Ehrenfest approach to open double-well dynamics

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We consider an Ehrenfest approximation for a particle in a double-well potential in the presence of an external environment schematized as a finite resource heat bath. This allows us to explore how the limitations in the applicability of Ehrenfest dynamics to nonlinear systems are modified in an open system setting. Within this framework, we have identified an environment-induced spontaneous symmetry breaking mechanism, and we argue that the Ehrenfest approximation becomes increasingly valid in the limit of strong coupling to the external reservoir, either in the form of an increasing number of oscillators or increasing temperature. The analysis also suggests a rather intuitive picture for the general phenomenon of quantum tunneling and its interplay with classical thermal activation processes, which may be of relevance in physical chemistry, ultracold atom physics, and fast-switching dynamics such as in superconducting digital electronics.

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I. INTRODUCTION

The analysis of the relationship between full quantum solutions and their semiclassical or classical counterparts has played an important role in modern physics both for promoting a better conceptual understanding of its foundations and for simplifying complex problems. Prominent among the many techniques introduced to treat quantum systems is the approximation known as Ehrenfest dynamics [1] extensively used both for its generality and for ready comparison to the classical limit. This latter feature has resulted in its use in the analysis of nonlinear dynamical systems, including those which exhibit chaotic properties [2].

Here we consider a one-dimensional dynamical system with nontrivial properties describing a particle moving under the simultaneous influence of a double-well potential and a heat bath defined in terms of its density of states and temperature. The case of a double well is very important for various reasons, and an Ehrenfest treatment is obviously going to be approximate, giving rise to misleading results on long time scales, as discussed in [3–6]. Nevertheless, this is a study of intrinsic value for various reasons.

First, the double well is a paradigmatic example of a dynamical system sufficiently different from that of the harmonic oscillator, with a rich interplay between intrawell and interwell dynamics. Second, the shortcomings of Ehrenfest dynamics may be mitigated in contexts in which the dynamics is frequently switched. Superconducting digital electronics provides an example for which the dynamics is frequently reset, and as such, no significant traces of the deviations between the Ehrenfest and the exact quantum dynamics should appear even on long time scales. Further, the double-well description here allows naturally for defining a dichotomous variable associated with quantities which are functions of the two potential minima. Third, there is a crossover regime between classical and quantum stochastic resonance in double-well systems [7], and the Ehrenfest dynamics may help fill this gap with simple, easily recognizable, dynamical structures, a sort of "coarse graining" of the full Hilbert space. In particular, we are able to interpolate between the situation of a purely quantum closed system dynamics, with the possibility of tunneling, and an open semiclassical or classical system in which hopping between the two minima of the double-well potential is instead achieved via thermal activation. In the open system case, the reservoir is schematized as a number of harmonic oscillators, with uniform density of frequencies in a finite bandwidth, linearly coupled to the particle in the double well. This represents the closest approximant to the ideal case of the Caldeira-Leggett model with infinite oscillators [8,9] and can also be considered as a general model for describing solid-state devices.

This paper is structured as follows. In Sec. II we introduce the model and discuss a feature which emerges already at the classical level, a sort of phase transition from a bistable to a monostable potential with an increasing number of bath oscillators. In Sec. III we discuss the Ehrenfest approximation to the double-well potential and introduce the extended phase space structure. Numerical results for the Ehrenfest model of a double well are reported in Sec. IV, including the classical limit and the Fourier analysis of the particle motion. In the concluding section we discuss the relevance of our considerations for the debate surrounding the Ehrenfest construction as applied to the double-well model, arguing that in the macroscopic limit, i.e., with a large number of oscillators, and/or in the high-temperature regime, the Ehrenfest approximation becomes more valid.

II. CLASSICAL CONSIDERATIONS

The double-well external potential acting on a particle of mass M is assumed to be of the form

$$V(Q) = -\mu Q^2 + \lambda Q^4, \tag{1}$$

where the two minima of the potential occur at $\pm\sqrt{\mu/(2\lambda)}$ and an energy barrier, relevant to the interwell dynamics, equal to $\Delta E = -\mu^2/(4\lambda)$. In the presence of a translationally invariant heat bath, the total Hamiltonian becomes

$$H_{\rm tot} = \frac{P^2}{2M} - \mu Q^2 + \lambda Q^4 + \sum_{n=1}^{N}\left[\frac{p_n^2}{2m} + \frac{1}{2}m\omega_n^2(q_n - Q)^2\right], \tag{2}$$

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which can be easily regrouped as

$$H_{\rm tot} = \frac{P^2}{2M} - \left(\mu - \frac{m}{2}\sum_{n=1}^{N}\omega_n^2\right)Q^2 + \lambda Q^4 - m\left(\sum_{n=1}^{N}\omega_n^2 q_n\right)Q + \sum_{n=1}^{N}\left(\frac{p_n^2}{2m} + \frac{1}{2}m\omega_n^2 q_n^2\right). \quad (3)$$

Notice that in the large N limit, if the initial positions of all particles are all randomly distributed (with their sign too), the linear term in Q is negligible. In the realistic case of finite N, this term will induce an external fluctuating, spatially independent force on the test particle resulting from the finite number of kicks due to the bath particles. Also, the effect of the heat bath is a renormalization of the quadratic term of the test particle potential energy, i.e., $\mu \mapsto \mu - m\langle\omega^2\rangle N/2$, where $\langle\omega^2\rangle = \sum_{n=1}^{N}\omega_n^2/N$. This is important because the heat bath renormalization term may change qualitatively the behavior which, in the large N limit, will end up in a quadratic plus quartic oscillator with no bistability. The average critical number of oscillators which defines the transition from a bistable to a monostable potential is easily written as $\langle N_{\rm crit}\rangle = 2\mu/(m\langle\omega^2\rangle)$. By assuming a uniform distribution of the frequencies in the interval $[\omega_{\rm min},\omega_{\rm max}]$, we obtain

$$\langle N_{\rm crit}\rangle = 6\mu\big/\left[m\left(\omega_{\rm max}^2 + \omega_{\rm max}\omega_{\rm min} + \omega_{\rm min}^2\right)\right] = \frac{2\mu}{m\omega_0^2}\frac{1}{1+\Delta\omega^2/\left(12\omega_0^2\right)}, \quad (4)$$

where in the last expression we have introduced the average angular frequency $\omega_0 = (\omega_{\rm max}+\omega_{\rm min})/2$ and the bandwidth $\Delta\omega = \omega_{\rm max} - \omega_{\rm min}$. In Fig. 1 we plot the absolute value of the minimum of the double-well potential, an indicator of the transition from bistable to monostable behavior, versus the number of oscillators. The presence of an increasing bath frequency bandwidth decreases the critical value of the number of oscillators for the transition, and the fact that these frequencies are randomly selected results in a stochastic component to the potential as well as in the location of its minimum. This provides an example of spontaneous symmetry breaking in which the environment restores the full symmetry of the ground state when the number of harmonic oscillators effectively interacting with the particle exceeds a threshold. The situation discussed here is similar to the one discussed in Ref. [10], in which localization is induced in a gas of pyramidal molecules (for which the interatomic interaction can be also approximated by a double-well potential) by the presence of the other molecules. Both situations can be considered cases in which there is a density-dependent phase transition. Notice that once the double-well potential is introduced, regardless of its microscopic origin, the phase transition occurs already in the classical limit, as our discussion thus far does not involve any consideration of quantum effects.

III. EHRENFEST APPROACH TO THE DOUBLE-WELL POTENTIAL

In order to analyze the quantum case within the Ehrenfest framework, we introduce the Heisenberg equations associated

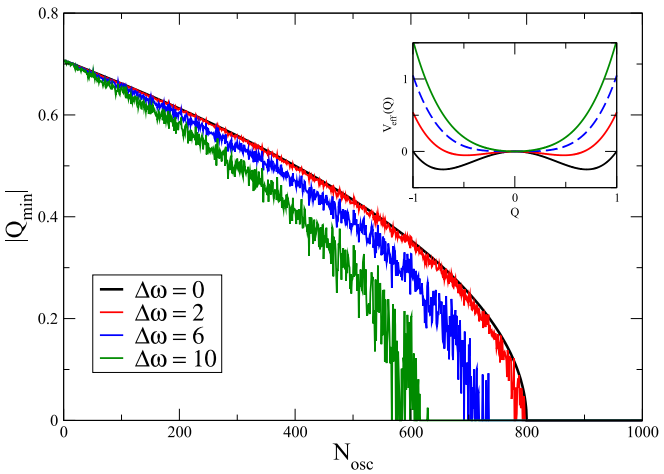

FIG. 1. (Color online) Transition from bistable to monostable behavior induced by the external environment. The absolute value of the minimum of the potential is plotted vs the number of involved harmonic oscillators with uniform distribution of their angular frequency in a range centered around $\omega_0 = 5$ for different values of their bandwidth $\Delta\omega = 0,2,6,10$ (from top to bottom, black, red, blue, and green curves, respectively). The case of a monochromatic bath yields the only deterministic curve and gives a value of $N_{\rm crit} = 2\mu/(m\omega_0^2) = 800$ for $\mu = 1$ and $m = 10^{-4}$, both in arbitrary units. The opposite case of maximal bandwidth yields instead a value which is smaller by a factor of $1 + \Delta\omega^2/(12\omega_0^2) = 4/3$, i.e., $N_{\rm crit} = 600$, in good agreement with the observed behavior of the curve at the end point. The inset explicitly shows the changing effective potential for an increasing number of oscillators, $N_{\rm osc} = 1,300,600,900$ (from bottom to top, black, red, blue, and green curves, respectively), with the dashed curve indicating the threshold case of $\langle N_{\rm crit}\rangle = 600$. The bandwidth in this case was taken to be the maximum value, $\Delta\omega = 10$.

with the closed system Hamiltonian $\hat{H} = \hat{P}^2/2M + V(\hat{Q},t)$,

$$\frac{d\hat{Q}}{dt} = \frac{\hat{P}}{M}, \quad \frac{d\hat{P}}{dt} = -\frac{\partial V(\hat{Q},t)}{\partial Q}. \quad (5)$$

Each operator can be written as $\hat{A} = \langle\hat{A}\rangle + \Delta\hat{A}$, where $\langle\cdots\rangle$ denotes the expectation value so that $\langle\Delta\hat{A}\rangle = 0$. Taylor expanding the potential $V(\hat{Q},t)$ about $\langle\hat{Q}\rangle$ leads to the Ehrenfest equations

$$\frac{d\langle\hat{Q}\rangle}{dt} = \frac{\langle\hat{P}\rangle}{M}, \quad (6)$$

$$\frac{d\langle\hat{P}\rangle}{dt} = -\sum_{n=0}^{\infty}\frac{1}{n!}V^{(n+1)}(\langle\hat{Q}\rangle)\langle\Delta\hat{Q}^n\rangle, \quad (7)$$

where $V^{(n)} = \partial^n V/\partial Q^n$. Writing down the corresponding evolution equations for $\langle\Delta\hat{Q}^n\rangle$ leads to an infinite hierarchy of equations [3–6]. While we summarize below the formalism with the goal of applying it to a double-well potential, we refer the interested reader to our recent discussion in Sec. II of [11] for a detailed description of the delicate interplay between the Heisenberg equations, the infinite hierarchy of Ehrenfest equations, and their truncation by using the Gaussian approximation.

Due to the presence of a quartic term in the polynomial form of the potential energy for a double well, moments across

different orders are coupled to each other, and higher-order moments grow and eventually become significant even if they were initially zero.

The Ehrenfest expansion in the Gaussian approximation becomes

$$\frac{dQ}{dt} = \frac{P}{M}, \tag{8}$$

$$\frac{dP}{dt} = -\sum_{n=0}^{\infty} V^{(2n+1)}(Q)\frac{\rho^{2n}}{n!2^n}, \tag{9}$$

$$\frac{d\rho}{dt} = \frac{\Pi}{M}, \tag{10}$$

$$\frac{d\Pi}{dt} = \frac{\hbar^2}{4M\rho^3} - \sum_{n=0}^{\infty} V^{(2n+2)}(Q)\frac{\rho^{2n+1}}{n!2^n}, \tag{11}$$

where $Q \equiv \langle \hat{Q} \rangle$ and $P \equiv \langle \hat{P} \rangle$ are the expectation values of position and momentum, respectively. Here odd cumulants are identically zero, and even cumulants can be written in terms of the variable ρ as $\langle \Delta \hat{Q}^{2n} \rangle = \rho^{2n}/[(2n!)2^n n!]$. As done earlier [3–5], we also introduce a new variable, $\Pi = \langle \Delta\hat{Q}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{Q} \rangle / 2\rho$, which, as is clear from its definition, reflects the correlation between $\Delta\hat{Q}$ and $\Delta\hat{P}$. Hamilton's equations for the extended open system are now written as

$$\frac{dQ}{dt} = \frac{P}{M}, \tag{12}$$

$$\frac{dP}{dt} = 2(\mu - 6\lambda\rho^2)Q - 4\lambda Q^3 + m\sum_{n=1}^{N}\omega_n^2(q_n - Q), \tag{13}$$

$$\frac{d\rho}{dt} = \frac{\Pi}{M}, \tag{14}$$

$$\frac{d\Pi}{dt} = \frac{\hbar^2}{4M\rho^3} + 2\left(\mu - 6\lambda Q^2 - \frac{m}{2}\sum_{n=1}^{N}\omega_n^2\right)\rho - 12\lambda\rho^3, \tag{15}$$

$$\frac{dq_n}{dt} = \frac{p_n}{m}, \tag{16}$$

$$\frac{dp_n}{dt} = -m\omega_n^2(q_n - Q), \tag{17}$$

corresponding to an extended Hamiltonian for the total system of the form

$$H(Q,P;\rho,\Pi;q_n,p_n) = \frac{P^2}{2M} - \mu Q^2 + \lambda Q^4 + 6\lambda Q^2\rho^2 + \frac{1}{2}m\sum_{n=1}^{N}\omega_n^2(q_n - Q)^2 + \frac{\Pi^2}{2M} + \frac{\hbar^2}{8M\rho^2} - \mu\rho^2 + 3\lambda\rho^4 + \frac{1}{2}m\left(\sum_{n=1}^{N}\omega_n^2\right)\rho^2 + \sum_{n=1}^{N}\frac{p_n^2}{2m}. \tag{18}$$

Together, the $4 + 2N$ Hamilton equations fully describe the evolution of both the centroid and the spreading of the wave packet associated with the particle experiencing both the double-well potential and the classical motion of each of the particles making the heat bath, including the back-action effect of the particle on the heat bath as seen in Eq. (17). With respect to the classical Hamilton's equation, there is a renormalization term in the quadratic coefficient and cross terms relating fluctuational and configurational dynamics, written as $-12\lambda\rho^2 Q$ and $-12\lambda Q^2\rho$, respectively. The presence of these cross terms (absent in the case of a harmonic oscillator, which may be obtained as a specific case for $\mu < 0$, $\lambda = 0$) shows that configurational and fluctuation dynamics are reciprocally interrelated, at variance with the case of the harmonic oscillator.

IV. NUMERICAL RESULTS

We have numerically integrated the equations of motion under a wide range of bath and test particle conditions using a variable-step, predictor corrector integration scheme, with absolute and relative errors in the 10^{-10}–10^{-12} range. In particular, we have explored the test particle dynamics for different initial conditions in the presence of the bath. The bath oscillators had energy drawn from a Boltzmann distribution, and the number of oscillators, the mass of the harmonic oscillators, the angular frequency bounds, and the spectral distribution between these two extrema were all varied. For the results shown, we have chosen, for simplicity and to contrast with the findings in [12], a uniform distribution of frequencies. However, more generic distributions [13] are easily analyzed with our numerical tools, although we do not expect any new or novel qualitative features to emerge [14].

We begin with the absence of a bath and consider a particularly intriguing choice of initial conditions corresponding to the test particle being located at rest at the minimum of the double-well potential. In this case we expect no dynamics in the classical limit. The corresponding extended space evolution is shown in the top panels of Fig. 2 and is considerably different. This results from the purely quantum evolution in the (ρ,Π) space and can be thought of as arising from auxiliary kinetic energy associated with quantum fluctuations and resulting from the enforcement of the Heisenberg principle. This emphasizes the crucial role of the (ρ,Π) space in providing fluctuation energy to the regular phase space. It is also evident that there are initial conditions in the former space for which no tunneling dynamics is achieved, i.e., in a region centered around $\rho = 1, \Pi = 0$. As far as we know, there has been little discussion of this interpretation of quantum tunneling resulting from the energy available in pure quantum fluctuations, even if the system is nominally, on average, at rest at a classical minimum of the potential. If the particle starts from initial conditions corresponding to higher energy, the dynamics changes accordingly, as seen in the bottom panels of Fig. 2. The particle dynamics starting with the same initial conditions is qualitatively quite different when the bath oscillators are included in the dynamics. In Figs. 3 and 4, we show cases in which the bath temperature is varied.

If the temperature is not very large, as in Fig. 3, the generalized phase space trajectories of the two different initial conditions are clearly distinct. By contrast, as seen in Fig. 4, coupling to a high-temperature bath gives rise to nearly indistinguishable trajectories in the generalized phase space. The interpretation is that Fig. 3 results from the combined action of the fluctuational quantum energy and the thermal energy, while in Fig. 4 the thermal activation

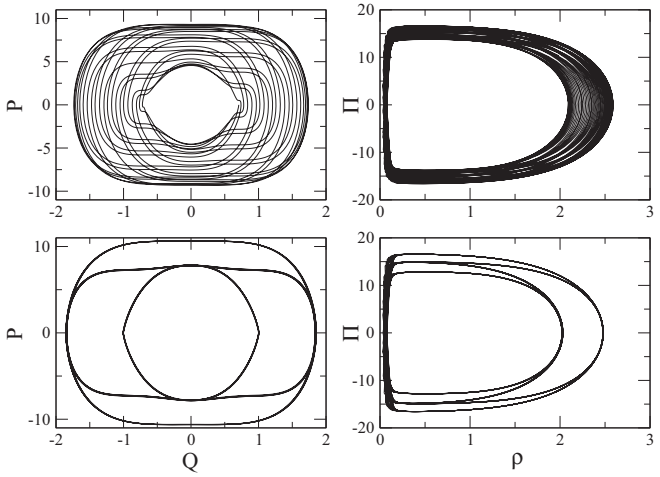

FIG. 2. The test particle's phase space (Q,P) and fluctuational phase space (ρ,Π) for the case of $\mu = 1, \lambda = 1$, $M = 1$. Initial conditions in (Q,P,ρ,Π) space are $(1/\sqrt{2},0,0.03,0)$, corresponding to the test particle sitting in the minimum of the potential well with zero initial momentum (top plots), and $(1,0,0.03,0)$, for which the test particle starts away from the minimum of the double well (bottom plots). The time evolution lasts for five periods, where the cycle time is defined by the effective period of the intrawell oscillations, $\tau = 2\pi/\sqrt{2\mu}$. Even starting at the minimum of the potential well with zero kinetic energy in real phase space, the kinetic energy associated with fluctuations feeds back on the phase space, giving rise to quantum tunneling.

processes dominate the dynamics. Also, the back-action in the fluctuation sector of phase space is more clearly manifest in Fig. 4, as seen from the wider excursions in Π due to the stronger correlation to the configurational dynamics due to the cross terms discussed after Eq. (8). By way of contrast, we show in Fig. 5 the strictly classical counterpart of the motion in the absence and presence of a heat bath with increasing temperature and the associated growth in thermal activation.

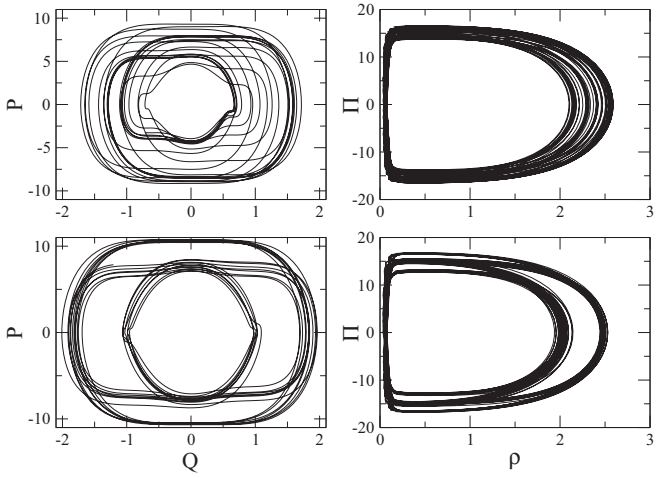

FIG. 3. Same as Fig. 2, but with the test particle in contact with a bath with a temperature $T = 10^{-6}$. The heat bath is made of 2000 oscillators with frequencies uniformly distributed in the range $[0,10]$ and mass $m = 10^{-9}$.

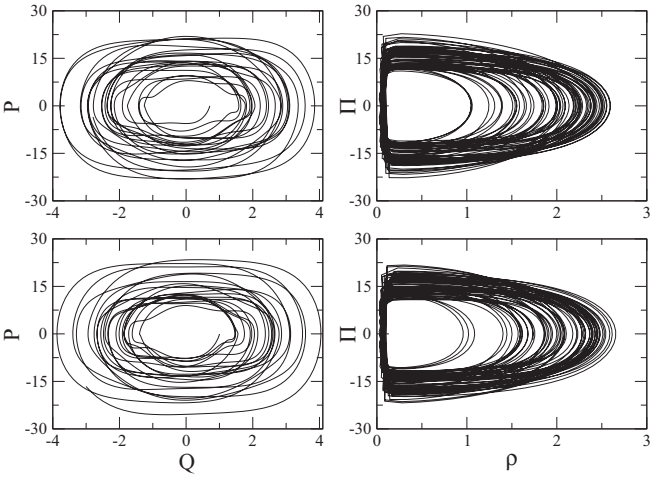

FIG. 4. Same as Fig. 3, but with the test particle in contact with a bath with a temperature $T = 10^{-3}$, corresponding to thermally activated hopping processes.

The dependence of the interwell period of oscillation upon the initial position of the test particle with a zero-temperature heat bath is shown in Fig. 6, where the interwell period grows quickly in a narrow region of $Q/Q_{\min}$ around unity. On closer inspection, it is evident that the maximum slope of the interwell period with respect to $Q/Q_{\min}$ occurs away from the actual minimum of the potential, consistent with the fact that energy coming from the (ρ,Π) space is also available for hopping. Moreover, the presence of the bath, even at zero temperature, renormalizes the bare potential of Eq. (1), as discussed in Fig. 1, resulting in a minimum at a value smaller than $Q/Q_{\min} = 1$. The increase in period is a manifestation of a lower transition probability, and one may extend this inference to baths at different temperatures. Thus, the crossover regime between hopping due to individual or collective mechanisms of quantum and thermal fluctuations can be explored. However, a closer inspection of the time dependence of Q with an fast Fourier transform (FFT) analysis shows that, even for a closed system, there are various frequencies contributing to the motion.

Therefore the interwell period of oscillation has to be read as a coarse-grained observable associated with the Fourier

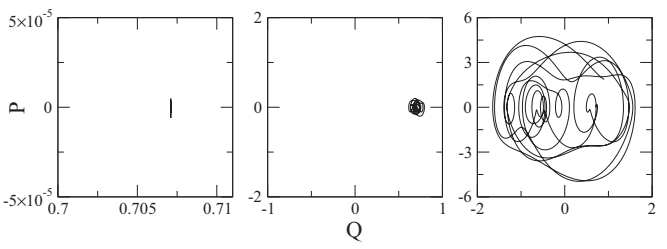

FIG. 5. Trajectories in phase space for the classical case $(\hbar = 0, \rho = 0)$ for a particle (left) in a double well without external heat bath and in contact with a heat bath at (middle) $T = 10^{-6}$ and (right) $T = 10^{-3}$, with initial conditions very close to the minimum of the potential, such that there is nearly no evolution in the first case, some random motion without hopping in the second case, and effective thermal activation in the third case.

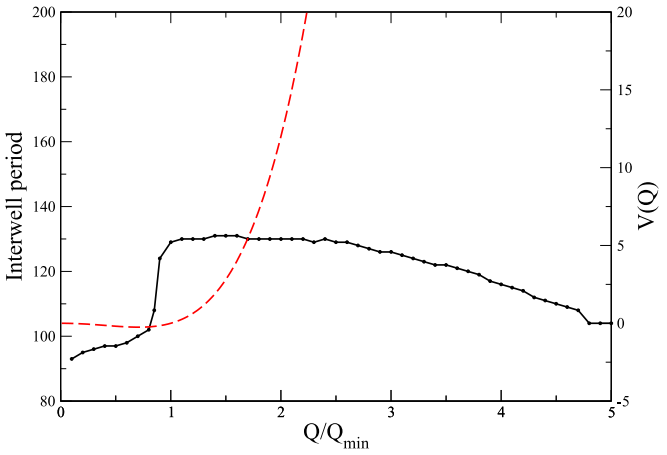

FIG. 6. (Color online) Interwell period vs the initial position of the test particle (black solid line, left vertical scale) and corresponding potential energy (red dashed line, right vertical scale). A dramatic increase in period corresponds to the initial position at the minimum of the potential, with the interpretation of a lower transition probability. Here the temperature of the bath is assumed to be zero and $\mu = \lambda = 1$, and the initial conditions for P, ρ, Π are the same as in Fig. 2.

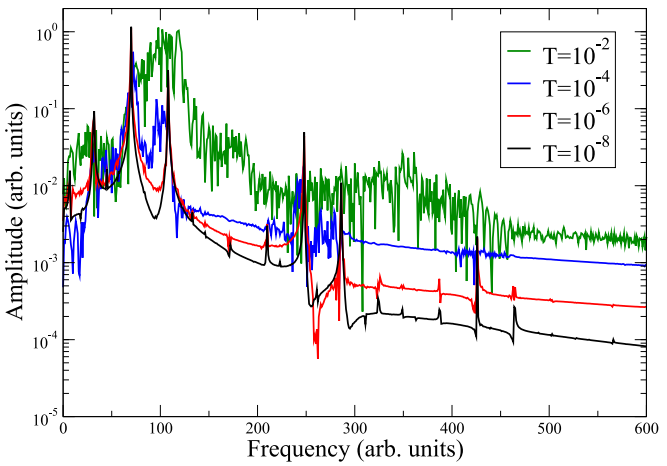

FIG. 7. (Color online) Fast Fourier transform of the time-dependent position of the particle in the double-well potential for different temperatures of the reservoir; from top to bottom on the right side of the plot: $T = 10^{-2}$ (green curve), $T = 10^{-4}$ (blue), $T = 10^{-6}$ (red), and $T = 10^{-8}$ (black), in arbitrary units. The regime of thermal activation is marked by the presence of a broad spectrum of frequencies, while at low temperatures the motion has a small set of dominant frequencies, peaked around curves progressively broadened as the temperature is increased. The motion is sampled for $2^{13} = 8192$ time steps, with initial conditions and the same system parameters as in the bottom plots of Fig. 2, and 2000 oscillators in the heat bath.

spectrum of $Q(t)$. The FFT spectrum is a more detailed indicator and strongly depends on the temperature of the heat bath, as shown in Fig. 7. At low temperature there are few prominent peaks, while at the highest explored temperature the peaks are barely recognizable as they are submerged in a broadband spectrum of frequencies.

This suggests that precision thermometry of tunneling electrons should be achieved by looking at the FFT spectrum of the Josephson current, or related quantities such as the integral of the FFT spectrum over the frequency range or the frequency of the peak value in the FFT spectrum, which is manifestly broadened as the temperature increases. Thermometry based on FFT analysis of transport properties could be more robust than the one based on decoherence of the interference fringes and could be implemented in the same measurement run by nondestructive imaging techniques in the case of ultracold atoms [15]. While thermometry using Josephson junctions is already available in the solid state [16], thermometry of ultracold atoms is a crucial issue [17], and the influence of a finite-temperature environment on quantum tunneling in a double-well trap [18,19] has been observed [20,21].

V. CONCLUSIONS

In conclusion, we have discussed within a unified framework the interplay between quantum and classical fluctuations in the nontrivial case of an open double-well system using the Ehrenfest approach implemented through the Gaussian approximation. We have provided plausibility arguments according to which this approximation should become increasingly reliable as the heat bath dominates the dynamics, which happens in the limits of high temperature and/or a large number of bath oscillators. At finite times, the method may include the case of resetting dynamics like the one of superconducting electronics or, possibly, time-dependent situations such as the ones related to stochastic resonance [7], readily achievable by adding a time-dependent tilting force term to the double-well potential. From the Ehrenfest viewpoint, stochastic resonance appears to be a specific example of control of the test particle Hamiltonian parameters in order to maximize performance, as measured by the signal-to-noise ratio when the test particle is subjected to an external perturbation and the fluctuations induced by the heat bath.

The inclusion of the external environment shows also that the Ehrenfest approximation is expected to be more reliable or better able to approximate the quantum dynamics for a longer time as we increase the coupling to the environment by increasing the heat bath temperature. This is not surprising as in this limit the dynamics of the double well is increasingly ruled by the (classical) heat bath. This is also suggestive of a possible solution to the controversial issue of the validity of the Ehrenfest approximation to systems other than harmonic ones. As neatly pointed out in [22], "when closed-system classical and quantum dynamics are treated in Gaussian approximation, they are in fact identical," which simplifies the situation as any difference in their average behavior is then due to the breakdown of the Ehrenfest approximation. Therefore in the case of an *open* system the situation is more forgiving, and the breakdown of the Ehrenfest approximation occurs either on much longer time scales, of the order of $\hbar/\Delta E$ (where ΔE is the minimum spacing of the relevant energies significantly contributing to the motion), or never for conditions under which the effective potential becomes monostable. As seen in Fig. 7 for higher temperatures, an increasing number of harmonics contribute, therefore making ΔE smaller with respect to the closed system. This should alleviate both the issues raised for a closed system in [23,24] and the need for using a superposition of Gaussian wave packets suggested in [25].

Less trivial is the dependence of the dynamics on N in the macroscopic limit, $N \to \infty$. Based on the considerations on the phase transition occurring when the number of oscillators exceeds $N_{\rm crit}$ as seen in Eq. (4), we notice that the same effect is applicable to Eqs. (13) and (15), where the cross term $6\lambda Q^2\rho^2$ no longer affects the qualitative analysis. This means that for the number of oscillators $N > N_{\rm crit}$ the effective potentials in both Q and ρ are monostable. Thus the dynamical behavior will be qualitatively quite similar to the case of a harmonic oscillator for which the Ehrenfest dynamics is valid for all times. Since, typically, the number of degrees of freedom of a bath also depends on temperature, with more degrees of freedom becoming unfrozen with increasing temperatures, there is a range of possible behavior in this macroscopic limit, depending on the concrete model schematizing the heat bath. We also note that the Caldeira-Leggett model, originally introduced to describe the influence of a high-temperature, infinite-bandwidth bath on a double-well system, is expected to fail at low temperatures for which ultraviolet cutoffs in the frequency spectrum of the bath oscillators are necessarily operative. This expected failure is consistent with our finding of a phase transition to a monostable potential for the case of a finite bandwidth and a number of oscillators in the bath above a well-defined threshold, a situation which could be indeed realized in low-temperature setups.

From the experimental perspective, in the framework of Bose condensates the system can be further generalized by including interatomic interactions [26,27]. For instance, in the experiment [20], the double well is created from a preexisting harmonic potential via a slow ramping up of a barrier. It remains to be understood if residual nonadiabaticity in the ramp-up may be a cause for temperature increase of the sample. The formalism developed here allows the implementation of strategies, such as frictionless cooling [28–32], capable of minimizing this heating source. Work on extending the Caldeira-Leggett model in the framework of trapped atoms is ongoing [33]. However, the functional dependence of the interacting Hamiltonian, necessary for making the test particle bath-particle interaction local, precludes closure of the system of equations even in the Gaussian approximation, so all current indications are that this formalism may not be trivially extended to this important class of many-body systems.

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