Dynamical class of a two-dimensional plasmonic Dirac system

Érica de Mello Silva*
Instituto de Física, Universidade Federal de Mato Grosso, Av. Fernando Corrêa da Costa, 2367, Bloco F, Sala 209, 78060-900 Cuiabá, MT, Brazil
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A current goal in plasmonic science and technology is to figure out how to manage the relaxational dynamics of surface plasmons in graphene since its damping constitutes a hinder for the realization of graphene-based plasmonic devices. In this sense we believe it might be of interest to enlarge the knowledge on the dynamical class of two-dimensional plasmonic Dirac systems. According to the recurrence relations method, different systems are said to be dynamically equivalent if they have identical relaxation functions at all times, and such commonality may lead to deep connections between seemingly unrelated physical systems. We employ the recurrence relations approach to obtain relaxation and memory functions of density fluctuations and show that a two-dimensional plasmonic Dirac system at long wavelength and zero temperature belongs to the same dynamical class of standard two-dimensional electron gas and classical harmonic oscillator chain with an impurity mass.

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I. INTRODUCTION

Since the advent of graphene [1] as a tunable plasmonic material, the dynamics of surface plasmons became a hot research topic in nanophotonics (see, e.g., Refs. [2–13]). Graphene plasmons have a high capability of light confinement and have been considered feasible to mediate interactions between externally controlled signals and small quantum systems, e.g., quantum dots [3]. However, in spite of significant progress in the field, graphene plasmons damping is still a hindrance for the realization of graphene-based plasmonic devices [7]. Therefore one may think that a relativistic version of the Sawada model [14], a quantum field-theoretical approach largely employed to describe plasmons in standard electron gas [15], might be insightful for studying the relaxation dynamics of plasmonic Dirac systems. The relativistic counterpart of this well-known quasiboson model was studied for the first time by Jancovici [16], and more recently it was applied to investigate the properties of collective excitations in the following realizations of a two-dimensional gas of massless Dirac particles: electron gas in graphene and helical liquid on the surface of topological insulator [17]. As pointed out in Ref. [17], this quantum field-theoretical model can be useful for problems of quantum plasmonics since the classical electrodynamic approach cannot account for quantum effects due to individual plasmons emission and absorption. In addition, it can also describe theoretically the influence of impurities or external fields in plasmonic Dirac systems [17]. By having these considerations in mind and the intraband plasmon that occurs in doped monolayer graphene as a motivation, in the present work we study time evolution in a reduced relativistic Sawada model at long wavelength and zero temperature. There is sound evidence that for a two-dimensional massless Dirac gas at long wavelength only intraband transitions contribute for the excitations [18–20]. In addition, graphene plasmon has been experimentally observed at midinfrared and longer wavelengths [10], and, since the effective Fermi temperature is very high in graphene, the $T = 0$ theory is applicable even at room temperatures [18].

In order to obtain the relaxation and memory functions of the density fluctuations, we are going to employ the recurrence relations (RR) method proposed by Lee [21] some years ago. It is an orthogonalization approach that exploits geometric properties of the realized Hilbert space and enables one to solve the Heisenberg equation exactly [22]. The RR method has already been successfully applied to nonrelativistic electron gas systems at long wavelength and zero temperature in one [23], two [24], three [25], and D dimensions [26,27]. In the study of nonequilibrium relaxation processes, the applicability of the RR method covers both quantum and classical systems [28–33], e.g., spin chains [34–42] and classical harmonic oscillator chains [43–47]. It is important to point out that the orthogonalization process by the RR method is not generally applicable. However, the sacrifice in generality leads to physically meaningful recurrence relations since they represent realizations of an abstract Hilbert space [48,49].

Another valuable feature of the present approach concerns the possibility of verifying if different systems are dynamically equivalent. It is well known that dynamical equivalence between seemingly unrelated systems can provide useful insight, as shown by Lieb *et al.* for an antiferromagnetic spin-$\frac{1}{2}$ XY chain and a free fermion model [50]. In the RR framework, the dimensionality d and the shape σ of the realized Hilbert space are the static properties that characterize time correlation functions of a dynamical variable in a system towards relaxation process. Therefore one can state if different systems are dynamically equivalent if they have the same d and σ, i.e., identical relaxation functions [34,51–54]. We are going to show that two-dimensional massless Dirac gas at long wavelength and zero temperature belongs to the same dynamical class of standard two-dimensional electron gas (2DEG) and classical harmonic oscillator chain with an impurity mass.

The rest of the paper follows a preliminary study we have carried out in Ref. [54]. The model and the method are discussed in Sec. 2. Section 3 is devoted to relaxation and memory functions for both ideal and interacting cases. In Sec. 4 we present the dynamical equivalencies we have

*erica@fisica.ufmt.br

found, and in the last section some concluding remarks are made.

II. MODEL AND METHOD

A. Model

In order to investigate the time evolution in a two-dimensional massless Dirac gas, we shall consider the following two-dimensional relativistic electron gas (2DREG), which corresponds to the relativistic Sawada model accounting for intraband transitions only:

$$H = \sum_q \epsilon_q [c_q^\dagger c_q - b_q^\dagger b_q] + \frac{1}{2} V_q \sum_{p \neq p'} [b_p c_{p+q} + c_{-p-q}^\dagger b_{-p}^\dagger] \times \sum_{p'} [b_{-p'} c_{-p'-q} + c_{-p'-q}^\dagger b_{-p'}^\dagger], \quad (1)$$

where $V_q = 2\pi e^2/\kappa q$ is the Fourier transform of the Coulomb potential (κ is the relative, or effective, dielectric constant), $c_q^\dagger$ and c_q represent particle creation and annihilation operators, $b_q^\dagger$ and b_q are hole creation and annihilation operators, $\epsilon_q = |\mathbf{q}| v_F$ is the linear energy dispersion of massless Dirac particles (we consider $\hbar = 1$ hereafter), and v_F is the Fermi velocity. In Eq. (1), $p \neq p'$ guarantees that the Pauli principle is satisfied, and the sum over momentum $\mathbf{p}$ implies the condition $|\mathbf{p}| < q_F$ and $|\mathbf{p} + \mathbf{q}| > q_F$, where q_F is the Fermi momentum. Spin and pseudospin degeneracies will be taken into account later.

In accordance with the linear-response theory, the 2DREG will undergo a relaxation process if slightly perturbed by an external field of the form $H_{\text{ext}} = \sum_q \rho_q(t) p_q e^{i\omega t}$, where p_q and ω are, respectively, the Fourier component and the frequency of the field. The density-fluctuations operator is given by $\rho_q(t) = e^{iHt} \rho_q e^{-iHt}$, where $\rho_q = \sum_p [b_p c_{p+q} + c_{-p-q}^\dagger b_{-p}^\dagger]$ is such that electron-electron (hole-hole) scatterings are ignored. The time evolution of the density fluctuations satisfies the Heisenberg equation

$$\dot{\rho}_q(t) = i[H, \rho_q(t)], \quad (2)$$

and the RR method will be employed here to handle it from first principles.

B. Recurrence relations method

In the RR approach, one can build a Hilbert space $\mathcal{L}$ for $\rho_q(t)$ and study its time evolution in a geometric frame. For some time instant $t \geqslant 0$, we express $\rho_q(t)$ as an orthogonal expansion

$$\rho_q(t) = \sum_{\nu=0}^{d-1} f_\nu a_\nu(t), \quad (3)$$

where $\{a_\nu\}$ is a set of time-dependent autocorrelation functions and $\{f_\nu\}$ constitutes a set of time-independent orthogonalized basis vectors. We assume that $\mathcal{L}$ has dimensionality d and is realized by the Kubo scalar product [21]. For this problem in which $T = 0$ is considered, we choose $(X, Y) = \frac{1}{2}\langle XY^\dagger + Y^\dagger X\rangle - \langle X\rangle\langle Y^\dagger\rangle$, a special case of the Kubo scalar product [55]. The realization of $\mathcal{L}$ by this inner product leads to a recurrence relation for the basis vectors

$$f_{\nu+1} = i[H, f_\nu] + \Delta_\nu f_{\nu-1}, \quad 0 \leqslant \nu \leqslant d-1, \quad (4)$$

where

$$\Delta_\nu = \frac{(f_\nu, f_\nu)}{(f_{\nu-1}, f_{\nu-1})}, \quad \nu = 0,1,2,\ldots,d-1. \quad (5)$$

The quantity (5) is a relative norm such that $f_{-1} \equiv 0$ and $\Delta_0 \equiv 1$. From now on the recurrence relation (4) will be referred as RR1 and Δ_ν as ν-recurrant.

There is an arbitrary initial choice among $\{f_\nu\}$, and it is convenient to define the basal vector as $f_0 = \rho_q(0)$. Given this, the rest is allowed no more freedom by RR1, and it implies the boundary condition $a_0(t=0) = 1$, and $a_\nu(t=0) = 0$, $\nu \geqslant 1$.

The fact of $\mathcal{L}$ being a realized space leads to a simple and physically based orthogonalization process. The Hamiltonian H enters explicitly in the construction of basis vectors $\{f_\nu\}$, $\nu \geqslant 1$, which is helpful in the determination of the dimensionality d. The basis vectors $\{f_\nu\}$ are unnormalized, and such a condition determines the shape of the space. The trajectory of dynamical variable $\rho_q(t)$ is model dependent and is governed by the Heisenberg equation (2). Since $\{f_\nu\}$ satisfies both the equation of motion (2) and RR1 (4), the autocorrelations $\{a_\nu\}$ must satisfy the recurrence relation

$$\Delta_{\nu+1} a_{\nu+1}(t) = -\dot{a}_\nu(t) + a_{\nu-1}(t), \quad 0 \leqslant \nu \leqslant d-1, \quad (6)$$

where $\dot{a}_\nu(t) = da_\nu(t)/dt$ and $a_{\nu-1} \equiv 0$. The recurrence relation (6), which will be referred to as RR2, leads directly to the generalized Langevin equation (GLE) [56]. It reflects the geometric shape $\sigma = (\Delta_1 \Delta_2 \ldots \Delta_{d-1})$ of realized space $\mathcal{L}$ for a given Hamiltonian H. Once a_0 is defined, all a_ν, $\nu \geqslant 1$, can be obtained by the RR2, but a_0 is unknown *a priori*. However, by taking the Laplace transform of RR2, regarding that $a_0(t=0) = 1$ and $a_\nu(t=0) = 0$, $\nu \geqslant 1$, results in

$$1 = z\tilde{a}_0 + \Delta_1 \tilde{a}_1, \quad (7a)$$

$$\tilde{a}_{\nu-1} = z\tilde{a}_\nu + \Delta_{\nu+1}\tilde{a}_{\nu+1}. \quad (7b)$$

After some manipulations of Eqs. (7a)–(7b), we obtain the continued fraction representation

$$\tilde{a}_0(z) = \frac{1}{z + \Delta_1 \phi(z)}, \quad \phi(z) = \cfrac{1}{z + \cfrac{\Delta_2}{z + \Delta_3}}, \quad (8)$$

which implies the existence of a d-dimensional space defined by a set of Δs. For certain physical models the sequence of recurrents converges and one can obtain $\tilde{a}_0(z)$ and the relaxation function $a_0(t)$ as well. Hence all other autocorrelations $a_\nu(t)$, $\nu \geqslant 1$, are provided by RR2.

For the dynamical analysis, we follow the same recipe to construct subspaces of $\mathcal{L}$. We rewrite Eq. (7a) as

$$\frac{1}{\tilde{a}_0} = z + \Delta_1 \frac{\tilde{a}_1}{\tilde{a}_0} \quad (9)$$

and call $\tilde{a}_1/\tilde{a}_0 \equiv \tilde{b}_1$ such that

$$\tilde{b}_1(z) = \cfrac{1}{z + \cfrac{\Delta_2}{z + \Delta_3}}. \quad (10)$$

In Eq. (10), $\tilde{b}_1$ indicates the existence of a $(d-1)$-dimensional subspace $\mathcal{L}_1$, $\mathcal{L}_1 \subset \mathcal{L}$, spanned by the basis vectors $f_1, f_2, \ldots, f_d$. Therefore, regarding that $b_1(t=0)=1$ and $b_\nu(t=0)=0$, $\nu \geqslant 2$, the following relations are valid for autocorrelations $\tilde{b}_\nu$:

$$1 = z\tilde{b}_1 + \Delta_2\tilde{b}_2, \tag{11a}$$

$$\tilde{b}_{\nu-1} = z\tilde{b}_\nu + \Delta_{\nu+1}\tilde{b}_{\nu+1}. \tag{11b}$$

Under these conditions, the inverse Laplace transform of Eq. (11b) leads to

$$\Delta_{\nu+1}b_{\nu+1}(t) = -\dot{b}_\nu(t) + a_{\nu-1}(t), \quad 1 \leqslant \nu \leqslant d-1, \tag{12}$$

which means that $b_0 \equiv 0$ [note that Δ_1 does not enter into Eq. (12)]. The autocorrelations b_ν are real time-dependent projection functions for the time evolution of the random force

$$F_q(t) = \sum_{\nu=1}^{d-1} b_\nu(t) f_\nu, \tag{13}$$

and in the GLE framework [56] the memory function depends on $b_1(t)$ such as

$$M_q(t) = \Delta_1 b_1(t). \tag{14}$$

In summary, if $a_0(t)$ is known, RR2 provides the set $\{a_\nu\}$, and hence $\{b_\nu\}$ may also be calculated once these two families of autocorrelation functions are related by convolution

$$a_\nu(t) = b_\nu(t) * a_0(t) \equiv \int_0^t dt' b_\nu(t-t')a_0(t'), \quad 1 \leqslant \nu \leqslant d-1. \tag{15}$$

III. RELAXATION AND MEMORY FUNCTIONS

In order to find the relaxation and memory functions, we shall determine the behavior of the Δs, which are the key to dynamical theory. Then we obtain the families of autocorrelations $\{a_\nu\}$ and $\{b_\nu\}$ for both ideal and interacting 2DREG at zero temperature and long wavelength.

A. Ideal system

We start with $f_0 = \rho_q$ to carry out the RR1 (4). The norms $(f_\nu, f_\nu)^{(0)}$ for the ideal 2DREG are given bellow (see the Appendix for detailed calculations):

$$(f_0, f_0)^{(0)} = g q_F v_F^{-1}/2\pi, \tag{16a}$$

$$(f_1, f_1)^{(0)} = q^2 g q_F v_F/4\pi, \tag{16b}$$

$$(f_2, f_2)^{(0)} = [q^4 + O(q^6)] g q_F v_F^3/16\pi, \tag{16c}$$

$$(f_3, f_3)^{(0)} = [q^6 + O(q^8)] g q_F v_F^5/64\pi, \tag{16d}$$

$$(f_4, f_4)^{(0)} = [q^8 + O(q^{10})] g q_F v_F^7/256\pi, \tag{16e}$$

$$(f_5, f_5)^{(0)} = [q^{10} + O(q^{12})] g q_F v_F^9/1024\pi, \tag{16f}$$

etc., where g stands for the spin and pseudospin degeneracies. It is worth mentioning that $(f_0, f_0)^{(0)}$ is the static susceptibility $\chi_q^{(0)}$. In order to obtain the recurrants (5), we have calculated only the leading order of q, valid in the static regime of $q \ll q_F$,

$$\Delta_1^{(0)} = \tfrac{1}{2} q^2 v_F^2, \tag{17a}$$

$$\Delta_2^{(0)} = \Delta_3^{(0)} = \Delta_4^{(0)} = \cdots = \tfrac{1}{4} q^2 v_F^2 \equiv \Delta. \tag{17b}$$

Hence, the Hilbert space of $\rho_q(t)$ in the ideal 2DREG is infinite dimensional ($d = \infty$), and the norms are finite and nonvanishing for all ν.

The autocorrelations $\{a_\nu^{(0)}\}$ are obtained from RR2 (6) once we have recurrants (17a)–(17b):

$$\tfrac{1}{2} q^2 v_F^2 a_1^{(0)}(t) = -\dot{a}_0^{(0)}(t), \tag{18a}$$

$$\tfrac{1}{4} q^2 v_F^2 a_{\nu+1}^{(0)}(t) = -\dot{a}_\nu^{(0)}(t) + a_{\nu-1}^{(0)}(t), \quad \nu \geqslant 1. \tag{18b}$$

Since the above realized RR2 (18a)–(18b) is isomorphic to recurrence relations for Bessel function J_ν [57],

$$J_1 = -\dot{J}_0, \tag{19a}$$

$$J_{\nu+1} = -2\dot{J}_\nu + J_{\nu-1}, \tag{19b}$$

we have

$$a_\nu^{(0)}(t) = 2^\nu (q v_F)^{-\nu} J_\nu(q v_F t), \quad \nu \geqslant 1. \tag{20}$$

The solution (20) satisfies the boundary condition $a_\nu^{(0)}(t=0) = 0$, $\nu \geqslant 1$. Note that $a_0^{(0)}(t)$ is the relaxation function for the noninteracting system,

$$\Xi_q^{(0)}(t) = J_0(q v_F t), \tag{21}$$

where J_0 is the Bessel function of zeroth order. In the limit $t \to \infty$, the behavior of the relaxation function (21) follows asymptotic properties of Bessel functions [57]:,

$$\Xi_q^{(0)}(t) \sim t^{-1/2} \cos\left(q v_F t - \tfrac{1}{4}\pi\right).$$

Given this, the autocorrelation functions $\{b_\nu^{(0)}\}$ can be directly calculated from relation (15). Using solution (20) for $a_\nu(t)$, we obtain

$$2^\nu (q v_F)^{-\nu} J_\nu(q v_F) = \int_0^t dt' b_\nu(t-t') J_0(q v_F t'), \tag{22}$$

which corresponds to the Bessel integral [57]. Hence the autocorrelations $\{b_\nu^{(0)}\}$ are

$$b_\nu^{(0)}(t) = 2^\nu (q v_F)^{-\nu} J_\nu(q v_F t)/t, \quad \nu \geqslant 1, \tag{23}$$

and the memory function (14) reads as

$$M_q^{(0)}(t) = q v_F J_1(q v_F t)/t, \tag{24}$$

where J_1 is the Bessel function of first order. The asymptotic behavior of the memory function (24) is given by

$$M_q^{(0)}(t) \sim t^{-3/2} \cos\left(q v_F t - \tfrac{3}{4}\pi\right), \quad t \to \infty.$$

It is important to point out that such relaxation appears in processes characterized by damped harmonic oscillatory behavior and has the exact form of the velocity-time correlation function of a Brownian particle in a linear chain of harmonic oscillators (see Refs. [32,33] and references therein).

B. Interacting system

We apply the same procedure to calculate the norms (f_ν, f_ν) for interacting 2DREG, except for (f_0, f_0). In this case we invoke the following RPA formulation, which is accurate for a $q \to 0$ regime,

$$(f_0, f_0) = \frac{(f_0, f_0)^{(0)}}{1 - V_q (f_0, f_0)^{(0)}}, \tag{25}$$

where $V_q = 2\pi e^2/\kappa q$, and $(f_0, f_0)^{(0)}$ is the noninteracting susceptibility $\chi_q^{(0)}$ given by Eq. (16a). The Hilbert space structure for interacting 2DREG is identical to ideal gas except by the squared norm of zeroth axis [58]. Hence, $(f_0, f_0) \neq (f_0, f_0)^{(0)}$ and $(f_\nu, f_\nu) = (f_\nu, f_\nu)^{(0)}$ for all $\nu \geqslant 1$. Therefore we straightforwardly obtain the recurrants' behavior

$$\Delta_1 = \Delta_1^{(0)} + \Gamma = \tfrac{1}{2} q^2 v_F^2 + \Gamma, \tag{26a}$$

$$\Delta_\nu = \Delta_\nu^{(0)} = \Delta = \tfrac{1}{4} q^2 v_F^2, \quad \nu \geqslant 2, \tag{26b}$$

where Γ is the squared plasma frequency

$$\omega_{pl}^2 = (2\kappa)^{-1} e^2 g q_F v_F q.$$

Note that $\omega_{pl} \propto \sqrt{q}$, as expected for two-dimensional electron gas systems. It can also be written in terms of electron density, $n = g q_F^2/4\pi$, and dimensionless fine structure constant, $r_s = e^2/\kappa v_F$ (we are considering $\hbar = 1$):

$$\omega_{pl} = \sqrt{r_s} (g\pi n)^{1/4} v_F q^{1/2}. \tag{27}$$

It is worth mentioning that our plasma frequency (27) agrees exactly with the expression obtained by Das Sarma and Hwang [20] for the long wavelength massless Dirac plasma, where $g = g_s g_v$, $g_s(=2)$ is the spin degeneracy, and $g_v(=2)$ the valley or pseudospin degeneracy. Our plasma frequency is in agreement with the theory of two-dimensional plasmons in graphene since $\omega_{pl}(q) \propto n^{1/4}$ and contains Planck's constant [2]. This suggests that RR method can be successfully applied to investigate time evolutions in massless Dirac plasma that occurs in 2D graphene layers.

In order to obtain autocorrelations $\{a_\nu\}$ of interacting system, we start by calculating $a_0(t)$ from its Laplace transform (8) and recurrants (26a)–(26b) to write the integral [58]

$$a_0(t) = \frac{1}{2\pi i} \int_C \frac{e^{zt}\, dz}{z + \left(1 - \frac{\Delta_1}{2\Delta}\right) + \frac{\Delta_1}{2\Delta}(z^2 + 4\Delta)^{1/2}}, \tag{28}$$

where contour C runs along the imaginary axis. The branch cut occurs between $-iqv_F \leqslant z \leqslant iqv_F$, and the isolated poles $z = \pm i\omega_p$ give the plasmon dispersion, $\omega_p = \Delta_1/\sqrt{\Delta_1 - \Delta}$. The solution of Eq. (28) is the relaxation function of interacting 2DREG,

$$\Xi_q(t) = A_s \sum_{n=0}^{\infty} (-\alpha)^n [\partial/\partial(t)]^{2n} J_1(qv_F t)/qv_F t + A_p \cos(\omega_p t), \tag{29}$$

where $\alpha = 4\Delta(1 - \Delta/\Delta_1)/\Delta_1$, and J_1 is the Bessel function of first order. The coefficients $A_s = 1 - (1-\alpha)^{-1/2}$ and $A_p = [(1-\alpha)^{1/2} - (1-\alpha)]/(\frac{1}{2}\alpha)$ stand for the single particle and the collective (plasma) excitations, respectively. The parameter α is defined in $[0,1]$ such that for $\alpha = 1$ we recover the ideal relaxation (21). For $\alpha \to 0$,

$$\Xi_q(t) = \tfrac{1}{2}\alpha J_1(qv_F t)/qv_F t + \left(1 - \tfrac{1}{4}\alpha\right)\cos(\omega_p t) + O(\alpha^2).$$

In the limit $t \to \infty$,

$$\Xi_q(t) \sim t^{-3/2} \cos\left(qv_F t - \tfrac{3}{4}\pi\right) + A_p \cos(\omega_p t).$$

For autocorrelations $\{b_\nu\}$, let us recall the recurrants (26a)–(26b). In this case, $\Delta_1 \neq \Delta_1^{(0)}$ and $\Delta_\nu = \Delta_\nu^{(0)}$ for all $\nu \geqslant 2$. As already mentioned, RR2 (12) does not depend on Δ_1, hence $\{b_\nu\} = \{b_\nu^{(0)}\}$ for all $\nu \geqslant 1$. Therefore we have the same autocorrelations (23) and memory function (24)

$$M_q(t) = qv_F J_1(qv_F t)/t;$$

i.e., $\{b_\nu\}$ and $M_q(t)$ are not affected by interaction.

IV. DYNAMICAL EQUIVALENCIES

The surface shape of a realized Hilbert space is defined by the sequence $\sigma = (\Delta_1 \Delta_2 \ldots \Delta_{d-1})$, and the geometry of this shape draws the time evolution of a dynamical variable. Therefore different systems are said to be dynamically equivalent if they have the same Hilbert dimensionality $d = (f_0 f_1 f_2 \ldots f_{d-1})$ and shape σ, i.e., identical relaxation functions. Given the asymptotic behavior of recurrants for two-dimensional massless Dirac electron gas (that we refer as 2DREG for convenience), we show that the 2DREG at long wavelength and zero temperature is dynamically equivalent to the following systems whose Hilbert space dimensionality is also infinite.

A. Ideal 2DEG at $T = 0$

Some years ago, Lee and Hong studied the time evolution of density fluctuations in 2DEG at long wavelength [24]. For ideal 2DEG system, the recurrants they obtained are

$$\Delta_1 = 2q^2 \epsilon_F^2, \tag{30a}$$

$$\Delta_\nu = \Delta = q^2 \epsilon_F^2, \quad \nu \geqslant 2, \tag{30b}$$

where $\epsilon_F = q_F^2/2m$ is the Fermi energy. By comparing (17a)–(17b) to (30a)–(30b), we see that recurrants for ideal 2DREG and ideal 2DEG have identical structure besides the same Hilbert space dimensionality. Therefore a dynamical equivalence is verified for these systems.

B. Interacting 2DEG at $T = 0$

The recurrants for the density fluctuations in interacting 2DEG at long wavelength obtained by Lee and Hong [24] are

$$\Delta_1 = 2q^2 \epsilon_F^2 + \Gamma, \tag{31a}$$

$$\Delta_\nu = \Delta = q^2 \epsilon_F^2, \quad \nu \geqslant 2, \tag{31b}$$

where Γ is the squared classical plasma frequency $(\omega_{pl}^{cl})^2 = 2\pi\rho e^2 q/m$. By comparison of (26a)–(26b) and (31a)–(31b), we can state a dynamical equivalence between the interacting versions of 2DREG and 2DEG systems [54].

C. Classical harmonic oscillator chain with equal mass (HO)

The classical harmonic oscillator chain with equal mass model, HO, useful for modeling acoustic phonons in crystalline solids at room temperature, was studied by Lee *et al.* via RR method [51]. The system has periodic boundary conditions and an even number N of oscillators. The tagged oscillator is labeled by the lattice site o and has momentum P_o. By considering P_o as the dynamical variable and $N \to \infty$, the authors found

$$\Delta_1 = 2\kappa/m, \tag{32a}$$

$$\Delta_\nu = \Delta = \kappa/m, \quad \nu \geqslant 2, \tag{32b}$$

where m is the mass of an oscillator and κ is the coupling constant. The recurrants (32a)–(32b) have the same structure of (30a)–(30b) obtained for the density fluctuations in ideal 2DEG [51,52]. Therefore, according to the result stated in **A**, a dynamical equivalence is established between ideal 2DREG, ideal 2DEG, and HO systems.

D. Classical harmonic oscillator chain with an impurity mass (HOI)

The HOI model is the same as HO model regarding that in the former the tagged oscillator has mass $m_o \neq m$. It describes a gas-solid interaction and is very practical for surfaces problems [59]. For $N \to \infty$, the recurrants are [51]

$$\Delta_1 = 2\lambda\kappa/m, \quad \lambda = m/m_o, \tag{33a}$$

$$\Delta_\nu = \Delta = \kappa/m, \quad \nu \geqslant 2, \tag{33b}$$

where $\lambda = 0$ and $\lambda = \infty$ represent heavy and light impurity mass limits, respectively. For $\lambda > 1$ (light regime), up to some scale factors [51], recurrants (33a)–(33b) become the same as (31a)–(31b) for density fluctuations in interacting 2DEG [51,52]. Hence, according to the result showed in **B**, the interacting 2DREG, interacting 2DEG, and HOI systems are dynamically equivalent.

V. CONCLUDING REMARKS

The time evolution of the density fluctuations in a two-dimensional plasmonic Dirac system at long wavelength and zero temperature was investigated via recurrence relations method. We used a quantum field-theoretical approach and considered only intraband transitions. We obtained relaxation and memory functions for ideal and interacting systems and found a plasma frequency that brings out the features of two-dimensional plasmons observed in graphene. We also verified that the ideal two-dimensional massless Dirac gas is dynamically equivalent to the ideal 2DEG and the classical harmonic oscillator chain with equal mass, a useful model for acoustic phonons in crystalline solids at room temperature. A dynamical equivalence was also established between the interacting two-dimensional massless Dirac system, the interacting 2DEG, and the classical harmonic oscillator chain with an impurity mass, a practical model for a gas-solid interaction. We believe that these dynamical equivalencies between the mentioned nonrelativistic many-body systems and the two-dimensional plasmonic Dirac system at long wavelength and zero temperature can lead to insightful connections which might be worth of further investigations.

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APPENDIX

We choose $f_0 = \rho_q(0)$ to carry out the RR1 (4) and write the basis vectors $\{f_\nu\}$ that span the Hilbert space $\mathcal{L}$:

$$\begin{aligned}
f_0 &= \rho_q, \\
f_1 &= \dot{\rho}_q, \\
f_2 &= \ddot{\rho}_q + \Delta_1\rho_q, \\
f_3 &= \overset{(3)}{\rho}_q + (\Delta_1 + \Delta_2)\dot{\rho}_q, \\
f_4 &= \overset{(4)}{\rho}_q + \big(\Delta_1 + \Delta_2 + \Delta_3\big)\ddot{\rho}_q + \Delta_1\Delta_3\rho_q, \\
f_5 &= \overset{(5)}{\rho}_q + (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)\overset{(3)}{\rho}_q \\
&\quad + (\Delta_1\Delta_3 + \Delta_1\Delta_4 + \Delta_2\Delta_4)\dot{\rho}_q,
\end{aligned}$$

etc. The number of dots or the number (.) above ρ_q counts for the number of times the commutator of ρ_q with H has been applied, e.g.,

$$\overset{(3)}{\rho}_q = i[H,\ddot{\rho}_q] = i^3[H,[H,[H,\rho_q]]],$$

and so on. In order to get the Δs, we calculate the norms of the basis vectors

$$\begin{aligned}
(f_0,f_0)^{(0)} &= (\rho_q,\rho_q), \\
(f_1,f_1)^{(0)} &= (\dot{\rho}_q,\dot{\rho}_q), \\
(f_2,f_2)^{(0)} &= (\ddot{\rho}_q,\ddot{\rho}_q) - \Delta_1(\dot{\rho}_q,\dot{\rho}_q), \\
(f_3,f_3)^{(0)} &= \big(\overset{(3)}{\rho}_q,\overset{(3)}{\rho}_q\big) - (\Delta_1 + \Delta_2)(\ddot{\rho}_q,\ddot{\rho}_q), \\
(f_4,f_4)^{(0)} &= \big(\overset{(4)}{\rho}_q,\overset{(4)}{\rho}_q\big) - (\Delta_1 + \Delta_2 + \Delta_3)(\overset{(3)}{\rho}_q,\overset{(3)}{\rho}_q) \\
&\quad + \Delta_1\Delta_3(\ddot{\rho}_q,\ddot{\rho}_q), \\
(f_5,f_5)^{(0)} &= \big(\overset{(5)}{\rho}_q,\overset{(5)}{\rho}_q\big) - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)\big(\overset{(4)}{\rho}_q,\overset{(4)}{\rho}_q\big) \\
&\quad + (\Delta_1\Delta_3 + \Delta_1\Delta_4 + \Delta_2\Delta_4)\big(\overset{(3)}{\rho}_q,\overset{(3)}{\rho}_q\big),
\end{aligned}$$

etc., by using the identity

$$(\overset{(n)}{\rho},\overset{(m)}{\rho}) = -(\overset{(n-1)}{\rho},\overset{(m+1)}{\rho}) = -(\overset{(n+1)}{\rho},\overset{(m-1)}{\rho}),$$

for n or $m \geqslant 1$. The scalar products of derivatives can be represented in terms of ensemble averages of commutators as

$$\begin{aligned}
(\dot{\rho}_q,\dot{\rho}_q) &= \langle[\rho_q^\dagger, L\rho_q]\rangle, \\
(\ddot{\rho}_q,\ddot{\rho}_q) &= \langle[L^2\rho_q, L\rho_q^\dagger]\rangle, \\
\big(\overset{(3)}{\rho}_q,\overset{(3)}{\rho}_q\big) &= \langle[L^2\rho_q^\dagger, L^3\rho_q]\rangle, \\
\big(\overset{(4)}{\rho}_q,\overset{(4)}{\rho}_q\big) &= \langle[L^4\rho_q, L^3\rho_q^\dagger]\rangle, \\
\big(\overset{(5)}{\rho}_q,\overset{(5)}{\rho}_q\big) &= \langle[L^4\rho_q^\dagger, L^5\rho_q]\rangle,
\end{aligned}$$

such that $L\rho_q = [H_0, \rho_q]$. The general expression for the Liouvillian action is $L^\nu \rho_q = \sum_p (\epsilon_p - \epsilon_{p+q})^\nu c_p^\dagger c_{p+q}$, which leads to the general inner product formula [58]

$$(L^\nu \rho_q, L^\nu \rho_q) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)^{2\nu-1} \langle n_p \rangle.$$

In order to express the energy dispersion $\epsilon_{p+q} - \epsilon_p$, regarding the linear spectrum of massless Dirac particles $\epsilon_q = |\mathbf{q}| v_F$, note that

$$\epsilon_{p+q} - \epsilon_p = (|\mathbf{p} + \mathbf{q}| - p) v_F,$$

where

$$|\mathbf{p} + \mathbf{q}| = \sqrt{p^2 + 2pq\cos\theta + q^2}$$
$$= p\sqrt{1+x}, \quad x = \left(\frac{q^2}{p^2} + \frac{2q}{p}\cos\theta\right).$$

By approximation and retaining only terms up to $O(q^2)$ since we are in the small q regime,

$$|\mathbf{p} + \mathbf{q}| = p\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right),$$
$$\approx p + \frac{q^2}{2p}\sin^2\theta + q\cos\theta,$$

which follows that

$$\epsilon_{p+q} - \epsilon_p = \left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right) v_F.$$

To calculate the momenta $(L^\nu \rho_q, L^\nu \rho_q)$, we change the summation to an integral, $\sum_p \to \frac{A}{(2\pi)^2}\int d\mathbf{p}$ (A has unit area and $\mathbf{p}$ is a two-dimensional vector), and use the fermion distribution at $T = 0$:

$$\langle n_p \rangle = \langle c_p^\dagger c_p \rangle = \begin{cases} 1 & \text{if } p \leqslant q_F, \\ 0 & \text{if } p > q_F. \end{cases}$$

Therefore, regarding the multiplication by the spin and pseudospin degeneracy factor g, we have

$$(\dot\rho_q, \dot\rho_q) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)\langle n_p \rangle$$
$$= 2g\frac{v_F}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right)$$
$$= \frac{1}{4\pi} g q_F v_F q^2,$$

$$(\ddot\rho_q, \ddot\rho_q) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)^3 \langle n_p \rangle$$
$$= 2g\frac{v_F^3}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right)^3$$
$$= \frac{3}{16\pi} g q_F v_F^3 q^4 + O(q^6),$$

$$\left(\overset{(3)}{\rho}_q, \overset{(3)}{\rho}_q\right) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)^5 \langle n_p \rangle$$
$$= 2g\frac{v_F^5}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right)^5$$
$$= \frac{5}{32\pi} g q_F v_F^5 q^6 + O(q^8) + O(q^{10}),$$

$$\left(\overset{(4)}{\rho}_q, \overset{(4)}{\rho}_q\right) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)^7 \langle n_p \rangle$$
$$= 2g\frac{v_F^7}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right)^7$$
$$= \frac{35}{256\pi} g q_F v_F^7 q^8 + O(q^{10}) + O(q^{12}) + O(q^{14}),$$

$$\left(\overset{(5)}{\rho}_q, \overset{(5)}{\rho}_q\right) = 2\sum_p (\epsilon_{p+q} - \epsilon_p)^9 \langle n_p \rangle$$
$$= 2g\frac{v_F^9}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\left(\frac{q^2}{2p}\sin^2\theta + q\cos\theta\right)^9$$
$$= \frac{63}{512\pi} g q_F v_F^9 q^{10} + O(q^{12}) + O(q^{14}) + O(q^{16}).$$

The static susceptibility can be defined as follows [58]:

$$(\rho_q, \rho_q) = \sum_p \frac{\langle n_p \rangle - \langle n_{p+q} \rangle}{\epsilon_{p+q} - \epsilon_p},$$

and, for small q, one may write [60]

$$\langle n_p \rangle - \langle n_{p+q} \rangle = \frac{dn}{d\epsilon_p}(\epsilon_{p+q} - \epsilon_p),$$

which leads to $(\rho_q, \rho_q) = \sum_p \delta(\epsilon_{p+q} - \epsilon_p)$. Taking care of the degeneracy and changing the summation to an integration as before, we obtain

$$(\rho_q, \rho_q) = \sum_p \delta\big(\epsilon_{p+q} - \epsilon_p\big)$$
$$= g\frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} pdp\ \delta\big(\epsilon_{p+q} - \epsilon_p\big)$$
$$= g\frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{q_F} p\frac{dp}{d\epsilon}\ \delta\big(\epsilon_{p+q} - \epsilon_p\big)d\epsilon$$
$$= \frac{1}{2\pi} g q_F v_F^{-1}.$$

Hence, norms (16a)–(16f) and recurrants (17a)–(17b) for ideal system are readily calculated.

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