estimation.

III. CONCLUSIONS

In this article we revisit the equations that govern soap film dynamics in the conditions often observed in vertical soap film tunnels [$M_e \sim O_{(10^{-1})}$]. We consider here the eventual link that may exist between the dynamics observed and the analogous 2D flow. The flows analyzed fall into a range not previously covered by theories based on the asymptotic expansions of the parameter M_e.

The main differences with previous works are that in these conditions the analogous flow corresponds to a 2D divergence free flow with small variations of the equivalent density (associated to the film thickness variations). This flow is in some sense analogous to a 2D Boussinesq flow but the buoyancy term in general is not very important for this kind of flow. However, the nonalignment of the gradient of thickness variations and surfactant concentration at the surface acts in this case as a source of vorticity. When these density variations are small a second expansion can be proposed in terms of the parameter (ZM_e^2) and, as expected, we recover the classical incompressible 2D flow equations.

In this situation we could link the rate of thickness variation of the fluid particle to the local value of the second invariant of the velocity gradient tensor. Interpretation of fringe patterns dynamics obtained by interferometry technique reveals therefore a difficult task. It depends on the local value of the invariant and on the time of residence of the particles within regions where this invariant is relatively important. However, the analysis of thickness variations gives information of the dynamics of this invariant that in general can only be determined when having access to the precise velocity field.

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