Diffusion for ensembles of standard maps

Or Alus* and Shmuel Fishman†
Physics Department, Technion–Israel Institute of Technology, Haifa 32000, Israel
(Received 6 June 2015; published 2 October 2015)

Two types of random evolution processes are studied for ensembles of the standard map with driving parameter K that determines its degree of stochasticity. For one type of process the parameter K is chosen at random from a Gaussian distribution and is then kept fixed, while for the other type it varies from step to step. In addition, noise that can be arbitrarily weak is added. The ensemble average and the average over noise of the diffusion coefficient are calculated for both types of processes. These two types of processes are relevant for two types of experimental situations as explained in the paper. Both types of processes destroy fine details of the dynamics, and the second process is found to be more effective in destroying the fine details. We hope that this work is a step in the efforts for developing a statistical theory for systems with mixed phase space (regular in some parts and chaotic in other parts).

DOI: 10.1103/PhysRevE.92.042904 PACS number(s): 05.45.Ac, 05.40.−a

I. INTRODUCTION

Typically physical systems are modeled by Hamiltonians or maps leading to dynamics in mixed phase space [1–3]. In such a phase space, the dynamics in some parts is chaotic and in other parts it is regular. Here we consider the case of conservative dynamics (area in phase space is conserved). The phase space exhibits structures on all scales. These fine details are typically very sensitive to the values of parameters, far beyond experimental resolution. In mixed systems, transport is affected by "sticking" to regular structures such as islands chains; see also Refs. [4,5]. By sticking we mean that a trajectory is trapped for a very long time near some structure. The regular structures are typically surrounded by broken invariant circles called "cantori" which function as barriers to the flux of chaotic trajectories. Chirikov and Shepelyansky [6] studied the decay of correlations near the critical point K_c of the standard map, where chaos becomes unbounded [see Eq. (1) and discussion that follows]. They observed an algebraic decay of correlations for long times. This calls for a statistical description of such systems [7].

Although time correlations in a specific region of phase space may decay algebraically, rapid decay of time correlations between different parts of phase space should be considered. Therefore, a statistical approach may be applicable to a single system. A comprehensive model for transport in such systems was proposed by Meiss and Ott [8], where a construction of a distribution of fluxes through different structures was introduced. In this way a complicated deterministic process was replaced by relatively simple random one. The distribution of flux ratios relevant for this process was calculated for the Hénon map in Ref. [7]. Recently an important contribution was made by Cristadoro and Ketzmerick, who demonstrated the universality of the decay of correlations in the framework of the model of Ref. [8]. They examined an ensemble of such systems by using an arbitrary distribution of transition probabilities in phase space [9]. Guided by similar ideas, Ceder and Agam [10] used diagrammatic methods to calculate the exponent of the decay of correlations and its fluctuations and found that the fluctuations are large. A summary of the exponents of the decay of correlations and relations to exponents characterizing the spreading is presented by Venegeroles [11].

Another approach to treat statistically this effect in such systems without modeling phase space was suggested in the pioneering work of Rechester, Rosenbluth, and White (RRW) [12,13], where noise was used. This enables one to define and calculate the diffusion coefficient in phase space. In particular the limit of vanishing noise was found to be meaningful. In these calculations the main effect of noise is to suppress long time "sticking" and to enable diffusion. Sometimes it is referred to as regularization of the diffusion process. On the other hand, it was found that noise may enhance trapping [14]. This motivates calculation of various scenarios and classification of "noisy" phenomena. A characterization of this type was introduced by Romeiras, Grebogi, and Ott [15] (see also Ref. [16]):

Problem 1: Noisy map. For an ensemble of trajectories each encounters different random perturbations.

Problem 2: Random map. For an ensemble of trajectories all encounter the same random perturbation.

In the present work we apply noise as in Problem 1, of small variance, and in some cases we consider a random map as in Problem 2. The processes we study can be classified into two types:

Type I is an ensemble of systems, each with a different parameter value that is constant in time.

Type II is an ensemble of systems where the parameters vary randomly with time, where each member is like Problem 2 (random map).

Randomness of type I is relevant for ensembles of devices such as driven Josephson junctions and Superconducting Quantum-Interference Devices [17]. Randomness of type II may be relevant for atomic billiards [18,19], where in spite of the experimental efforts the walls of the billiard move during the experiment.

The noise introduced by RRW (Problem 1) results most naturally in experiments, as a result of the interaction with the environment. The purpose of the present paper is to explore the statistical effects of the randomness of types I and II in the presence of weak noise of the type introduced by RRW. We

*oralus@tx.technion.ac.il
†fishman@physics.technion.ac.il

explore in particular the question of the effect of the two types of randomness on the fine details of the system. The noise was introduced by RRW to regularize the map and to be able to define the diffusion coefficient. The reason the result is meaningful, also in the absence of noise, is the fact that for short times correlations fall off exponentially, as in the case of idealized fully chaotic systems in the asymptotic infinite time limit [20–22]. This leads to a result similar to the one of RRW [23]. The main goal of introducing randomness in the present work is to study ensembles of mixed systems rather than achieving regularization for a specific system. The calculations in the present paper are performed for the standard map [2,3,24]. This map is given in terms of the variables θ and J:

$$\theta_{t+1} = \theta_t + J_{t+1}, \quad J_{t+1} = J_t - K \sin(\theta_t), \tag{1}$$

where the parameter K controls the level of chaos. For $K > K_c = 0.971635\ldots$ it exhibits diffusionlike dynamics in momentum J for most values of K. The diffusion coefficient is

$$D(K) = \lim_{t\to\infty} \frac{1}{2t}\langle (J_t - J_0)^2\rangle, \tag{2}$$

where $\langle\,\rangle$ denotes the average over initial conditions in the chaotic component. If RRW's type of noise is added also averaging over the noise is understood. Fine details of the system are the accelerator modes; that is, for K in the vicinity of $K = 2\pi n$, where n is an integer, acceleration is found for some initial conditions. Effects of noise (Problem 1) were studied specifically for accelerator modes [25]. Deviations from diffusion as a result of sticking were studied by Zaslavsky and Edelman [5] and later by Venegeroles [22]. A process of type II leads to diffusion into islands of the standard map [26].

In the present work, suppression of accelerator modes by randomness of types I and II are studied. The outline of this paper is as follows. In Sec. II approximate analytical expressions for the diffusion coefficient in phase space of the standard map are derived for processes of types I and II. These are compared and tested numerically in Sec. III. The results are summarized and discussed in Sec. IV.

II. ENSEMBLES OF STANDARD MAPS

The standard map is defined by Eq. (1). For any $K > 0$, chaotic regions in phase space are found. There is a critical value $K = K_c \simeq 0.9716$ so that for $K < K_c$ the various chaotic regions are separated by invariant circles, while for $K > K_c$ chaotic regions merge so that there is an infinite chaotic component and diffusion in momentum is found in numerical calculations for many values of K.

The diffusion coefficient was calculated as an expansion series in powers of $\frac{1}{\sqrt{K}}$ [12]. To define and calculate the diffusion coefficient, noise was introduced by the distribution of the random variable $\delta\theta$ [12,13,25],

$$\eta(\delta\theta, J) = \frac{1}{\sqrt{2\pi}\sigma}\sum_{n=-\infty}^{\infty} e^{-\frac{(\delta\theta - J + 2\pi n)^2}{2\sigma^2}}, \tag{3}$$

which centers $\delta\theta$ around a mean value equal to $J + 2\pi n$ as in Eq. (1). It may be replaced with help of the Poisson summation formula by

$$\eta(\delta\theta, J) = \frac{1}{\sqrt{2\pi}\sigma}\sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2 m^2 + im(\delta\theta - J)}. \tag{4}$$

The deterministic evolution (1) in time is then replaced by a probabilistic one given by the distribution

$$P(\theta, J, t) = \int_0^{2\pi} d\theta' \eta(\theta - \theta', J) P(\theta', J + K\ \sin(\theta'), t-1), \tag{5}$$

where θ, J are the values at time t and $\theta', J + K\sin\theta'$ are the values at time $t-1$. Considering the initial probability density $P(\theta, J, t=0) = \frac{1}{2\pi}\delta(J - J_0)$, the diffusion coefficient in momentum space to leading order in $\frac{1}{\sqrt{K}}$ is

$$D_{\rm SM}(K) \approx \left(\frac{K}{2}\right)^2 \left[1 - 2\mathcal{J}_2(K)e^{-\sigma^2}\right], \tag{6}$$

where $\mathcal{J}_n(K)$ are the Bessel functions of order n, as found by RRW [12]. For large K it reduces approximately to

$$D_{\rm QL} := \frac{K^2}{4}. \tag{7}$$

In the limit $\sigma \to 0$, Eq. (6) approximates extremely well the momentum diffusion coefficient for nearly all $K > K_c$, excluding small intervals near $K \approx 2\pi n$ (with integer n) where the acceleration modes are found. Although the limit $\sigma \to 0$ of Eq. (6) exists, for the standard map (1) without noise the diffusion coefficient is not defined, as a result of "sticking" to regular structures. In the present work ensembles of standard maps are introduced in terms of the distribution of the parameter K. As explained in the introduction it describes physical situations where either the value of the parameter K is not known exactly but is fixed or it is not known and it is varying. We assume that the parameter K is taken from a Gaussian distribution with average $\bar{K}$ and standard deviation σ_K:

$$P(K) = \frac{1}{\sqrt{2\pi}\sigma_K} e^{-\frac{(K-\bar{K})^2}{2\sigma_K^2}}. \tag{8}$$

This randomness is applied in two ways, defining two types of processes and ensembles.

A. Type I: The parameter K is fixed but not known

In this type of dynamics each system is evolved with the parameter K taking the *same value*, which is chosen at random from the Gaussian distribution (8). The diffusion coefficient for each system is given by Eq. (6) and the average diffusion coefficient is found by

$$D_F(\bar{K}, \sigma, \sigma_K) = \int_{-\infty}^{\infty} dK\, P(K) D_{\rm SM}(K). \tag{9}$$

In the leading order in σ_K it reduces to

$$D'_{\rm QL} := \left(\frac{\bar{K}}{2}\right)^2 + \frac{\sigma_K^2}{4}. \tag{10}$$

The symbol F stands for averaging over final results of $D_{\rm SM}(K)$.

B. Type II: Changing parameter at each step

In this type each system is evolved in a way that K changes at random at each step and is chosen at random with the distribution (8). In this case the distribution (5) is replaced by

$$P(\theta',J+K\sin(\theta'),t-1)\to\langle P(\theta',J+K\sin(\theta'),t-1)\rangle_K, \tag{11}$$

where $\langle\,\rangle_K$ denotes averaging with respect to Eq. (8), leading to

$$P(\theta,J,t)=\int_0^{2\pi}d\theta'\eta(\theta-\theta',J)\int_{-\infty}^{\infty}dK\,P(K)\times P(\theta',J+K\sin(\theta'),t-1). \tag{12}$$

Taking the initial probability

$$P(\theta,J,0)=\frac{1}{2\pi}\delta(J-J_0) \tag{13}$$

and calculating the paths in Fourier space following Ref. [12] one obtains an expression in next to leading order in $\frac{1}{\sqrt{K}}$ for the diffusion coefficient:

$$D_E(\bar{K},\sigma,\sigma_K)\approx\left(\frac{\bar{K}}{2}\right)^2+\frac{\sigma_K^2}{4}-2\left(\frac{\bar{K}}{2}\right)^2\sum_{l'=-\infty}^{\infty}\mathcal{J}_{2l'+2}(\bar{K})\mathcal{I}_{l'}\times\left(\frac{\sigma_K^2}{4}\right)e^{-\sigma^2}e^{-\frac{\sigma_K^2}{4}}, \tag{14}$$

where $\mathcal{J}_l$ and $\mathcal{I}_l$ are the first order Bessel functions and first order modified Bessel functions, respectively. E stands for changing parameter at each step. The details of this calculation are presented in the Appendix. In the leading order in $\frac{1}{\sqrt{K}}$ where the sum can be neglected, one obtains for Eq. (14) the same result as Eq. (10). That is, in the leading order in $\frac{1}{\sqrt{K}}$, $D_F=D_E$.

III. COMPARISON OF THE RESULTS FOUND FOR THE TWO DIFFERENT ENSEMBLES

In this section the values of the diffusion coefficient D_F [Eq. (9)] and D_E [Eq. (14)] are compared, and comparison to the results of numerical simulations are presented. For both processes, types I and II, $N=10\,000$ initial conditions were used for each value of K and the map was iterated $N_t=100$ steps. The number of values of K chosen from the ensemble defined by Eq. (8) is $N'=10\,000$. The map (1) was evolved in the presence of noise with the distribution (4). The variance σ of the noise was set to a small value, namely 10^{-5}, and was kept constant, while the variance of K, σ_K, was varied and the diffusion coefficient was calculated by Eqs. (9) and (14). The corresponding numerical calculations were done for 200 different values of $\bar{K}$ in the interval $\bar{K}_{\min}=5$ and $\bar{K}_{\max}=30$. For a given K the diffusion coefficient was calculated numerically by

$$D(\bar{K},\sigma,\sigma_K)=\frac{\langle(J-\langle J\rangle)^2\rangle}{2N_t}. \tag{15}$$

Figure 1 presents the numerical results of D_E and D_F for various values of σ_K. It is shown that while the oscillations in D_E decay with increasing the value of σ_K, the values of D_F exhibit enhanced diffusion in the entire range compared to D_E. This is due to the effect of convolution of the Gaussian probability with the accelerator modes. Furthermore, the accelerator modes disappear in D_E for smaller values of σ_K than they do for D_F; therefore, D_F is time dependent.

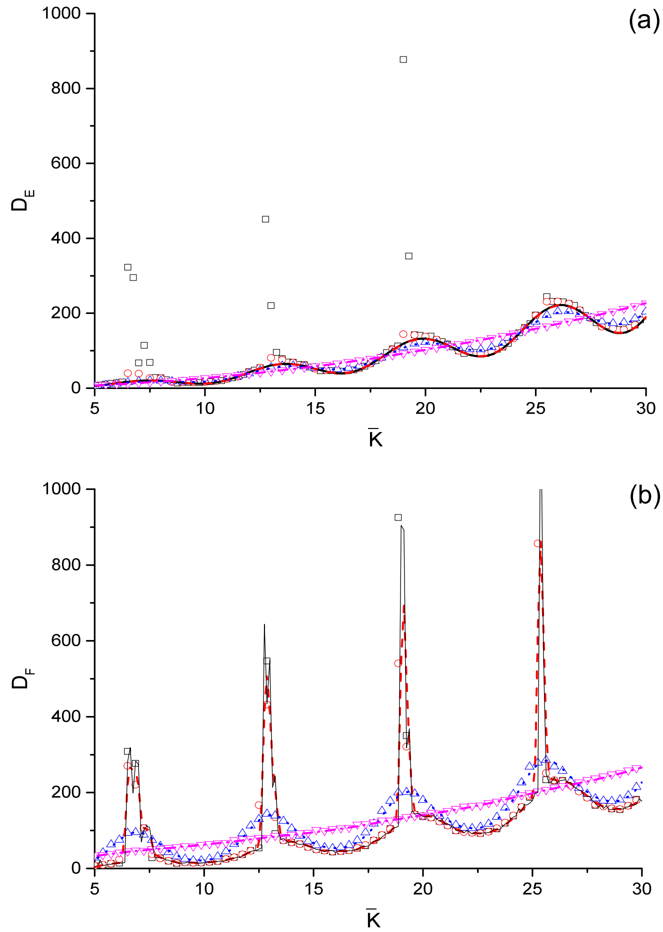

FIG. 1. (Color online) Diffusion coefficients found by numerical simulations [analytic results of Eqs. (14) and (9)]: (a) D_E and (b) D_F for problems of type II and type I, respectively, as a function of $\bar{K}$ for various values of σ_K. $\sigma_K=0.1$ in red ○ (−−), $\sigma_K=1$ in blue △ (···), $\sigma_K=3$ in magenta ▽ (− · −), and $\sigma_K=1\times10^{-5}$ in black □ (−). The black line in (b) was calculated from Eq. (2) contrary to other cases. In the calculation of D_F by Eq. (9), $D_{\rm SM}$ was obtained from a numerical simulation

In Fig. 2 the diffusion coefficient is presented for moderate $\sigma_K=1$. Numerical and analytical calculations are shown for the two types and compared with Eq. (6) where K is replaced by $\bar{K}$. Depletion or blurring of the accelerator modes is clearly shown for type II processes, such that for this value of σ_K the diffusion is well described by Eq. (14). RRW [12] found that the peak of the accelerator mode is obtained by summation Fourier paths of very high order (Fig. 9 in Ref. [12]). Since there is no exact analytic expression for the accelerator modes, we found numerically $D(K)$ of Eq. (2) with only RRW type of noise and used it in Eq. (9). For $\sigma_K=10^{-5}$ we assumed $D_F(\bar{K})=D(\bar{K})$. The time dependence of D_F (resulting from the accelerator modes) is demonstrated in the inset of Fig. 2.

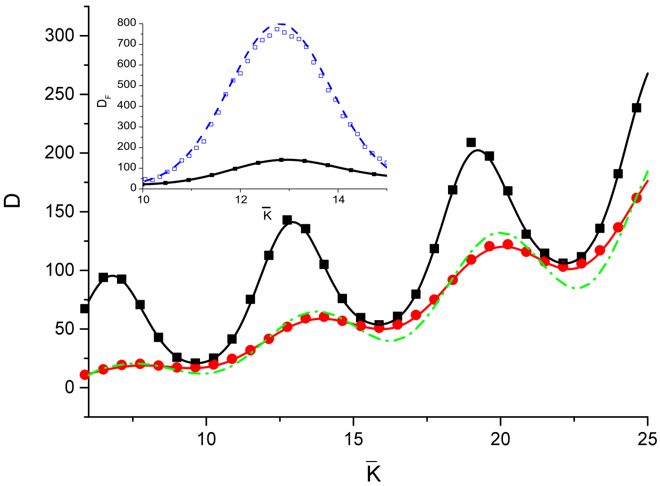

FIG. 2. (Color online) Comparison between analytical and numerical calculations of D_F and D_E for $\sigma_K = 1$. Numerical results for D_F are shown in black ■, and for D_E are in red ●. Analytic calculation for D_F [Eq. (9)] is represented by black solid line, and for D_E [Eq. (14)] in gray (red). The analytical results of $D_{\rm SM}$ [Eq. (6)] are represented by the dash-dotted (green) line. The inset shows the numerical results for D_F for $N_t = 1000$ in blue □, and for $N_t = 100$ in black ■. The analytical results are in blue dashed and black solid lines, correspondingly.

The elimination of the accelerator modes for the type II system can be explained by the following. Because contribution to it comes from Fourier paths of high order, multiplying each step of the path by $e^{-k'^2\sigma_K^2/4}$ in Eq. (A4) affects the contribution strongly. In this sense this type of randomness eliminates the accelerator modes more effectively than the RRW type of noise. Increasing time (not shown here) leads to an even further diminishing effect of the accelerator modes. Although longer paths give a contribution, it is negligible due to division by $1/2N_t$.

IV. SUMMARY

In the present work two ensembles of standard maps of the form (1) with a Gaussian distribution of the driving parameter K were studied. In one case (type I) the K was kept fixed and finally the average over the diffusion coefficient was taken, while in the other process (type II) K was varied at each step. Noise with standard deviation $\sigma = 10^{-5}$ was added to make the diffusion well defined. The resulting diffusion coefficients differ, as discussed in what follows. Increasing σ_K^2, the variance of K, tends to wash out details; hence, the effect of acceleration modes weakens, as can be seen from Figs. 1 and 2. We see that relatively weak randomness is sufficient to wash out the fine details of the dynamics generated by the map. The most important is the effect of this on accelerator modes. We find that the averaged diffusion coefficient exhibits oscillations as a function of the averaged driving parameter $\bar{K}$ similar to the situation found for fixed K [12]. The type I processes result in replacement of the acceleration modes by enhanced diffusion for a wide range of $\bar{K}$. The type II process where the driving parameter K is chosen at each step is more effective in washing out the accelerator modes than the type I processes where K is chosen at random but is kept fixed for the entire evolution time. This leads us to conjecture that processes of type II are more efficient in eliminating fine details of mixed systems.

The processes of types I and II are paradigms of various physical situations and the difference found here may give hope to develop statistical theories for mixed systems as proposed by Refs. [8–10] and others, and to classify relevant averaging processes into few classes.

ACKNOWLEDGMENTS

The work was supported in part by the Israel Science Foundation (ISF) Grant No. 1028/12, and by the US-Israel Binational Science Foundation (BSF) Grant No. 2010132, by the Shlomo Kaplansky academic chair. We would like to thank James Meiss, Edward Ott, and Hagar Veksler for fruitful discussions.

APPENDIX: CALCULATING THE DIFFUSION COEFFICIENT D_E

We start the calculation from Eq. (12). The Fourier expansion is

$$P(\theta,J,t) = \frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, a_m^t(k)e^{i(m\theta+kJ)}. \tag{A1}$$

We now turn to find the recursion relation of the expansion coefficient $a_m^t(k)$. Substituting the Fourier expansion for the distribution at time $t-1$, one finds

$$\begin{aligned} P(\theta,J,t) = {} & \int_0^{2\pi} d\theta' \frac{1}{2\pi}\sum_{\tilde{m}=-\infty}^{\infty} e^{-\frac{\sigma^2\tilde{m}^2}{2}+i\tilde{m}(\theta-\theta'-J)}\frac{1}{(2\pi)^2} \\ & \times \sum_{m'=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, a_{m'}^{t-1}\int_{-\infty}^{\infty} dK \\ & \times \left[\frac{1}{\sqrt{2\pi}\sigma_K}e^{-\frac{(K-\bar{K})^2}{2\sigma'^2}}e^{i(m'\theta'+kJ+kK\sin\theta')}\right]. \end{aligned} \tag{A2}$$

Integrating over the noise and averaging over K one finds

$$\begin{aligned} P(\theta,J,t) = {} & \sum_{\tilde{m}=-\infty}^{\infty} e^{-\frac{\sigma^2\tilde{m}^2}{2}+i\tilde{m}\theta}\sum_{m'=-\infty}^{\infty}\int_{-\infty}^{\infty} dk'\, a_{m'}^{t-1}(k')e^{-k'^2\frac{\sigma_K}{4}} \\ & \times \sum_{l=-\infty}^{\infty}\mathcal{J}_l(|k|\bar{K})e^{i(k'-\tilde{m})J}\sum_{l'=-\infty}^{\infty} I_{l'}\left(\frac{k^2\sigma_K^2}{4}\right) \\ & \times \delta(m'-\tilde{m}+l\,\mathrm{sign}k'+2l'). \end{aligned} \tag{A3}$$

Taking the sum over m' and making use of the orthogonality of the Fourier components one finds the recursion relation of the Fourier components,

$$\begin{aligned} a_m^t(k) = {} & e^{-\frac{\sigma^2\tilde{m}^2}{2}}e^{-k'^2\frac{\sigma_K^2}{4}}\sum_{l=-\infty}^{\infty}\mathcal{J}_l(|k|\bar{K}) \\ & \times \sum_{l'=-\infty}^{\infty}\mathcal{I}_{l'}\left(\frac{k^2\sigma_K^2}{4}\right)a^{t-1}_{\tilde{m}-l\,\mathrm{sign}k'-2l'}(k') \end{aligned} \tag{A4}$$

with the following relations:

$$\tilde{m} = m, \quad k' = k+m, \quad m' = m - l\,\mathrm{sign}k' - 2l'. \tag{A5}$$

Next using paths in Fourier space the leading order in the Bessel functions can be evaluated. The diffusion coefficient is

$$D = \lim_{t\to\infty} \frac{\langle (J_t - J_0)^2 \rangle}{2t}, \tag{A6}$$

where [12]

$$\langle (J_t - J_0)^2 \rangle = i^2 \lim_{k\to 0^+} \left(\frac{\partial^2}{\partial k^2}\right) a_0^t(k). \tag{A7}$$

The leading order term is the path which stays at the origin for t steps; that is,

$$a_0^t(k) = \left[e^{-k^2 \frac{\sigma_K^2}{4}} \sum_{l=-\infty}^{\infty} \mathcal{J}_l(|k|\bar{K}) \sum_{l'=-\infty}^{\infty} \mathcal{I}_{l'}\left(\frac{k^2\sigma_K^2}{4}\right)\right]^t a_0^0(0). \tag{A8}$$

Since when differentiating twice and taking the $k \to 0^+$ limit only the $\mathcal{J}_0, \mathcal{I}_0$ contribute, one finds

$$a_0^t \simeq \left[e^{-k^2 \frac{\sigma_K}{4}} \left[1 - \left(\frac{k\bar{K}}{2}\right)^2\right]\left[1 + \left(\frac{k^2\sigma_K^2}{8}\right)^2\right]\right]^t. \tag{A9}$$

The leading term is Eq. (10).

The next term is calculated using a path that leaves the origin. Leaving and returning to the origin have to be done using two different points. The shortest path possible is of the form

$$0,0 \to k, \quad m \to k', \quad m' \to 0,0 \tag{A10}$$

using the relations (A5), when leaving the origin

$$-m = -l - 2l' \tag{A11}$$

and when entering the origin

$$m' = 0 - l - 2l'. \tag{A12}$$

The only paths making a contribution to the diffusion are as in Refs. [12,13],

$$\begin{aligned} &(0,0) \to (1,-1) \to (0,1) \to (0,0),\\ &(0,0) \to (-1,1) \to (0,-1) \to (0,0), \end{aligned} \tag{A13}$$

and each has an equal contribution. For the second transition in the first path we get

$$1 = -1 + l - 2l' \tag{A14}$$

or

$$l = 2 + 2l', \tag{A15}$$

leading to

$$\begin{aligned} a_0^t(k) = {} & 2(t-2)\left(\sum_{l'=-\infty}^{\infty} \mathcal{J}_{2l'}(|k|\bar{K})\mathcal{I}_{l'}\left(\frac{k^2\sigma_K^2}{4}\right)\right)^{t-3} \sum_{l'=-\infty}^{\infty} \mathcal{J}_{-2l'-1}(|k|\bar{K})\mathcal{I}_{l'}\left(\frac{k^2\sigma_K^2}{4}\right) \\ & \times \sum_{l'=-\infty}^{\infty} \mathcal{J}_{2l'+2}(|-1+k|\bar{K})\mathcal{I}_{l'}\left(\frac{(-1+k)^2\sigma_K^2}{4}\right) e^{-\frac{\sigma_K^2}{2}} e^{-(-1+k)^2\frac{\sigma_K^2}{4}} \sum_{l'=-\infty}^{\infty} \mathcal{J}_{-2l'-1}(|k|\bar{K})\mathcal{I}_{l'}\left(\frac{k^2\sigma_K^2}{4}\right) e^{-\frac{\sigma}{2}}. \end{aligned} \tag{A16}$$

Using Eqs. (A6) and (A7) and including the leading term one finds Eq. (14).

[1] J. D. Meiss, Symplectic maps, variational principles, and transport, Rev. Mod. Phys. **64**, 795 (1992).

[2] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction* (Wiley, New York, 1989).

[3] A. J. Lichtenberg and M. A. Lieberman, Regular and chaotic dynamics, in *Applied Mathematical Sciences*, Vol. 38 (Springer, New York, 1983).

[4] G. M. Zaslavsky, M. Edelman, and B. A. Niyazov, Self-similarity, renormalization, and phase space nonuniformity of Hamiltonian chaotic dynamics, Chaos **7**, 159 (1997).

[5] G. M. Zaslavsky and M. Edelman, Hierarchical structures in the phase space and fractional kinetics: I. Classical systems, Chaos **10**, 135 (2000).

[6] B. V. Chirikov and D. L. Shepelyansky, Correlation properties of dynamical chaos in Hamiltonian systems, Phys. D (Amsterdam, Neth.) **13**, 395 (1984).

[7] O. Alus, S. Fishman, and J. D. Meiss, Statistics of the island-around-island hierarchy in Hamiltonian phase space, Phys. Rev. E **90**, 062923 (2014).

[8] J. D. Meiss and E. Ott, Markov-Tree Model of Intrinsic Transport in Hamiltonian Systems, Phys. Rev. Lett. **55**, 2741 (1985).

[9] G. Cristadoro and R. Ketzmerick, Universality of Algebraic Decays in Hamiltonian Systems, Phys. Rev. Lett. **100**, 184101 (2008).

[10] R. Ceder and O. Agam, Fluctuations in the relaxation dynamics of mixed chaotic systems, Phys. Rev. E **87**, 012918 (2013).

[11] R. Venegeroles, Universality of Algebraic Laws in Hamiltonian Systems, Phys. Rev. Lett. **102**, 064101 (2009).

[12] A. B. Rechester, M. N. Rosenbluth, and R. B. White, Fourier-space paths applied to the calculation of diffusion for the Chirikov-Taylor model, Phys. Rev. A **23**, 2664 (1981).

[13] A. B. Rechester and R. B. White, Calculation of Turbulent Diffusion for the Chirikov-Taylor Model, Phys. Rev. Lett. **44**, 1586 (1980).

[14] E. G. Altmann and A. Endler, Noise-Enhanced Trapping in Chaotic Scattering, Phys. Rev. Lett. **105**, 244102 (2010).

[15] F. J. Romeiras, C. Grebogi, and E. Ott, Multifractal properties of snapshot attractors of random maps, Phys. Rev. A **41**, 784 (1990).

[16] T. Bódai, E. G. Altmann, and A. Endler, Stochastic perturbations in open chaotic systems: Random versus noisy maps, Phys. Rev. E **87**, 042902 (2013).

[17] M. Trepanier, D. Zhang, O. Mukhanov, and S. M. Anlage, Realization and Modeling of Metamaterials Made of rf Superconducting Quantum-Interference Devices, Phys. Rev. X **3**, 041029 (2013).
[18] N. Friedman, A. Kaplan, D. Carasso, and N. Davidson, Observation of Chaotic and Regular Dynamics in Atom-Optics Billiards, Phys. Rev. Lett. **86**, 1518 (2001).
[19] V. Milner, J. L. Hanssen, W. C. Campbell, and M. G. Raizen, Optical Billiards for Atoms, Phys. Rev. Lett. **86**, 1514 (2001).
[20] M. Khodas and S. Fishman, Relaxation and Diffusion for the Kicked Rotor, Phys. Rev. Lett. **84**, 2837 (2000).
[21] M. Khodas, S. Fishman, and O. Agam, Relaxation to the invariant density for the kicked rotor, Phys. Rev. E **62**, 4769 (2000).
[22] R. Venegeroles, Calculation of Superdiffusion for the Chirikov-Taylor Model, Phys. Rev. Lett. **101**, 054102 (2008).
[23] J. R. Cary, J. D. Meiss, and A. Bhattacharjee, Statistical characterization of periodic, area-preserving mappings, Phys. Rev. A **23**, 2744 (1981).
[24] B. V. Chirikov, A universal instability of many-dimensional oscillator systems, Phys. Rep. **52**, 263 (1979).
[25] C. F. F. Karney, A. B. Rechester, and R. B. White, Effect of noise on the standard mapping, Phys. D (Amsterdam, Neth.) **4**, 425 (1982).
[26] A. Kruscha, R. Ketzmerick, and H. Kantz, Biased diffusion inside regular islands under random symplectic perturbations, Phys. Rev. E **85**, 066210 (2012).