Relative criterion for validity of a semiclassical approach to the dynamics near quantum critical points

Qian Wang,[1] Pinquan Qin,[2] and Wen-ge Wang[1,*]
[1]*Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China*
[2]*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
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Based on an analysis of Feynman's path integral formulation of the propagator, a relative criterion is proposed for validity of a semiclassical approach to the dynamics near critical points in a class of systems undergoing quantum phase transitions. It is given by an effective Planck constant, in the relative sense that a smaller effective Planck constant implies better performance of the semiclassical approach. Numerical tests of this relative criterion are given in the XY model and in the Dicke model.

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I. INTRODUCTION

Quantum phase transitions (QPTs) have attracted a great deal of attention in recent years [1–13], with most of the studies focusing on equilibrium properties. Recently, due to significant progress in experimental techniques, simulation of the time evolution of models undergoing QPTs is becoming realizable [14,15], and their dynamical properties are receiving increasing attention. Only a few analytical methods have been shown to be useful in the study of the dynamics near QPTs, for example the so-called Kibble-Zurek mechanism for slow variation of a controlling parameter λ passing the critical point [16–24], and the adiabatic perturbation theory [25–28].

For quench dynamics, a semiclassical approach has been found to be useful in several models in predicting decaying behaviors of the survival probability (SP) near quantum critical points at which the ground levels have infinite degeneracy [29,30]. Numerical results suggest that the semiclassical approach may work well even in the neighborhood of the critical points [31]. However, not much is known about the condition under which this semiclassical approach is applicable; in fact, only qualitative arguments have been given [29], based on properties such as a high density of states and relatively high initial energy.

In this paper, we study the above-mentioned condition in a more quantitative way. Based on analysis of Feynman's path-integral formulation of the propagator, we propose that an effective Planck constant can be introduced and used to give a relative criterion for the validity of the semiclassical approach. That is, a smaller effective Planck constant should imply a better performance of the semiclassical approach. To numerically test this prediction, we utilize the SP, which has been shown recently, both theoretically [7–9,32–35] and experimentally [36,37], to exhibit a highly enhanced decay near critical points and can be used as an indicator of QPTs. We test the criterion in two models, namely a one-dimensional (1D) XY model [38–40] and the Dicke model [41], which belong to two different universal classes, respectively [29,35].

The paper is organized as follows. In Sec. II, we introduce the above-mentioned effective Planck constant and present arguments that lead to the relative criterion. In Sec. III, we discuss the properties of the effective Planck constant in the two models mentioned above. Then, in Sec. IV, we numerically study the deviation of the SP decay from predictions of the semiclassical theory in the two models. The numerical results show that the relative criterion works well. Finally, conclusions and discussions are presented in Sec. V.

*wgwang@ustc.edu.cn

II. A RELATIVE CRITERION FOR THE VALIDITY OF THE SEMICLASSICAL APPROACH

In this section, we present arguments that lead to a relative criterion for the validity of the semiclassical approach to dynamics near quantum critical points, which was discussed in the preceding section. We consider systems whose Hamiltonians are written as $H(\lambda)$, with a critical point at $\lambda = \lambda_c$ of the controlling parameter λ. We assume that the ground level of $H(\lambda_c)$ has infinite degeneracy in the thermodynamic limit and that each system $H(\lambda)$ with λ close to λ_c has a classical counterpart at least in the low-energy region. For the sake of convenience in presentation, here we introduce some notation that is to be used in what follows, namely

$$\delta = \lambda_c - \lambda, \quad \delta_0 = \lambda_c - \lambda_0, \quad \epsilon = \lambda - \lambda_0. \tag{1}$$

A. Basic ideas

Feynman's path-integral formulation of quantum mechanics supplies the most convenient framework for discussing the semiclassical approach. For a quantum system whose classical counterpart lies in a d-dimensional configuration space with points denoted by q, in terms of Feynman's path integral, the time evolution of an initial packet $\psi_0(q_0,t_0)$ is written as

$$\psi(q_b,t_b) = \int dq_0 K(q_b,t_b;q_0,t_0)\psi_0(q_0,t_0). \tag{2}$$

Here, $K(q_b,t_b;q_0,t_0)$ is the propagator [42,43],

$$K(q_b,t_b;q_0,t_0) = \int_\Lambda \mathcal{D}[C]e^{iS/\hbar}, \tag{3}$$

where Λ denotes the set of paths that carries an initial point q_0 at an initial time t_0 to a final point q_b at a final time t_b, $C = \{q(t)\}$ indicates an element of Λ, and $S = \int_{t_0}^{t_b} dt\, L(q(t), \frac{d}{dt}q(t))$ is the action along a path, with L denoting the Lagrangian function of the system. In 1D configuration space, the path-integral symbol $\int_\Lambda \mathcal{D}[C]$ is defined by

$$\int_\Lambda \mathcal{D}[C] = \lim_{n\to\infty} \sqrt{\frac{1}{2i\hbar\pi\varepsilon}} \prod_{k=1}^{n-1} \left[\int_{-\infty}^{+\infty} \sqrt{\frac{1}{2i\hbar\pi\varepsilon}} dq_{(k)} \right], \quad (4)$$

where $\varepsilon = (t_b - t_0)/n$.

In the case in which the action S changes sufficiently fast with a variation of the paths and its amplitude is sufficiently large compared with $\hbar$, the stationary-phase approximation may become valid. In this case, the main contribution to the propagator should be given by those paths not far from the stationary solutions satisfying $\delta S = 0$, leading to the well-known semiclassical Van Vleck–Gutzwiller propagator,

$$K_{\rm sc} = \sum_s \frac{D_s^{1/2}}{(2\pi i\hbar)^{d/2}} \exp\left[\frac{i}{\hbar} S_s - \frac{i\pi}{2}\mu_s\right], \quad (5)$$

where the label s [more exactly $s(q_b,t_b;q_0,t_0)$] indicates a classical trajectory starting from q_0 and ending at q_b within a time period $[t_0,t_b]$, S_s is the action along the trajectory s, $D_s = |\det(\partial^2 S_s/\partial q_{0i}\partial q_{bj})|$, and μ_s is the Maslov index.

Due to complexity of the paths, it is usually not an easy task to show directly whether the action S varies sufficiently fast with a sufficiently large amplitude to guarantee the applicability of the stationary-phase approximation. Therefore, we adopt an alternative approach to the problem of the validity of semiclassical treatment. Our approach is based on the observation that, at least for an important class of systems undergoing QPTs, their classical counterparts have the following rescaling property in the neighborhood of the critical points. That is, for each pair (λ,λ') close to λ_c, some rescaling procedure can carry classical trajectories of the system $H(\lambda)$ to those of the system $H(\lambda')$, at least in an approximate way. (A class of systems possessing this property will be discussed in Sec. II C.)

The basic ideas of our approach are as follows: First, for systems possessing the above-discussed rescaling property, the integrand $e^{iS/\hbar}$ in Eq. (3) for a classical trajectory of each system $H(\lambda)$ can be written in terms of the related quantity for a fixed system $H(\lambda_m)$ of a fixed value λ_m. Specifically, with the dependence of S on λ written explicitly, one may write

$$\frac{1}{\hbar} S(\lambda) = \frac{g}{\hbar_{\rm eff}(\lambda)} S(\lambda_m), \quad (6)$$

where $\hbar_{\rm eff}$ is some effective Planck constant as a function of λ, and g is a quantity that does not change much with variation of λ, say, within one order of magnitude. [The explicit form of $\hbar_{\rm eff}(\lambda)$ will be discussed in the following subsection.] Second, (nonclassical) paths not far from the classical trajectories in the two systems $H(\lambda)$ and $H(\lambda_m)$ should have a relation similar to that discussed above for classical trajectories. As a result, $S(\lambda)$ along these paths can also be written in terms of those for λ_m. Third, since the action $S(\lambda_m)$ is independent of λ and the quantity g can be regarded approximately as a constant, variation of $S(\lambda)/\hbar$ lies mainly in the variation of the quantity $\hbar_{\rm eff}(\lambda)$. This suggests that a smaller $\hbar_{\rm eff}(\lambda)$ should imply a better performance of the stationary phase approximation, and thus one has a relative criterion for the validity of the semiclassical approach.

B. Detailed discussions for a relative criterion

In this subsection, we discuss the details of the relative criterion discussed above. In many cases, one is interested in a time period $[t_0,t_b]$ whose length is proportional to some characteristic time scale τ of the system $H(\lambda)$, namely, $T = (t_b - t_0) \propto \tau$. Near the critical point, the characteristic time scales as $\tau \sim |\lambda - \lambda_c|^{-\alpha}$, with some exponent $\alpha > 0$, and it diverges at the critical point. For brevity, we do not write explicitly the λ dependence of τ.

Since $T \propto \tau$, we can perform a time rescaling, such that the rescaled time periods of interest in systems $H(\lambda)$ lie in the same region. Specifically, we do this by

$$t \to \widetilde{t} = t/\eta \quad \text{with } \eta = \tau/\tau_m, \quad (7)$$

where τ_m is the characteristic time scale of the system $H(\lambda_m)$. Here, we use a tilde to indicate rescaled quantities. For example, the rescaled time region is written as $[\widetilde{t_0},\widetilde{t_b}]$ with a length $\widetilde{T} = \widetilde{t_b} - \widetilde{t_0} \propto \tau_m$.

Let us consider systems possessing the rescaling property discussed in the previous subsection, with the time rescaling in Eq. (7) being part of it. The action along a trajectory s of the system $H(\lambda)$ has the expression $S(s) = \int_{t_0}^{t_b} L\, dt$. Using $\overline{L}$ to denote the average value of L along the trajectory s and making use of the time rescaling in Eq. (7), this action is written as $S(s) = \overline{L}\eta\widetilde{T}$. Using the rescaling procedure mentioned above, the trajectory s is mapped to a rescaled trajectory $\widetilde{s}$ that is close to a trajectory s_m of the system $H(\lambda_m)$. Using $\overline{L_m}$ to denote the average value of the Lagrangian along the trajectory s_m, the action for s_m is written as $S(s_m) = \int L_m dt_m = \overline{L_m}\widetilde{T}$. If we write $\overline{L} = \gamma\overline{L_m}$, then we have

$$S(s) = \gamma\eta S(s_m). \quad (8)$$

Furthermore, we note that Eq. (8) also holds, approximately, for Feynman paths close to the classical trajectories s and s_m in the two system, respectively. For such paths in the system $H(\lambda)$, the integrand in Eq. (3) can be written in terms of S_m of the fixed system $H(\lambda_m)$, namely $e^{i\gamma\eta S_m/\hbar}$. The point is that, with variation of λ, S_m remains unchanged and it is $(\gamma\eta)$ that changes. It is not difficult to see that, if $(\gamma\eta)$ may become sufficiently large, it should be reasonable to expect that the stationary approximation can be applicable and the semiclassical approach can be valid. Clearly, the quantity $\hbar/(\gamma\eta)$ plays the role of a rescaled, effective Planck constant.

With the system $H(\lambda_m)$ fixed, to get a criterion in a *relative* sense for the validity of the semiclassical approach, one can use τ to replace η. Similarly, one can use $\overline{L}$ to replace γ. Moreover, in most cases, apart from some constant, the value of $\overline{L}$ lies in the same order of magnitude as the initial energy. Hence, instead of $\overline{L}$, we can consider the expectation value of the energy of the initial state with respect to the ground level, which we denote by Δ_E. For example, for an initial state such as the ground state of $H(\lambda_0)$, denoted by $|\Psi_0(\lambda_0)\rangle$, Δ_E is given by

$$\Delta_E = \langle\Psi_0(\lambda_0)|H(\lambda)|\Psi_0(\lambda_0)\rangle - E_0(\lambda), \quad (9)$$

where $E_0(\lambda)$ denotes the ground level of $H(\lambda)$. Finally, in the relative criterion discussed above, one can use the following form of the effective Planck constant:

$$\hbar_{\text{eff}} = \frac{\hbar}{\tau \Delta_E}. \tag{10}$$

Hereafter, we set $\hbar$ unit.

To summarize, based on the arguments given above, we propose the following relative criterion for the validity of the semiclassical approach to dynamics near a quantum critical point. That is, if (i) trajectories of the classical counterparts of the system $H(\lambda)$ have similar shapes after a certain rescaling procedure, and (ii) the contribution from Feynman paths far from classical trajectories can be neglected, then a smaller effective Planck constant $\hbar_{\text{eff}}$ should imply a better performance of the semiclassical approach.

C. Classical trajectories in a class of systems possessing the required rescaling property

In this subsection, we show that a prerequisite of the relative criterion given above, namely the closeness of rescaled classical trajectories, is satisfied in a class of systems undergoing QPTs. In fact, in many cases, in the low-energy region of a system close to a critical point, the Hamiltonian can be approximately written as harmonic oscillators, i.e.,

$$H(\lambda) \simeq \sum_{i=1}^{n} \omega_i(\lambda) c_i^\dagger(\lambda) c_i(\lambda), \tag{11}$$

where $c_i^\dagger(\lambda)$ and $c_i(\lambda)$ are *bosonic* creation and annihilation operators for the ith mode with frequency $\omega_i(\lambda)$. In terms of the action-angle variables, the classical motion is given by

$$\theta_i(t) = \omega_i(\lambda)t + \theta_i(0), \quad I_i(t) = I_i(0). \tag{12}$$

In the vicinity of the critical point, one may focus on the zero modes. For these modes, the frequencies $\omega_i(\lambda)$ have the following form:

$$\omega_i(\lambda) \simeq \nu_i(\lambda_c)|\delta|^{\alpha_i}, \tag{13}$$

where δ is defined in Eq. (1). If these zero modes belong to the same class of universality, their values of α_i should be equal, say, being α. Then, one has

$$\frac{\omega_i(\lambda)}{\omega_i(\lambda_m)} \simeq \left|\frac{\delta}{\delta_m}\right|^{\alpha}, \tag{14}$$

independent of the mode i. Making use of the relation in Eq. (14), it is not difficult to find a rescaling procedure, under which trajectories of the classical system $H(\lambda)$ are mapped to those of $H(\lambda_m)$.

III. PROPERTIES OF THE EFFECTIVE PLANCK CONSTANT IN TWO MODELS

To test the above-discussed relative criterion for the validity of the semiclassical approach, we first need to study the properties of the effective Planck constant $\hbar_{\text{eff}}$ in Eq. (10) in some concrete models. In this section we do this in two models, namely the XY model [38–40] and the Dicke model [41].

A. Effective Planck constant in an XY chain

We first discuss an XY chain of N $\frac{1}{2}$-spins in a transverse external field. The Hamiltonian is written as

$$H = -\sum_{i=1}^{N}\left(\frac{1+\gamma}{2}\sigma_i^x\sigma_{i+1}^x + \frac{1-\gamma}{2}\sigma_i^y\sigma_{i+1}^y + \lambda\sigma_i^z\right), \tag{15}$$

where γ gives a measure to the anisotropy of the in-plane interaction, and λ denotes the intensity of the external magnetic field applied along the z axis. We use the periodic boundary condition. This model reduces to an Ising chain in a transverse field at $\gamma = 1$ and gives the XX model for $\gamma = 0$. This XY model has two critical points $\lambda_c = \pm 1$, independent of the value of γ, with the correlation length scaling as $|\lambda - \lambda_c|^{-\nu}$ [6,8,44,45]. Without loss of generality, we discuss only the critical point $\lambda_c = 1$.

The Hamiltonian in Eq. (15) can be diagonalized analytically. To do this, one may first map the system to that of spinless fermions through the Jordan-Wigner transformation [11,38–40],

$$\begin{aligned}
\sigma_i^x &= \prod_{j<i}(1 - 2d_j^\dagger d_j)(d_i^\dagger + d_i),\\
\sigma_i^y &= -i\prod_{j<i}(1 - 2d_j^\dagger d_j)(d_i^\dagger - d_i),\\
\sigma_i^z &= 1 - 2d_i^\dagger d_i,
\end{aligned} \tag{16}$$

where $d_i^\dagger$ and d_i are fermionic creation and annihilation operators for a site i. In terms of $d_i^\dagger$ and d_i, the Hamiltonian is written as

$$\begin{aligned}
H = &-\sum_{i=1}^{N}[d_{i+1}^\dagger d_i + d_i^\dagger d_{i+1} + \gamma(d_i^\dagger d_{i+1}^\dagger + d_i d_{i+1})]\\
&- \lambda\sum_{i=1}^{N}(1 - 2d_i^\dagger d_i).
\end{aligned} \tag{17}$$

Next, transform to momentum space by the transformation

$$d_k^\dagger = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} d_i^\dagger e^{kr_i}, \quad d_k = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} d_i e^{-kr_i}, \tag{18}$$

where $d_k^\dagger$ and d_k are fermionic creation and annihilation operators for a wave vector k, and r_i represents the position at a site i. Setting as a unit the distance between neighboring sites, the wave vectors k are given by $k = 2\pi m/N$, where $m = -N/2, \ldots, (N/2 - 1)$ for an even N. Then, the Hamiltonian is written as

$$H = \sum_k [2(\lambda - \cos k)d_k^\dagger d_k + i\gamma \sin k(d_{-k}^\dagger d_k^\dagger + d_{-k}d_k) - \lambda]. \tag{19}$$

This Hamiltonian can be diagonalized by the following Bogoliubov transformation:

$$\begin{aligned}
f_k &= u_k(\lambda)d_k - i v_k(\lambda)d_{-k}^\dagger,\\
f_k^\dagger &= u_k(\lambda)d_k^\dagger + i v_k(\lambda)d_{-k},
\end{aligned} \tag{20}$$

where

$$u_k(\lambda) = \cos[\theta_k(\lambda)/2], \quad v_k(\lambda) = \sin[\theta_k(\lambda)/2], \tag{21}$$

with θ_k defined by

$$\tan[\theta_k(\lambda)] = \frac{\gamma \sin k}{\lambda - \cos k}. \tag{22}$$

The result is

$$H = \sum_{k>0} \varepsilon_k \left(f_k^\dagger f_k - \frac{1}{2} \right), \tag{23}$$

where

$$\varepsilon_k = 2\sqrt{(\lambda - \cos k)^2 + \gamma^2 \sin^2 k}. \tag{24}$$

When λ is sufficiently close to $\lambda_c = 1$, in the low-energy region with $|m| \ll N$, Eq. (24) gives

$$\varepsilon_k \simeq 4\pi\gamma |m|/N. \tag{25}$$

Due to this linear dependence of ε_k on m, these fermionic modes can be mapped to bosonic modes via the method of bosonization [11]. The bosonic modes, labeled by r with $r = 1,2,\ldots$, have energies $E_{\rm bos}^{(r)} = \omega_r \hbar_{\rm bos}$, where

$$\omega_r \approx r \tag{26}$$

and $\hbar_{\rm bos} = 4\pi\gamma/N$. A characteristic time of the system is given by

$$\tau = \frac{2\pi}{E_{\rm bos}^{(1)}} \simeq \frac{N}{2\gamma}. \tag{27}$$

To compute Δ_E, we express $|\Psi_0(\lambda_0)\rangle$ in terms of the eigenmodes diagonalizing $H(\lambda)$ in Eq. (23). Using η_k to represent the annihilation operators for the eigenmodes of $H(\lambda_0)$, η_k and f_k are connected by the relation [7,8]

$$\eta_{\pm k} = \cos\alpha_k f_{\pm k} - i \sin\alpha_k f_{\mp k}^\dagger, \tag{28}$$

where

$$\alpha_k = [\theta_k(\lambda) - \theta_k(\lambda_0)]/2. \tag{29}$$

Note that $\eta_k|\Psi_0(\lambda_0)\rangle = 0$ for all k. The ground state $|\Psi_0(\lambda_0)\rangle$ can be rewritten as [7,8]

$$|\Psi_0(\lambda_0)\rangle = \prod_k [\cos\alpha_k - i \sin\alpha_k f_k^\dagger f_{-k}^\dagger] |\Psi_0(\lambda)\rangle, \tag{30}$$

where $|\Psi_0(\lambda)\rangle$ is the ground state of $H(\lambda)$. Let us compute

$$\begin{aligned} &\langle \Psi_0(\lambda_0)|H(\lambda)|\Psi_0(\lambda_0)\rangle \\ &= \langle \Psi_0(\lambda)| \prod_k [\cos\alpha_k + i \sin\alpha_k f_{-k} f_k] \\ &\times H(\lambda) \prod_k [\cos\alpha_k - i \sin\alpha_k f_k^\dagger f_{-k}^\dagger] |\Psi_0(\lambda)\rangle. \end{aligned} \tag{31}$$

Making use of the relation

$$H(\lambda) f_k^\dagger = f_k^\dagger [\varepsilon_k(\lambda) + H(\lambda)], \tag{32}$$

after some algebra, we get

$$\langle \Psi_0(\lambda_0)|H(\lambda)|\Psi_0(\lambda_0)\rangle = E_0(\lambda) + \sum_{k>0} 2\varepsilon_k(\lambda) \sin^2\alpha_k. \tag{33}$$

Finally, we have

$$\Delta_E = \sum_{k>0} 2\varepsilon_k(\lambda) \sin^2\alpha_k. \tag{34}$$

To see the main properties of the energy scale Δ_E in Eq. (34), let us compute $\sin^2\alpha_k$. Making use of Eqs. (22) and (29), after some algebra, we get

$$\sin^2\alpha_k = \frac{1}{2} - \frac{1}{2} \frac{p_k}{\sqrt{p_k^2 + q_k^2}}, \tag{35}$$

where

$$p_k = (\lambda_0 - \cos k)(\lambda - \cos k) + \gamma^2 \sin^2 k, \tag{36}$$

$$q_k = (\lambda_0 - \lambda)\gamma \sin k. \tag{37}$$

For fixed values of δ and δ_0, numerically we found that, when N is sufficiently large, the main contribution to Δ_E comes from those k for which $|k| \ll |\delta|$ and $|k| \ll |\delta_0|$ and the number of these k is proportional to N. Making use of this fact, and the fact that $|\delta| \ll \gamma$ and $|\delta_0| \ll \gamma$, we get $\varepsilon_k \approx 2\delta$ and

$$\sin^2\alpha_k \approx \frac{(\delta - \delta_0)^2}{4\delta^2\delta_0^2} \gamma^2 k^2. \tag{38}$$

Therefore, for sufficiently large N,

$$\Delta_E \propto \frac{\gamma^2 N}{\delta \delta_0^2} (\delta - \delta_0)^2, \tag{39}$$

proportional to both N and $(\delta - \delta_0)^2$ (see Fig. 1 for direct numerical simulations).

Now, we can discuss the behaviors of the effective Planck constant $\hbar_{\rm eff}$ in Eq. (10). Substituting Eqs. (27) and (34) into Eq. (10), we get

$$\hbar_{\rm eff} = \frac{\gamma}{N \sum_k \varepsilon_k(\lambda) \sin^2\alpha_k}. \tag{40}$$

Variation of $\hbar_{\rm eff}$ with δ for a fixed δ_0 and a fixed N is shown in Fig. 2. It is seen that $\hbar_{\rm eff}$ reduces to a quite small but nonzero value when $\delta \to 0$. On the other hand, when $\delta \to \delta_0$, one has

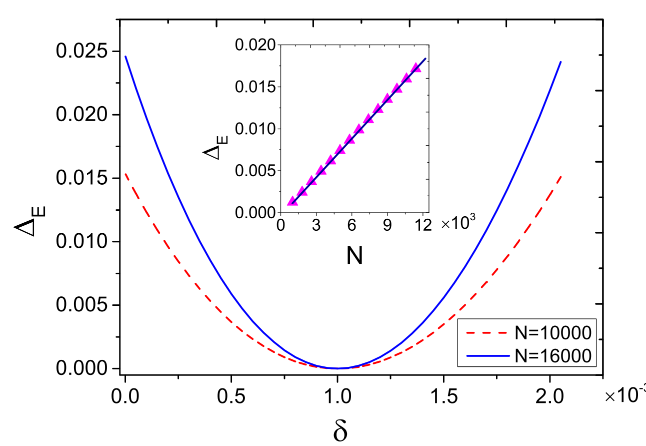

FIG. 1. (Color online) Variation of the energy scale Δ_E in Eq. (34) with δ in the XY model for $\delta_0 = 10^{-3}$ and $\gamma = 0.7$. Inset: Δ_E (triangles) increases linearly with the particle number N [see Eq. (39)], with $\delta = 10^{-5}$. The solid line is a fitting straight line.

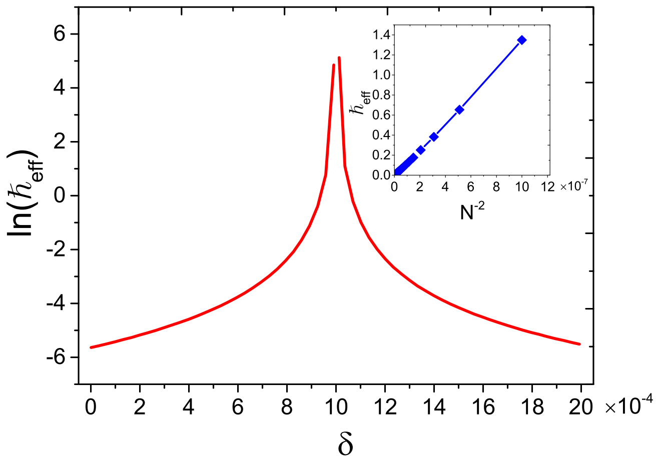

FIG. 2. (Color online) The effective Planck constant $\hbar_{\rm eff}$ (in the logarithm scale) vs δ in the XY model, for $N = 16\,000$, $\delta_0 = 10^{-3}$, and $\gamma = 0.7$. Inset: $\hbar_{\rm eff}$ is approximately proportional to N^{-2} ($\delta = 4 \times 10^{-4}$).

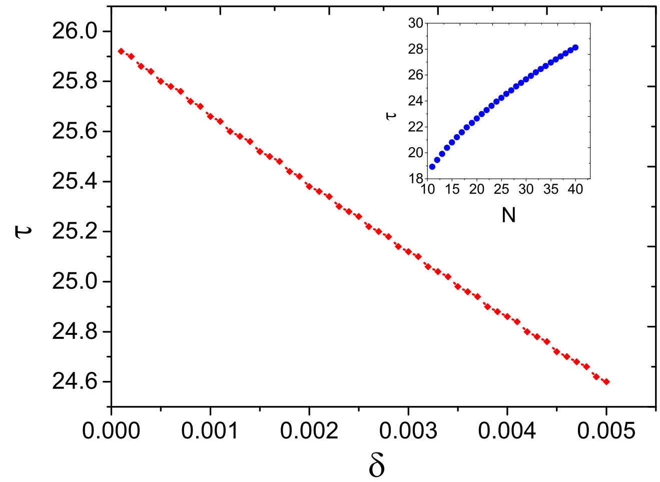

FIG. 3. (Color online) Variation of a characteristic time τ with δ in the Dicke model, for $N = 30$, $\delta_0 = 10^{-2}$, and $\omega = \omega_0 = 1$. Inset: τ vs the system's size N for $\delta = 10^{-3}$.

$\Delta_E \to 0$; as a result, $\hbar_{\rm eff} \to \infty$. Moreover, since both τ and Δ_E are linear in the system's size N, the effective Planck constant decreases as $1/N^2$ (see the inset of Fig. 2).

B. Effective Planck constant in the Dicke model

The Dicke model describes the interaction of N atoms with a number of bosonic modes via dipole interaction within an ideal cavity [41]. In our study, we consider a single-mode bosonic field. In terms of a collective operator $\mathbf{J}$ for the N atoms, whose components satisfy the same commutation relations as those of the angular momentum, the single-mode Dicke Hamiltonian is written as [46]

$$H(\lambda) = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(J_+ + J_-). \quad (41)$$

Making use of the Holstein-Primakoff representation [47–49],

$$\begin{aligned} J_+ &= b^\dagger \sqrt{2j - b^\dagger b}, \\ J_- &= \sqrt{2j - b^\dagger b}\, b, \\ J_z &= b^\dagger b - j, \end{aligned} \quad (42)$$

where b and $b^\dagger$ satisfy $[b,b^\dagger] = 1$ and $j = N/2$, the Hamiltonian can be diagonalized in the thermodynamic limit [46], giving

$$H(\lambda) = \sum_{n=1,2} \varepsilon_n(\lambda) c_n^\dagger(\lambda) c_n(\lambda) + f, \quad (43)$$

where $c_n^\dagger(\lambda)$ and $c_n(\lambda)$ are bosonic creation and annihilation operators, $\varepsilon_n(\lambda)$ are the quasiparticle energies in increasing order, and f is a c-number function. In this limit, the system undergoes a QPT at a critical value $\lambda_c = \sqrt{\omega\omega_0}/2$, with a normal phase for $\lambda < \lambda_c$ and a superradiant phase for $\lambda > \lambda_c$. In the normal phase,

$$\varepsilon_{1,2}(\lambda) = \left\{\tfrac{1}{2}\left[(\omega^2 + \omega_0^2) \pm \sqrt{(\omega_0^2 - \omega^2)^2 + 16\lambda^2\omega\omega_0}\right]\right\}^{1/2}, \quad (44)$$

and in the superradiant phase,

$$\varepsilon_{1,2}(\lambda) = \left\{\frac{1}{2}\left[\omega^2 + \frac{\omega_0^2}{\mu^2} \pm \sqrt{\left(\frac{\omega_0^2}{\mu^2} - \omega^2\right)^2 + 4\omega^2\omega_0^2}\right]\right\}^{1/2}, \quad (45)$$

where $\mu = \omega\omega_0/(4\lambda^2)$. It is easy to check that $\varepsilon_1(\lambda) = 0$ at λ_c in both phases.

In the thermodynamic limit, the Dicke model has a characteristic time scale τ given by

$$\tau = \frac{2\pi}{\varepsilon_1(\lambda)}, \quad (46)$$

diverging in the limit $\lambda \to \lambda_c$, and the SP has a period $\tau/2$ [50]. For finite N, we utilize the revival of the SP to compute numerically the characteristic time τ. In Fig. 3, it is seen that τ increases with decreasing δ and remains finite in the limit $\delta \to 0$ at finite N.

The energy scale Δ_E in the Dicke model cannot be calculated analytically, and to understand its properties we have to resort to numerical simulations (see Fig. 4 for an example). Making use of the obtained Δ_E, the effective Planck constant $\hbar_{\rm eff}$ can be computed. Variation of $\hbar_{\rm eff}$ with δ is shown in Fig. 5. It is seen that $\hbar_{\rm eff}$ in the Dicke model behaves in a way that is qualitatively similar to that in the XY model discussed above, e.g., it also decreases with increasing N (inset of Fig. 5).

IV. NUMERICAL TEST OF THE RELATIVE CRITERION IN TWO MODELS

In this section, we discuss a numerical test for the relative criterion proposed in Sec. II, that is, a smaller effective Planck constant $\hbar_{\rm eff}$ implying a better performance of the semiclassical approach. The two models discussed in the previous section satisfy a prerequisite of the relative criterion, which is discussed in Sec. II C (for the XY model, see discussions in Sec. IV A for details), hence they are suitable for this purpose.

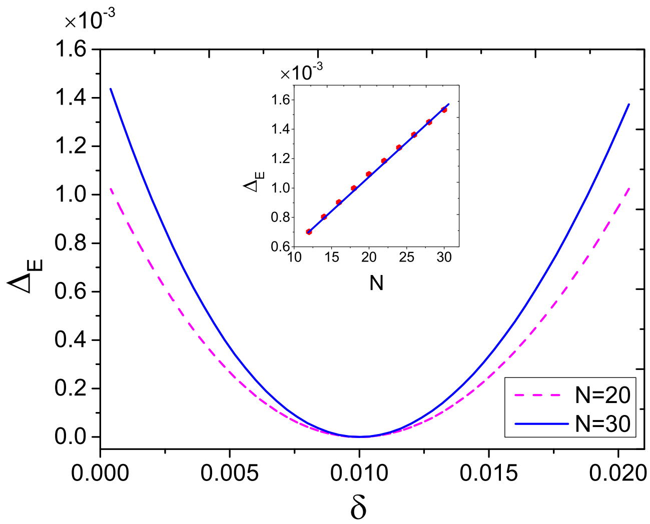

FIG. 4. (Color online) Variation of Δ_E with δ in the Dicke model for $\delta_0 = 10^{-2}$ and $\omega = \omega_0 = 1$. Inset: Δ_E increases almost linearly with the system's size N, for $\delta_0 = 10^{-2}$ and $\delta = 10^{-4}$.

Specifically, we study the SP decay for an initial state as the ground state of $H(\lambda_0)$, written as

$$M(t) = |\langle\Psi_0(\lambda_0)|e^{-iH(\lambda)t}|\Psi_0(\lambda_0)\rangle|^2. \quad (47)$$

The SP is a special case of the so-called quantum Loschmidt echo (LE) [51–53], defined by the overlap of the time evolution of the same initial state under two slightly different Hamiltonians, namely

$$M_{\mathrm{LE}}(t) = |\langle\Psi(0)|\exp[iH(\lambda)t]\exp[-iH(\lambda_0)t]|\Psi(0)\rangle|^2. \quad (48)$$

In the study of the LE, usually one writes $H(\lambda) = H(\lambda_0) + \epsilon V$. In recent years, the LE has been studied extensively in the field of quantum chaos [54–69].

The semiclassical approach has been shown to be useful in predicting decaying behaviors of the SP for times that are not long in the two models discussed above [29]. Here, we are interested in the deviation of the semiclassical prediction from the exact SP. Suppose we consider a time period from $t = 0$ to $t = T_L$. We divide this time period into m intervals, separated by instants $t_i = iT_L/m$ of $i = 1,2,\ldots,m-1$. The following quantity gives a measure to the deviation mentioned above:

$$D \equiv \sqrt{\frac{1}{m}\sum_{i=1}^{m}(x_i - \bar{x})^2}, \quad (49)$$

where

$$x_i \equiv \left|\frac{\ln M(t_i) - \ln M_{\mathrm{sc}}(t_i)}{\ln M(t_i)}\right| \quad (50)$$

and $\bar{x}$ indicates the average value of x_i. Here, $M_{\mathrm{sc}}(t)$ represents the semiclassical prediction of the SP. To test the relative criterion, we must compare the behaviors of the derivation D and those of the effective Planck constant $\hbar_{\mathrm{eff}}$ with the variation of the parameter δ.

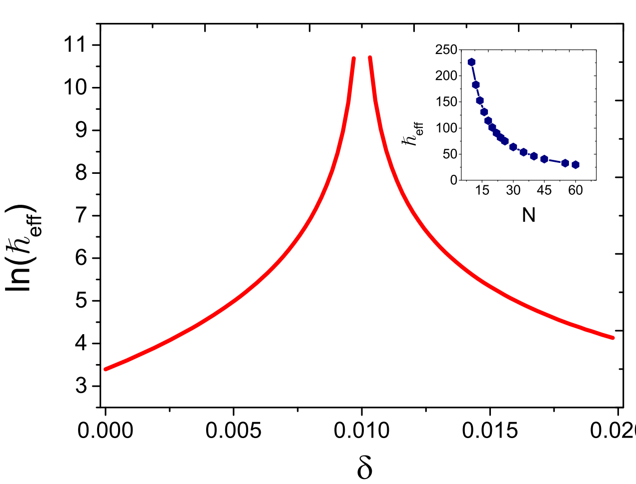

FIG. 5. (Color online) Variation of $\hbar_{\mathrm{eff}}$ (in the logarithm scale) with δ in the Dicke model, for $N = 30$, $\delta_0 = 10^{-2}$, and $\omega = \omega_0 = 1$. Inset: $\hbar_{\mathrm{eff}}$ vs the system's size N for $\delta = 10^{-4}$.

A. Test in the XY model

In the XY model, the SP has the following expression [8]:

$$M(t) = \prod_{k>0} F_k, \quad (51)$$

where

$$F_k = 1 - \sin^2(2\alpha_k)\sin^2(\varepsilon_k t) \quad (52)$$

evaluated at the parameter value λ. In this model, when δ is small, the SP shows a periodic behavior in time, with a period proportional to the system size N [7,32]. In our numerical simulations, we found that the period is given by $\tau/2$, where τ is the characteristic time given in Eq. (27). However, for δ large the SP exhibits an irregular behavior [32].

The model does not have any classical counterpart in the whole energy region. However, in the close neighborhood of the critical point and in the low-energy region, as discussed in Ref. [29], a classical counterpart can be constructed. In fact, as discussed in Sec. III A, near the critical point λ_c, low-lying states of the system can be mapped to a bosonic system, such as harmonic oscillators with frequencies ω_r. In the case in which the frequencies ω_r are sufficiently incommensurable, the classical counterpart has a motion resembling a chaotic one when the time is not long. In this case, the semiclassical theory predicts an exponential decay of the SP [29],

$$M_{\mathrm{sc}}(t) \simeq e^{-2R\epsilon^2 t/\hbar^2}, \quad (53)$$

where

$$R = \frac{1}{2t}\left(\left\langle\left[\int_0^t V(t)dt\right]^2\right\rangle - \left\langle\int_0^t V(t)dt\right\rangle^2\right). \quad (54)$$

Direct computation of the value of R in the classical system is very difficult, hence, instead, we evaluate it numerically by fitting $M_{\mathrm{sc}}(t)$ to the exact SP at a very large value of N.

The SP may follow the semiclassically predicted exponential decay for a finite time period only. Examples are given in Fig. 6. We first take $T_L = 60$, which is the approximate time at which obvious deviation appears for $\delta = 2\times10^{-4}$. In Fig. 7, we plot the deviation D, computed with this T_L, as a function of the parameter δ for a fixed δ_0. It is seen that the value of D remains small outside the region of δ close to δ_0,

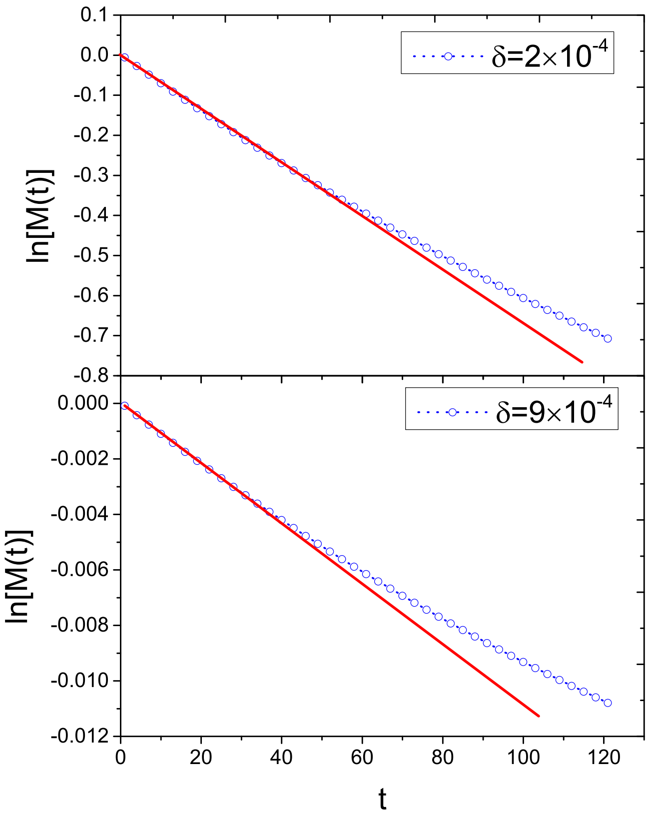

FIG. 6. (Color online) Decay of the SP (circles) (in the logarithm scale) in the XY model. Parameters: $\delta_0 = 10^{-3}$, $N = 16\,000$, $\gamma = 0.7$; $\delta = 2 \times 10^{-4}$ in the upper panel and $\delta = 9 \times 10^{-4}$ in the lower panel. The solid line indicates the semiclassical prediction.

and it decreases with increasing N (right inset of Fig. 7). It is important to note that the shape of the curve of the distance D in Fig. 7 is similar to that of the effective Planck constant $\hbar_{\text{eff}}$ in Fig. 2. This confirms the relative criterion proposed in Sec. II that $\hbar_{\text{eff}}$ can serve as a relative measure for the validity of the

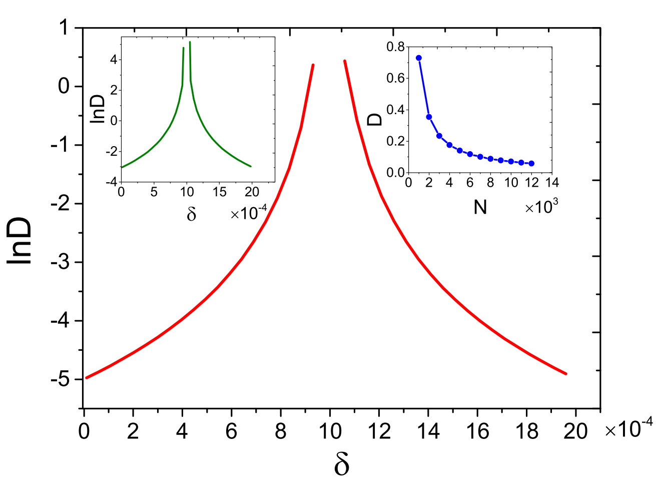

FIG. 7. (Color online) The deviation D (in the logarithm scale) vs δ in the XY model. Parameters: $N = 16\,000$, $\delta_0 = 10^{-3}$, $\gamma = 0.7$, and $T_L = 60$. Left inset: D vs δ for $T_L = 120$. Right inset: Variation of D with the system's size N for $\delta = 4 \times 10^{-4}$.

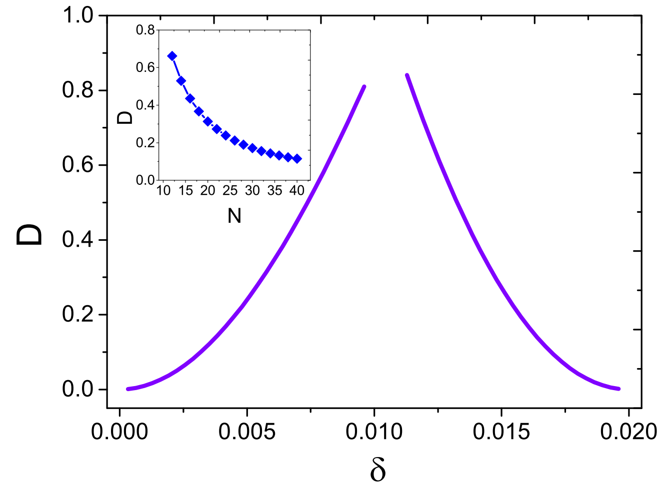

FIG. 8. (Color online) The deviation D vs δ in the Dicke model for $N = 30$, $\delta_0 = 10^{-2}$, $\omega = \omega_0 = 1$, and $T_L = \tau/5$. Inset: variation of D with the system's size N for $\delta = 2 \times 10^{-3}$.

semiclassical approach. Furthermore, we found that variation of the computed time period T_L does not alter this conclusion (see the left inset of Fig. 7 for results obtained with another value of T_L).

B. Dicke model

In the vicinity of the critical point of the Dicke model, the semiclassical approach predicts the following decay behavior of the SP [29]:

$$M_{\text{sc}}(t) \simeq c_0(1 + \xi^2 t^2)^{-1/2} e^{-\Gamma t^2/(1+\xi^2 t^2)}, \tag{55}$$

where $c_0 \sim 1$ and

$$\Gamma = \frac{1}{2}\left(\frac{\epsilon w_p}{\hbar}\frac{\partial U}{\partial p_0}\right)^2, \quad \xi = \left|\frac{\epsilon w_p^2}{2\hbar}\frac{\partial^2 U}{\partial p_0^2}\right|, \tag{56}$$

with the derivatives evaluated at the center of the initial state of the system. Here, ω_p is the width of the initial Gaussian wave packet in momentum space, $U = \int_0^{T_p} V\,dt$, and T_p is the period of the classical motion corresponding to the nonzero mode. Equation (55) predicts an initial Gaussian decay $e^{-\Gamma t^2}$, followed by a long-time power-law decay $1/t$.

In our computation, as in the XY model, the values of Γ and ξ are evaluated by fitting $M_{\text{sc}}(t)$ to the exact SP at large N. In the computation of the deviation D, we take $T_L = \tau/5$. As seen in Fig. 8, variation of D with δ shows a feature similar to that of the effective Planck constant $\hbar_{\text{eff}}$ given in Fig. 5. In addition, as $\hbar_{\text{eff}}$ shown in the inset of Fig. 5, D decreases with increasing N. These results confirm the validity of the above-discussed relative criterion in the Dicke model.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, a relative criterion is proposed for the validity of a semiclassical approach to the dynamics near quantum critical points, which should work at least for an important class of systems undergoing QPTs. Specifically, an effective

Planck constant $\hbar_{\text{eff}}$ is introduced that can give a relative measure to the extent of the validity; that is, a smaller $\hbar_{\text{eff}}$ should imply a better performance of the semiclassical approach. This criterion implies that, under a fixed initial condition and in the thermodynamic limit, the closer the controlling parameter λ is to the critical value λ_c, the better the semiclassical approach should work. This relative criterion has been tested numerically in two models, namely the XY model and the Dicke model.

Still, several problems remain concerning the above-discussed semiclassical approach to the dynamics near QPTs. First, the arguments given for the above-mentioned relative criterion are based on the closeness between classical trajectories of different systems $H(\lambda)$ under a certain rescaling procedure. The closeness has been shown to really exist in a class of systems, however a generic condition for the existence of the closeness is still lacking. Second, the contribution is still unclear for the paths that are not close to any classical trajectory. Third, the proposed criterion has a semiquantitative feature in the sense that it gives a relative measure only, and several approximations have been used in the introduction of the effective Planck constant.

Finally, it should be reasonable to expect that the semiclassical approach discussed above may be useful in the study of other properties of the nonequilibrium dynamics of a system near a quantum critical point. This nonequilibrium dynamics is an important topic and is under investigation both numerically [70] and experimentally [71]. For example, it is relevant to the problem of defect creation in an approximately adiabatic process [27].

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