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Active-to-absorbing-state phase transition in an evolving population with mutation

Niladri Sarkar[*]

Condensed Matter Physics Division, Saha Institute of Nuclear Physics, Calcutta 700064, India

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We study the active to absorbing phase transition (AAPT) in a simple two-component model system for a species and its mutant. We uncover the nontrivial critical scaling behavior and weak dynamic scaling near the AAPT that shows the significance of mutation and highlights the connection of this model with the well-known directed percolation universality class. Our model should be a useful starting point to study how mutation may affect extinction or survival of a species.

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I. INTRODUCTION

Active to absorbing state phase transition (AAPT) forms a paradigmatic example of nonequilibrium critical phenomena [1,2]. In models exhibiting AAPT [3–5], a species can exist in both the active and absorbing states, such that once it is in the latter state, the transition probability to the active state vanishes. Simplest models that exhibit AAPT often belong to the well-known directed percolation (DP) universality class. Some popular examples of systems showing DP universal scaling behavior [6,7] are the epidemic process with recovery or the *Gribov process* [8] and the predator prey cellular automation models [9–12]. In predator prey models for example [13,14], the growth (birth) and decay (death) of particles or species competes and thus there may be a finite density of the species in the steady state ("active state") or extinction of the species ("inactive or absorbing state"). Under the DP hypothesis [15], a system with a single absorbing state undergoing AAPT, shows critical behavior belonging to the DP universality class in the absence of any special symmetry, long-range interactions, conservation law, or quenched disorder. Else, non-DP-like critical behavior cannot be ruled out. In some cases, many absorbing states in an AAPT have also been found; see, e.g., in Refs. [16,17].

Continuum descriptions of AAPT in models displaying DP universality are based on the Reggeon field theory [18–20], which is a stochastic multiparticle process used to describe the local growth of populations near their extinction threshold in an uniform environment [3,21]. The parameters of the model depends on the embedding environment, which are taken as constants and their fluctuations are ignored. If the fluctuations of the environment are taken into consideration, then whether the DP hypothesis and the DP universality class survive remains a question of general interest. Studies dealing with the effect of environmental fluctuations on a species undergoing AAPT has been made in Refs. [22–25]. It is now believed that nontrivial environmental dynamics and its feedback on the species undergoing an AAPT substantially alter the critical exponents at the AAPT leading to new universal behavior different from the DP universality class. For instance, by considering the environment to follow its own fluctuating scale invariant dynamics, Refs. [22,23] generically found non-DP-like critical scaling at the AAPT, often associated with *weak dynamic scaling*, where the species undergoing AAPT and the environment have unequal dynamic exponents. Reference [23] also finds feedback of the species on the environment to be relevant in determining the ensuing universal behavior. These are in general modeled by coupling a second auxiliary dynamical field having its own dynamics with the species that undergoes an AAPT.

Mutation of microbes and bacterial colonies has been an active area of research for quite some time now. Mutation in an evolving and growing population of a species can significantly alter its long-time state. Mutation in microbial colonies are important to understanding how the microbial population differentiates along the growing front in time and gives rise to well-defined domains of different colonies [26–28]. For instance, if the mutation rate that sets the rate of creation of the mutant of the original species is large, but the back mutation is small, it is conceivable that the original species will eventually go extinct, leaving only the mutant population as active. On the other hand, if the mutation rate is small compared to back mutation, the original species should continue to thrive with a small population of the mutant floating around in an otherwise pure species dominated world. Thus, depending upon the relative magnitudes of the mutation and the back-mutation rate, the original species may become extinct by undergoing an AAPT [27,29]. This has important consequences specially in the formation of cancer and tumour cells in tissues. If mutation gives rise to deleterious population, a proliferation of the mutants might result in cancerous growth in a healthy tissue [30–33]. So to contain the deleterious mutation, one can theoretically argue that the back-mutation rate into the original species should be larger than the forward-mutation rate. Since survival of the original species depends upon the suppressing mutation or facilitating back mutation, it is conceivable that a back mutation that is triggered by the presence of the original species may serve as a simple model of defense mechanism against proliferation of the mutant population, e.g., mutant cells in a body. We incorporate this in a simple way in our model below.

In this article, we propose a two-species nonconserved reaction-diffusion model that describes the competing population dynamics of species A and and its mutant B, where the mutant B is allowed to back mutate into the pure species A. We study the AAPT displayed by it. Our model is distinguished by the feature that the density of the mutant species B does not obey any conservation law in the active state of the model, as a result of its interaction with the species A, unlike the

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[*] niladri2002in@gmail.com

models in Refs. [22,23]. Our model is well-suited to study whether or not the lack of conservation laws for the mutant B dynamics due to its coupling with the pure species A undergoing AAPT affects the critical scaling of the AAPT. Apart from that, a more practical motivation of our model is definitely the production of mutants during the growth of a bacterial colony [27,34–36]. Despite the simplicity of our model, we obtain a set of interesting results. For example, we find one physically stable fixed point with different dynamic exponents corresponding to the two species, respectively, when the diffusion coefficient of the species undergoing AAPT is much greater than the diffusion coefficient of the mutant species, which in turn follows a conservation law in the absence of the pure species. This phenomena is commonly referred to as weak dynamic scaling in phase transition literature. We find that the fixed point exhibiting weak dynamic scaling shows DP-like universal behavior, with exactly the same critical exponents as the DP universality class, which we argue as purely coincidental and is a consequence of our one loop dynamic renormalization group (DRG) analysis. Strong dynamic scaling with same dynamic exponent of the two fields are expected when the diffusion coefficient of the two species are of the same order, a feature we have not discussed in this article. The rest of the paper is organized as follows: In Sec. II we introduce our model following a brief review of the DP universality class. In Sec. III, we do a dynamic renormalization of our model using the DRG procedure. In Sec. IV, we find out the fixed points and the corresponding critical exponents in the weak dynamic scaling regime. In Sec. VI we conclude our study with a summary of our results.

II. THE MODEL AND EQUATIONS OF MOTION

In this section we introduce our model of population dynamics of species A and its mutant B with densities $\rho(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, respectively. Due to the mutation of A to B and the latter's back mutation to A lead to nonconservation of B in the active state of A. Before going to the details of our model, we recall the DP model in brief.

A. Directed percolation model

Consider a population dynamics model in which the growth of the population is linearly dependent on the local species density given by $\phi(\mathbf{x},t)$, and the death is proportional to the square of the species density, which describes death due to overcrowding. The species density undergoes a nonequilibrium AAPT, whose long-wavelength, large-time behavior is described by the DP universality class. The Langevin equation for such a population dynamics model can be written in terms of species density as [37]

$$\frac{\partial \phi}{\partial t} = D\nabla^2\phi + \lambda_g\phi - \lambda_d\phi^2 + \sqrt{\phi}\zeta, \tag{1}$$

where the first term on the right-hand side is the diffusion term with D as the diffusion coefficient, λ_g is the birth rate, and λ_d is the rate of death due to overcrowding. The stochastic function $\zeta(\mathbf{x},t)$ is a Gaussian distributed white noise with zero mean and a variance:

$$\langle\zeta(\mathbf{x},t)\zeta(0,0)\rangle = 2D_2\delta(\mathbf{x})\delta(t). \tag{2}$$

The multiplicative nature of the effective noise ensures the existence of an absorbing state ($\phi = 0$). On dimensional ground, a characteristic length scale $\xi \sim \sqrt{D/|\lambda_g|}$ and a diffusive timescale $t_c \sim \xi^2/D \sim 1/|\lambda_g|$ can be derived from Eq. (1), with both diverging upon approaching the critical point $\lambda_g = 0$. The critical exponents may be defined in the usual way [37],

$$\begin{gathered}\langle\phi(\mathbf{x},t\to\infty)\rangle \sim \lambda_g^{\beta}, \quad \langle\phi(\mathbf{x},t)\rangle \sim t^{-\alpha}\ (\lambda_g = 0),\\ \xi \sim \lambda_g^{-\nu}, \quad t_c \sim \xi_\phi^{z}/D \sim \lambda_g^{-z_\phi\nu},\end{gathered} \tag{3}$$

with the mean-field scaling exponents given by

$$\beta = 1, \alpha = 1, \nu = 1/2, \text{ and, } z_\phi = 2. \tag{4}$$

The anomalous dimension η, which characterizes the scaling of the two-point correlation function, is zero [37] in the mean-field limit. To find out how the fluctuations affect the mean-field scaling exponents, a dynamic renormalization group (DRG) calculational scheme is used to find out the corrections to the bare correlation and vertex functions in the model. It should be noted that the Janssen-de Dominicis action functional, which corresponds to the Langevin Eq. (1), has an invariance under rapidity symmetry given by $\hat{\phi}(\mathbf{x},t) \leftrightarrow \phi(\mathbf{x},-t)$ [37], where $\hat{\phi}$ is the auxiliary field conjugate to ϕ [37]. Invariance under rapidity symmetry is a signature of the DP universality class and all models, whichever falls under the DP universality class, should be invariant under the rapidity symmetry asymptotically. By performing a perturbative expansion in $\epsilon = d_c - d$, $d_c = 4$ is the upper critical dimension for this model using the DRG scheme, one obtains [37],

$$z = 2 - \epsilon/12, \eta = -\epsilon/12, \text{ and } \frac{1}{\nu} = 2 - \epsilon/4. \tag{5}$$

These universal scaling exponents characterize the DP universality class. As the DP hypothesis [15] suggest that the DP universality class is very robust, any one of the conditions of the DP hypothesis are to be violated in order to find new scaling behavior. References [22,24,25] have shown that fluctuating environments with spatially long-ranged noises can modify the scaling behavior of the DP universality. In this article, we have introduced a reaction-diffusion model involving two species and studied how the interdependence of the two species on their mutual birth and death affects the scaling properties of the DP universality class.

B. Two species reaction diffusion model

Our two-species model consists of the species A and its mutant B. Species A reproduces at a given rate; it can also mutate to species B and also die due to overcrowding at fixed rates. Naturally, proliferation of the mutant B, if unchecked, should lead to eventual extinction of A. In order to enlarge the scope of our model, we allow back mutation from species B to A, ensuring a competition between the original species and the mutant. We consider the specific case where back mutation of B to A is triggered by the presence of A locally. Thus, species B can back mutate to A at a given rate, provided species A is available in its neighborhood. Our choice for the specific form of back mutation, though admittedly oversimplified, serves several purposes. For instance, since the back mutation is facilitated by the presence of A, it suggests that the original species has an ability to suppress effects of (unwanted or random) mutations, necessary for its survival as a species. In addition, it is consistent with an absorbing state transition of A with the system being filled up with B, which we are interested to study. Together with the other processes described above it provides a minimal model to study mutation and back mutation in population dynamics of an evolving species and their effects on the AAPT in the model. The two Langevin equations for the densities ρ and ϕ are

$$\frac{\partial\rho}{\partial t} = D_\rho\nabla^2\rho + (1-\lambda_1)\rho + \lambda_g\rho\phi - \lambda_d\rho^2 + \sqrt{\rho}\eta, \quad (6)$$

$$\frac{\partial\phi}{\partial t} = D_\phi\nabla^2\phi + \lambda_1\rho - \lambda_2\rho\phi + \sqrt{\rho}\xi. \quad (7)$$

In Eq. (6), the first term on the right-hand side represents diffusion of species A with a diffusion coefficient D_ρ. The second term with $1-\lambda_1 > 0$ represents growth (reproduction) of A at rate $1-\lambda_1$. The third term represents the growth in population of A due to back mutation of B with a rate $\lambda_g\phi$, $\lambda_g > 0$. The next term is a decay term ($\lambda_d > 0$), which represents the death of ρ due to overcrowding. The stochastic noise $\eta(\mathbf{x},t)$ is the Gaussian-distributed white noise with zero mean and a variance,

$$\langle\eta(\mathbf{x},t)\eta(0,0)\rangle = 2D_2\delta(\mathbf{x})\delta(t). \quad (8)$$

The multiplicative nature of the noise in Eq. (6) ensures the existence of an absorbing state ($\rho = 0$). The dynamics of species B, as given by Eq. (7), is a combination of diffusion with diffusion coefficient D_ϕ, production of B through mutation of A to B at rate λ_1, and back mutation of B by A at rate $\lambda_2\rho\phi$, $\lambda_1,\lambda_2 > 0$. Clearly, back mutation of B can take place only if there are some species A around locally. We assume for simplicity that the only source of stochasticity in the dynamics of ϕ is ρ, and hence, we model it by a multiplicative noise $\sqrt{\rho}\xi$, such that in the absorbing state, the dynamics of B is noise-free. We choose ξ to be a Gaussian-distributed white noise with zero mean and a variance,

$$\langle\xi(\mathbf{x},t)\xi(0,0)\rangle = 2D_1\delta(\mathbf{x})\delta(t). \quad (9)$$

Evidently, our model as given by Eqs. (6) and (7) admits $\rho = 0, \phi = \text{const.} \neq 0$ as an absorbing state. One may also add a conserving additive noise in Eq. (7), reflecting the thermal fluctuations of ϕ. This noise would then have survived in the absorbing state. We neglect this noise for simplicity, which is akin to assuming a "low-temperature limit" for species B. Interestingly, in the absence of an additive conserved noise in Eq. (7), $\rho = 0, \phi = 0$ is also an absorbing state. We ignore this and focus on the absorbing state $\rho = 0, \phi = \phi_0 = \text{const.} \neq 0$. It is instructive to compare Eqs. (6) and (7) with the model in Ref. [23]. In Ref. [23], the second field is a conserved field in both the active and absorbing states of the species, and hence is inappropriate to model a mutant. In contrast, ϕ here is *nonconserved* in the active state of A, appropriate to model population changes of the mutant due to mutation or back mutation. It is only in the absorbing state of A that ϕ is conserved; see also Ref. [27]. This feature clearly distinguishes our model from Ref. [23].

We write $\phi = \phi_0 + \delta\phi$. This modifies Eqs. (6) and (7) to

$$\frac{\partial\rho}{\partial t} = D_\rho\nabla^2\rho + r\rho + \lambda_g\rho\delta\phi - \lambda_d\rho^2 + \sqrt{\rho}\eta, \quad (10)$$

$$\frac{\partial\delta\phi}{\partial t} = D_\phi\nabla^2\delta\phi + \lambda_3\rho - \lambda_2\rho\delta\phi + \sqrt{\rho}\xi, \quad (11)$$

where $r = 1 - \lambda_1 + \lambda_g\phi_0$ and $\lambda_3 = \lambda_1 - \lambda_2\phi_0$. Coupling constant λ_3 should be positive so as to prevent ϕ from collapsing into an absorbing state in the presence of ρ, without passing through an active configuration. Now denoting $\delta\phi$ as ϕ so as to avoid notational complexity, the equations of motion for the two fields in the model can be written as

$$\frac{\partial\rho}{\partial t} = D_\rho\nabla^2\rho + r\rho + \lambda_g\rho\phi - \lambda_d\rho^2 + \sqrt{\rho}\eta, \quad (12)$$

$$\frac{\partial\phi}{\partial t} = D_\phi\nabla^2\phi + \lambda_3\rho - \lambda_2\rho\phi + \sqrt{\rho}\xi. \quad (13)$$

We redefine the coefficients $r = D_\rho\tau$ and $\lambda_d = D_\rho g_1/2$ for calculational convenience, so that Eq. (12) now takes the form

$$\frac{\partial\rho}{\partial t} = D_\rho(\tau + \nabla^2)\rho + \lambda_g\rho\phi - \frac{D_\rho g_1}{2}\rho^2 + \sqrt{\rho}\eta. \quad (14)$$

The critical point is given by renormalized $\tau = 0$. To what extent the nonlinear couplings in our model alter the mean-field DP exponents given by Eq. (5) may be answered systematically by using the standard one-loop dynamic renormalization group (DRG) framework. This requires calculating the primitively divergent vertex functions in the model up to the one-loop order in expansions in terms of the effective coupling constants and absorbing the divergences in redefined or *renormalized* parameters of the model. These allow us to obtain the renormalized vertex or correlation functions in the model, from which the critical scaling exponents may be obtained. See Ref. [38] for detailed technical discussions on the DRG technique.

Using the Langevin Eqs. (14) and (13), together with the corresponding noise variance Eqs. (8) and (9), the Janssen-De Dominics [39] generating functional can be constructed, which can be written as

$$\mathcal{Z} = \int D\rho D\hat{\rho} D\phi D\hat{\phi} \exp[-S], \quad (15)$$

where $\hat{\rho}$ and $\hat{\phi}$ are the auxiliary fields corresponding to the dynamical fields ρ and ϕ, respectively, which enters the Eq. (15) after elimination of the noises from the generating functional $\mathcal{Z}$. For calculational convenience we redefine $i\hat{\phi} \to \hat{\phi}, i\hat{\rho} \to \hat{\rho}$, and $D_2 = \frac{D_\rho g_2}{2}$ in the generating functional $\mathcal{Z}$. The dynamical action functional S corresponding to the model is then given by

$$\begin{aligned} S = &\int d^dx \int dt\hat{\rho}\{\partial_t + D_\rho(-\tau + \nabla^2)\}\rho \\ &+ \int d^dx \int dt\hat{\phi}\{\partial_t + D_\phi\nabla^2\}\phi - \lambda_3\int d^dx\int dt\hat{\phi}\rho \\ &- \lambda_g\int d^dx\int dt\hat{\rho}\rho\phi \\ &- D_1\int d^dx\int dt\hat{\phi}\hat{\phi}\rho + \lambda_2\int d^dx\int dt\hat{\phi}\phi\rho \\ &- \frac{D_\rho g_2}{2}\int d^dx\int dt\hat{\rho}\hat{\rho}\rho + \frac{D_\rho g_1}{2}\int d^dx\int dt\hat{\rho}\rho\rho. \end{aligned} \quad (16)$$

Unlike the pure DP problem, in Eq. (16), the last two terms have different coefficients $D_\rho g_1/2$ and $D_\rho g_2/2$, due to the breakdown of the rapidity symmetry by the couplings λ_g, λ_2, and D_1. In addition, time can be rescaled to absorb D_ϕ in Eq. (16).

In a naïve perturbative expansion, λ_g, D_1, λ_2, and $u \equiv g_1 g_2$ appear as the expansion parameters. Rescaling space and time, it is straightforward to show that the upper critical dimension $d_c = 4$ for the coupling constants λ_g, D_1, λ_2, and u. We also introduce a control parameter $\theta = D_\phi/D_\rho$ known commonly as the Schmidt number, which determines the ensuing nonequilibrium steady state (NESS) of our model. If the renormalized versions of λ_g, D_1, λ_2 are nonzero at the DRG fixed points (FP) at $d < d_c$, then new universal critical scaling behavior is expected to emerge at the DRG FPs, such that the critical exponents would pick up values different from their values for the DP universality class. In the DRG analysis of our model, $\epsilon = d_c - d = 4 - d$ appears as a small parameter; see Ref. [40] for the detailed technical discussions on DRG applications in the DP problem.

We begin by identifying the primitively divergent vertex functions in model Eq. (16). The vertex functions of our model are defined formally by taking the appropriate functional derivatives of the vertex generating functional $\Gamma[\rho,\hat{\rho},\phi,\hat{\phi}]$ with respect to the various fields ρ, $\hat{\rho}$, ϕ, and $\hat{\phi}$, with $\Gamma[\rho,\hat{\rho},\phi,\hat{\phi}]$ being the Legendre transform of $\log \mathcal{Z}$ [38,40]:

$$\Gamma_{a_1 a_2 \ldots a_n} \equiv \frac{\delta^n \Gamma}{\delta a_1 \delta a_2 \ldots \delta a_n}, \tag{17}$$

where $a_1, a_2, \ldots, a_n$ are the fields ρ, $\hat{\rho}$, ϕ, $\hat{\phi}$. The bare vertex functions in our model that have divergent one-loop corrections are listed in Appendix A.

III. RENORMALIZATION GROUP ANALYSIS

To renormalize the vertex functions we choose $\tau = \mu^2$ as the appropriate normalization point about which the vertex corrections are found out up to one-loop order, with μ being the intrinsic momentum scale of the renormalized theory. Next we need the multiplicative renormalization Z factors to determine the scale dependence of the renormalized vertex functions on the momentum scale μ. This is possible as the Z factors absorb the ultraviolet divergences arising out of the one-loop integrals, which makes the resulting theory finite. The Z factors for the various fields and parameters are defined as follows:

$$\begin{aligned}
&\phi = Z_\phi \phi^R, \rho = Z_\rho \rho^R, \hat{\rho} = Z_{\hat{\rho}} \hat{\rho}^R, \hat{\phi} = Z_{\hat{\phi}} \hat{\phi}^R,\\
&D_\rho = Z_{D_\rho} D_\rho^R, \lambda_g = Z_{\lambda_g} \lambda_g^R, g_1 = Z_{g_1} g_1^R,\\
&g_2 = Z_{g_2} g_2^R, \tau = Z_\tau \tau^R,\\
&\lambda_3 = Z_{\lambda_3} \lambda_3^R, \lambda_2 = Z_{\lambda_2} \lambda_2^R, D_\phi = Z_{D_\phi} D_\phi^R,\ D_1 = Z_{D_1} D_1^R,
\end{aligned} \tag{18}$$

where the superscript R refers to renormalized quantities. The various Z factors can be found out from the normalization conditions; see Appendix B. Thus, we have 11 renormalized vertex functions in comparison with the 13 Z factors defined here. Therefore, there are two redundant Z factors, which can be chosen arbitrarily. We hence use this freedom to set $Z_\rho = Z_{\hat{\rho}}$ and $Z_\phi = Z_{\hat{\phi}}$. The Z factors calculated from the one-loop irreducible diagrams are found to be

$$Z_\rho = Z_{\hat{\rho}} = 1 + \frac{g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{19}$$

$$Z_{D_\rho} = 1 - \frac{g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{20}$$

$$Z_\tau = 1 + \frac{3 g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{21}$$

$$Z_{g_1} = 1 + \frac{3 g_1 g_2}{4} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} + \frac{4 \lambda_g^2 D_1}{D_\rho g_1 D_\phi (D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{22}$$

$$Z_{g_2} = 1 + \frac{3 g_1 g_2}{4} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{23}$$

$$Z_{\lambda_2} = 1 - \frac{g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} + \frac{\lambda_2 g_2}{(D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{24}$$

$$Z_{\lambda_3} = Z_\phi^{-1} \left[1 - \frac{g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} - \frac{\lambda_2 g_2}{(D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} \right], \tag{25}$$

$$Z_{\lambda_g} = Z_\phi^{-1} \left[1 + \frac{g_1 g_2}{4} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} + \frac{\lambda_2 g_2}{(D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} \right]. \tag{26}$$

We also find that there are no one-loop corrections to $\Gamma_{\hat{\phi}\phi}$, which means that $Z_\phi Z_{\hat{\phi}} = 1$. Using the choice $Z_\phi = Z_{\hat{\phi}}$, we get $Z_\phi = 1 = Z_{\hat{\phi}}$. Thus, Eqs. (25) and (26) have only unity as contributions coming from Z_ϕ:

$$Z_{\lambda_3} = 1 - \frac{g_1 g_2}{8} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} - \frac{\lambda_2 g_2}{(D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}, \tag{27}$$

$$Z_{\lambda_g} = 1 + \frac{g_1 g_2}{4} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon} + \frac{\lambda_2 g_2}{(D_\rho + D_\phi)} \frac{\mu^{-\epsilon}}{16\pi^2 \epsilon}. \tag{28}$$

For the purpose of calculational convenience, we define three dimensionless constants α, γ, and ψ through

$$\lambda_2 = D_\rho g_1 \alpha, \lambda_g = D_\rho g_1 \gamma, D_1 = D_\rho g_2 \psi. \tag{29}$$

In what follows below, we treat α, γ, ψ as the coupling constants in the present problem without any loss of generality. From the physical interpretations of the different constants in the present model all of α, γ, and ψ should be positive. Clearly, if $\alpha = 0 = \gamma$, there is no back mutation. Equation (29) gives us the multiplicative Z factors of α, γ, and ψ as $Z_\alpha = Z_{\lambda_2} Z_{D_\rho}^{-1} Z_{g_1}^{-1}$, $Z_\gamma = Z_{\lambda_g} Z_{D_\rho}^{-1} Z_{g_1}^{-1}$, and $Z_\psi = Z_{D_1} Z_{D_\rho}^{-1} Z_{g_2}^{-1}$. Their explicit values in terms of the effective coupling constant,

$$u = g_1 g_2, \tag{30}$$

and Schmidt number $\theta = D_\phi / D_\rho$, take the form

$$Z_\alpha = 1 - \frac{3u\mu^{-\epsilon}}{4\epsilon} + \frac{\alpha u \mu^{-\epsilon}}{(1+\theta)\epsilon} - \frac{4u\gamma^2\psi\mu^{-\epsilon}}{\theta(1+\theta)\epsilon}, \tag{31}$$

$$Z_\gamma = 1 - \frac{3u\mu^{-\epsilon}}{8\epsilon} + \frac{u \alpha \mu^{-\epsilon}}{(1+\theta)\epsilon} - \frac{4u\gamma^2\psi\mu^{-\epsilon}}{\theta(1+\theta)\epsilon}, \tag{32}$$

$$Z_\psi = 1 - \frac{3u\mu^{-\epsilon}}{4\epsilon} + \frac{2u\alpha\mu^{-\epsilon}}{(1+\theta)\epsilon}, \tag{33}$$

where we have absorbed a factor of $1/16\pi^2$ in the definition of u. With $u = g_1 g_2$, the multiplicative Z factor for u is given by $Z_u = Z_{g_1} Z_{g_2}$ and $Z_\theta = Z_{D_\phi} Z_{D_\rho}^{-1}$. As $Z_{D_\phi} = 1$, due to lack of renormalization of $\Gamma_{\hat{\phi}\phi}$, $Z_\theta = Z_{D_\rho}^{-1} = 1 + \frac{u\mu^{-\epsilon}}{8\epsilon}$. Thus, we have identified the effective coupling constants for the model to be u, α, γ, and ψ. From the Z factors of these couplings, the β functions corresponding to the renormalized coupling constants u^R, α^R, γ^R, and ψ^R can be obtained, given by

$$\beta_u = u^R\left[-\epsilon + \frac{3u^R}{2} + \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R(1+\theta^R)}\right], \tag{34}$$

$$\beta_\alpha = \alpha^R\left[-\frac{3u^R}{4} + \frac{u^R\alpha^R}{(1+\theta^R)} - \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R(1+\theta^R)}\right], \tag{35}$$

$$\beta_\gamma = \gamma^R\left[-\frac{3u^R}{8} + \frac{u^R\alpha^R}{(1+\theta^R)} - \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R(1+\theta^R)}\right], \tag{36}$$

$$\beta_\psi = \psi^R\left[-\frac{3u^R}{4} + \frac{2u^R\alpha^R}{(1+\theta^R)}\right], \tag{37}$$

$$\beta_\theta = \theta_R\left[\frac{u^R}{8}\right]. \tag{38}$$

The zeros of the β functions gives us the fixed point (FP) solutions for the model, i.e., by setting the right-hand side of Eqs. (34)–(38). The FPs may be obtained in three different physical limits, *viz.*, $\theta_R \to 0,\infty$ and θ_R remaining finite. The first two cases should be characterized by *weak dynamic scaling*, i.e., by $z_\rho < z_\phi$ and $z_\rho > z_\phi$, respectively. In contrast, the third possibility corresponds to $z_\rho = z_\phi$, implying *strong dynamic scaling*. In the limit of $\theta_R \to \infty$, $\beta_\theta = 0$ yields $u_R = 0$, which though satisfies $\beta_u = 0$ is not a stable FP; or, equivalently, even for a very small u_R, β_θ shoots up to ∞. We thus discard this possibility. To find the fixed points for a finite θ value, the full β functions should be equated to zero, which is a highly daunting task given the complexity of the equations involved. But for $\theta^R \to 0$, the algebra is tractable. Hence, we settle for the tractable $\theta^R \to 0$ limit here and leave out the case of finite θ^R.

IV. FIXED POINT ANALYSIS AND CRITICAL EXPONENTS

In this section we perform a fixed point analysis corresponding to the $\theta^R \to 0$ limit. In this limit $D_\rho^R \gg D_\phi^R$, so that a weak dynamic scaling with $z_\rho < z_\phi$ is expected at the stable FP. The FPs are given by the solutions of the equations

$$\frac{3u^R}{2} + \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R} = \epsilon, \tag{39}$$

$$-\frac{3u^R}{4} + u^R\alpha^R - \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R} = 0, \tag{40}$$

$$-\frac{3u^R}{8} + u^R\alpha^R - \frac{4u^R(\gamma^R)^2\psi^R}{\theta^R} = 0, \tag{41}$$

$$-\frac{3u^R}{4} + 2u^R\alpha^R = 0, \tag{42}$$

from which nontrivial fixed points corresponding to $u^R \neq 0$, $\alpha^R \neq 0$, $\gamma^R \neq 0$, and $\psi^R \neq 0$ are obtained. As we can see, $u^R \sim O(\epsilon)$, but α^R, γ^R, and ψ^R are just numbers. To extract physically meaningful fixed points from Eqs. (39), (40), (41), and (42), the terms should be divergence free in the $\theta^R \to 0$ limit. But as can be seen, Eqs. (39)–(42) contain θ^R in the denominator making them diverge in the $\theta^R \to 0$ limit. To make them free from divergences, one should scale $(\gamma^R)^2\psi^R \sim \theta$. Assuming this scaling to hold good in our case, we take

$$m = u\alpha, \tag{43}$$

and

$$t = \frac{u\gamma^2\psi}{\theta}, \tag{44}$$

as the effective coupling constants in the limit $\theta^R \to 0$. Evidently, both m and t should be positive on physical ground. The renormalized effective couplings are hence $m^R = u^R\alpha^R$ and $t^R = \frac{u^R(\gamma^R)^2\psi^R}{\theta^R}$ and their corresponding Z factors given by $Z_m = Z_u Z_\alpha$ and $Z_t = Z_u Z_\gamma^2 Z_\psi Z_\theta^{-1}$, respectively. The explicit form of the Z factors for the effective coupling constants then turn out to be

$$Z_u = 1 + \frac{3u\mu^{-\epsilon}}{2\epsilon} + \frac{4t\mu^{-\epsilon}}{\epsilon}, \tag{45}$$

$$Z_m = 1 + \frac{3u\mu^{-\epsilon}}{4\epsilon} + \frac{m\mu^{-\epsilon}}{\epsilon}, \tag{46}$$

$$Z_t = 1 - \frac{u\mu^{-\epsilon}}{8\epsilon} + \frac{4m\mu^{-\epsilon}}{\epsilon} - \frac{4t\mu^{-\epsilon}}{\epsilon}. \tag{47}$$

The β functions evaluated from Eqs. (45), (46), and (47) are written as follows:

$$\beta_u = u^R\left[-\epsilon + \frac{3u^R}{2} + 4t^R\right], \tag{48}$$

$$\beta_m = m^R\left[-\epsilon + \frac{3u^R}{4} + m^R\right], \tag{49}$$

$$\beta_t = t^R\left[-\epsilon - \frac{u^R}{8} + 4m^R - 4t^R\right]. \tag{50}$$

The fixed points (FPs) are evaluated by equating the β functions to zero. Noting that all of $u^R, m^R, t^R > 0$ on physical ground, we discard any FP with negative values of the renormalized coupling constants. The positive semidefinite FPs are given by

1. FPI: Gaussian FP $u^R = 0, m^R = 0, t^R = 0$.
2. FPII: DP FP $u^R = \frac{2\epsilon}{3}, m^R = 0, t^R = 0$.
3. FPIII: $u_R = 0, m^R = \epsilon, t^R = 0$.
4. FPIV: $u^R = \frac{2\epsilon}{3}, m^R = \frac{\epsilon}{2}, t^R = 0$.
5. FPV: $u^R = 0, m^R = \epsilon, t^R = \frac{3\epsilon}{4}$.

Note that FPIV and FPV involves $m^R > 0$ and $(m^R > 0, t^R > 0)$, respectively. What this means physically is that there is back mutation involved from mutants B to species A in both the FPs as the parameters m and t involves the back-mutation coefficients λ_2 and λ_g. The stability of the FPs can be found by evaluating the eigenvalues Λ of the stability matrix corresponding to each FP. The eigenvalues are found to be

(1) FPI (Gaussian FP): The eigenvalues are $\Lambda = -\epsilon, -\epsilon, -\epsilon$. All the eigenvalues are negative indicating that the gaussian FP is unstable in all directions.

(2) FPII (DP FP): The eigenvalues are $\Lambda = \epsilon, \frac{-\epsilon}{2}, \frac{-13\epsilon}{12}$. Thus, it is stable only along the u^R axis and unstable in all the other directions.

(3) FPIII The eigenvalues are $\Lambda = -\epsilon, \epsilon, 3\epsilon$. Thus, FPIII is unstable only along the u^R direction.

(4) FPIV: The eigenvalues are $\Lambda = \epsilon, \frac{\epsilon}{2}, \frac{11\epsilon}{12}$. Thus, this FP is stable along all directions. This is an important observation considering the fact that it involves back mutation from species B to A.

(5) FPV: The eigenvalues are $\Lambda = 2\epsilon, \epsilon, -3\epsilon$. Thus, FPV is unstable along the t^R axis but stable in the other two directions.

The Wilson's flow are used to determine the critical exponents corresponding to the different FPs. They are evaluated as follows:

$$\zeta_\rho = \mu\frac{\partial}{\partial\mu}\ln Z_\rho^{-1}, \zeta_{\hat\rho} = \mu\frac{\partial}{\partial\mu}\ln Z_{\hat\rho}^{-1}, \zeta_{D_\rho} = \mu\frac{\partial}{\partial\mu}\ln Z_{D_\rho}^{-1}, \tag{51}$$

$$\zeta_\tau = \mu\frac{\partial}{\partial\mu}\ln Z_\tau^{-1} - 2.$$

The critical exponents are derived from the flow functions Eq. (51) as

$$\eta_\rho = \eta_{\hat\rho} = -\zeta_\rho, \tag{52}$$

$$\frac{1}{\nu} = -\zeta_\tau, \tag{53}$$

$$z_\rho = 2 + \zeta_{D_\rho}. \tag{54}$$

We obtain

(1) FPII (DP FP): The exponents are $\eta_\rho = \eta_{\hat\rho} = -\frac{\epsilon}{12}$, $\nu^{-1} = 2 - \frac{\epsilon}{4}$, dynamic exponent $z_\rho = 2 - \frac{\epsilon}{12}$.

(2) FPIV: $z = 2 - \frac{\epsilon}{12}$, $\eta_\rho = \eta_{\hat\rho} = -\frac{\epsilon}{12}$, and $\nu^{-1} = 2 - \frac{\epsilon}{4}$, which is exactly equal to the DP critical exponents. Thus, this FP behaves like a DP FP and displays weak dynamic scaling making it physically acceptable. Note that this FP involves back mutation from mutant B to species A.

Thus, we find that FPII and FPIV are the physically acceptable FPs displaying weak dynamic scaling as is expected in the $\theta^R \to 0$ limit (see below). Also, we see that $\frac{\partial\beta_\theta}{\partial\theta^R} = \frac{u^R}{8} > 0$, and $\frac{\partial\beta_\theta}{\partial A^R} = 0$, with $A = u, m, t$. This shows that the weak dynamic scaling shown by the FP FPIV is stable along the θ^R direction also. Surprisingly for FPIV, the critical exponents are all equal to the DP critical exponents. One cannot say for sure if it falls under the DP universality class, as we have calculated the exponents only up to one loop order. Higher loop corrections are necessary to settle the issue. The DP FP or the FPII also shows DP-like critical exponents, which is not surprising given that the couplings other than u^R are taken to be zero. But this FP is unstable, which is due to the birth-death couplings of ρ with ϕ in our model, unlike its analogue in the DP model. Clearly, thus on stability grounds, we accept FPIV as the FP that describes the scaling of the AAPT in the present model. A schematic diagram of the fixed points in the parameter space is given in Fig. 1.

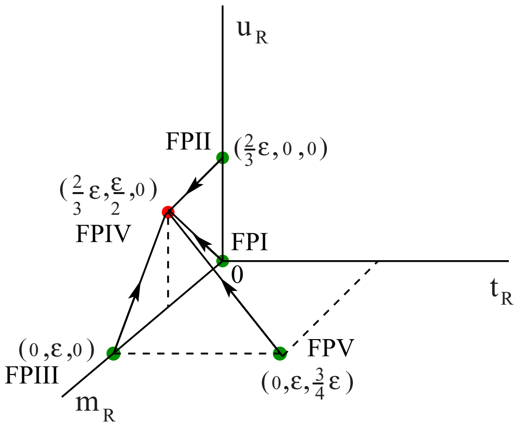

FIG. 1. (Color online) A schematic DRG flow diagram showing the stable and unstable fixed points (FPs) of our model in the parameter space u^R-m^R-t^R. The stable FP is FPIV given by the red thick dot and the unstable FPs are given by thick green dots. All the FPs correspond to weak dynamic scaling. Note that the flow is from the unstable FPs to the stable FPIV.

Finally, with the knowledge of $\eta_\rho, \eta_{\hat\rho}$, we now obtain the scaling of the renormalized equal-time correlator,

$$\langle|\rho(\mathbf{q},t)|^2\rangle \sim q^{2\eta_\rho}. \tag{55}$$

The fact that the critical exponents at the FPIV are identical to those at the DP FP may be heuristically argued as follows. Notice that at FPIV, $t^R = 0$, which means either ψ^R or γ^R or both of them are zero. Thus, either λ_g^R or D_1^R or both are zero in this FP. If $\lambda_g^R = 0$, the dynamics of ρ is autonomous in the renormalized theory; consequently, DP-like exponents should be expected for the scaling properties of ρ dynamics near AAPT. On the other hand, if $\lambda_G^R \neq 0$ but $D_1^R = 0$, then the renormalized dynamics of ϕ is effectively noise free. Thus, to the leading order, $\phi \approx \lambda_1^R/\lambda_2^R$ in the renormalized theory. If we substitute this in the renormlized version of Eq. (6), the effective renormalized equation for ρ has the form of the basic Langevin equation for DP in terms of shifted coefficients. This indicates the DP-like exponents for ρ near AAPT. This argument is, however, only suggestive and cannot be used to claim the equality of the scaling exponents at FPIV with the same for the DP problem at higher order in perturbation theory. Due to the difference in the stability properties of FPII and FPIV already at the one-loop order, we tend to speculate that these two FPs will correspond to different values for the scaling exponents at higher loop orders. This can only be checked by rigorous higher-order calculations, which are beyond the scope of the present work.

V. CORRELATIONS OF ϕ

With the knowledge of the scaling at the AAPT, we can now calculate the renormalized correlator $C_{\phi\phi}$ of ϕ. Notice that there are no anomalous dimension of ϕ and $\hat\phi$, nor is there any renormalization to D_ϕ. We linearize Eq. (13) to obtain

$$\frac{\partial\phi}{\partial t} = D_\phi\nabla^2\phi + (\lambda_3 - \lambda_2\phi_0)\rho + \sqrt{\rho}\zeta. \tag{56}$$

Noting that $\langle\rho\rangle \to 0$ near AAPT, the scaling of the renormalized correlation of ϕ may be obtained as

$$\langle|\phi(\mathbf{q},\omega)|^2\rangle = \frac{1}{\omega^2 + D_\phi^2 q^4}(\lambda_3 - \lambda_2\phi_0)^2\langle|\rho(\mathbf{q},\omega)|^2\rangle. \quad (57)$$

Now use the form of renormalized $\langle|\phi(\mathbf{q},\omega)|^2\rangle$ at the stable FPs,

$$\langle|\rho(\mathbf{q},\omega)|^2\rangle \sim \frac{q^{z+2\eta_\rho}}{\omega^2 + D_\rho q^{2z}}, \quad (58)$$

such that the equal-time correlator $\langle|\rho(\mathbf{q},t)|^2\rangle \sim q^{2\eta_\rho}$. This yields

$$\langle|\phi(\mathbf{q},t)|^2\rangle \sim \frac{1}{D_\phi} q^{2\eta_\rho - 2 - z} \exp(-D_\phi q^2 t), \quad (59)$$

revealing that (i) $z_\phi = 2 \neq z_\rho$ implying weak dynamic scaling and (ii) ϕ is spatially long-ranged correlated, since $2\eta_\rho - 2 - z < 0$.

VI. CONCLUSION

In this article we have proposed and studied a simple population dynamics model involving a species and its mutant: species A (with density $\rho(\mathbf{x},t)$) reproduces and dies, and also mutates to species B. We allow for a specific form of back mutation from B to A that allows for an AAPT of A that we study here. In the absorbing state, the mutant B is conserved. We perform a one-loop perturbative DRG analysis to extract the critical exponents of the AAPT. For reasons of analytical tractability, we analyze the model at low ($\theta_R \to 0$) Schmidt number. For $\theta_R \to 0$, we find weak dynamic scaling, i.e., $z_\rho < z_\phi = 2$ at two FPs: FPII or the DP FP and FPIV, consistent with $\theta_R \to 0$. Interestingly, FPIV exhibits scaling exponents that are the same as those in the DP universality class or FPII. We believe this surprising feature is fortuitous as it is not likely to be preserved when higher-order contributions are taken into account. In any case, FPII is unstable whereas FPIV turns out to be stable in all the directions of the parameter space. In this article we have not attempted to obtain the FPs for finite θ_R, for which strong dynamic scaling should follow, due to the algebraic complications involved. Nevertheless, from the overall stability of FPIV with $z_\rho < z_\phi$, together with $\beta_\theta > 0$ at this FP and the fact that no stable FP is obtained for $\theta_R \to \infty$, we speculate that the AAPT in our model is indeed characterized by weak dynamic scaling only ($z_\rho < z_\phi$), precluding no stable FP for strong dynamic scaling. Further conclusive evidence of these will, however, require numerically obtaining the FPs from the zeros of the β function Eqs. (34)–(38). Interestingly, the scaling of $C_{\phi\phi}$ at the AAPT displays long-ranged spatial correlation, which is a consequence of the long-ranged ρ fluctuations at the AAPT.

Our model has a highly simplified structure and is designed to study specific issues as discussed above. As a result, it lacks many details of a realistic population dynamics model. First of all, we have assumed an artificial form for back mutation, which may be generalized and included in our model in a straightforward way. In addition, we have excluded the effects of environment from our study, assuming a uniform surrounding in which the interactions take place. These may be included in our model in straightforward ways; see, e.g., Ref. [22]. In addition, the process of mutation is generally more complex than a simple conversion of one species to the other at a given fixed rate, as assumed here [41]. Nonetheless, our model should be useful as a starting point to understand the critical behavior of AAPT in generic population dynamics model with mutations or with multiple species. We expect our studies should be helpful in understanding the APPT in rock-paper-scissors-type systems [42–45], where one species feeds on a second species B, which in turn feeds on another third species that in turn feeds on the first species. We look forward to further theoretical studies along these lines.

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APPENDIX A: BARE VERTEX FUNCTIONS

The bare vertex functions for the model can be found out by taking functional derivatives of the generating functional $\Gamma[\rho,\hat{\rho},\phi,\hat{\phi}]$ with respect to the various fields in the model, i.e., ρ, $\hat{\rho}$, ϕ, $\hat{\phi}$. They are listed below:

$$\frac{\delta^2\Gamma}{\delta\rho(\mathbf{k},\omega)\delta\hat{\rho}(-\mathbf{k},-\omega)} = \Gamma_{\rho\hat{\rho}} = i\omega + D_\rho(-\tau + k^2), \quad (A1)$$

$$\frac{\delta^2\Gamma}{\delta\phi(\mathbf{k},\omega)\delta\hat{\phi}(-\mathbf{k},-\omega)} = \Gamma_{\phi\hat{\phi}} = (i\omega + D_\phi k^2), \quad (A2)$$

$$\frac{\delta^2\Gamma}{\delta\hat{\phi}(-\mathbf{k},-\omega)\delta\rho(\mathbf{k},\omega)} = \Gamma_{\hat{\phi}\rho} = -\lambda_3, \quad (A3)$$

$$\frac{\delta^3\Gamma}{\delta\hat{\rho}(\mathbf{q}_1,\omega_1)\delta\rho(\mathbf{q}_2,\omega_2)\delta\rho(-\mathbf{q}_1 - \mathbf{q}_2,-\omega_1-\omega_2)} = \Gamma_{\hat{\rho}\rho\rho} = \frac{D_\rho g_1}{2}, \quad (A4)$$

$$\frac{\delta^3\Gamma}{\delta\hat{\rho}(\mathbf{q}_1,\omega_1)\delta\hat{\rho}(\mathbf{q}_2,\omega_2)\delta\rho(-\mathbf{q}_1-\mathbf{q}_2,-\omega_1-\omega_2)} = \Gamma_{\hat{\rho}\hat{\rho}\rho} = -\frac{D_\rho g_2}{2}, \quad (A5)$$

$$\frac{\delta^3\Gamma}{\delta\hat{\rho}(\mathbf{k},\omega)\delta\rho(\mathbf{q},\Omega)\delta\phi(-\mathbf{k}-\mathbf{q},-\omega-\Omega)} = \Gamma_{\hat{\rho}\rho\phi} = -\lambda_g, \quad (A6)$$

$$\frac{\delta^3\Gamma}{\delta\hat{\phi}(\mathbf{k},\omega)\delta\phi(\mathbf{q},\Omega)\delta\rho(-\mathbf{k} - \mathbf{q},-\omega-\Omega)} = \Gamma_{\hat{\phi}\phi\rho} = \lambda_2, \quad (A7)$$

$$\frac{\delta^3\Gamma}{\delta\hat{\phi}(\mathbf{k},\omega)\delta\hat{\phi}(\mathbf{q},\Omega)\delta\rho(-\mathbf{k} - \mathbf{q},-\omega-\Omega)} = \Gamma_{\hat{\phi}\hat{\phi}\rho} = -D_1. \quad (A8)$$

APPENDIX B: NORMALIZATION CONDITIONS

The renormalized vertex functions when expressed in terms of the renormalized quantities can be written as follows:

$$\frac{\partial \Gamma_{\hat{\rho}\rho}}{\partial \omega}|_{(\mathbf{k}=0,\omega=0)} = i, \tag{B1}$$

$$\frac{\partial \Gamma_{\hat{\rho}\rho}}{\partial k^2}|_{(\mathbf{k}=0,\omega=0)} = D_\rho^R, \tag{B2}$$

$$\Gamma_{\hat{\rho}\rho}(\mathbf{k}=0,\omega=0) = D_\rho^R \tau^R, \tag{B3}$$

$$\frac{\partial \Gamma_{\hat{\phi}\phi}}{\partial \omega}|_{(\mathbf{k}=0,\omega=0)} = i, \tag{B4}$$

$$\frac{\partial \Gamma_{\hat{\phi}\phi}}{\partial k^2}|_{(\mathbf{k}=0,\omega=0)} = D_\phi^R, \tag{B5}$$

$$\Gamma_{\hat{\phi}\rho}(\mathbf{k}=0,\omega=0) = -\lambda_3^R, \tag{B6}$$

$$\Gamma_{\hat{\rho}\rho\rho}(\mathbf{k}=0,\mathbf{q}=0,\ \omega=0,\Omega=0) = \frac{D_\rho^R g_1^R}{2}, \tag{B7}$$

$$\Gamma_{\hat{\rho}\hat{\rho}\rho}(\mathbf{k}=0,\mathbf{q}=0,\omega=0,\Omega=0) = -\frac{D_\rho^R g_2^R}{2}, \tag{B8}$$

$$\Gamma_{\hat{\rho}\rho\phi}(\mathbf{k}=0,\mathbf{q}=0,\omega=0,\Omega=0) = -\lambda_g^R, \tag{B9}$$

$$\Gamma_{\hat{\phi}\phi\rho}(\mathbf{k}=0,\mathbf{q}=0,\omega=0,\Omega=0) = \lambda_2^R, \tag{B10}$$

$$\Gamma_{\hat{\phi}\hat{\phi}\rho}(\mathbf{k}=0,\mathbf{q}=0,\omega=0,\Omega=0) = -D_1^R. \tag{B11}$$

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