PHYSICAL REVIEW E **92**, 040102(R) (2015)

Crossing probability for directed polymers in random media

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(Received 6 July 2015; revised manuscript received 27 August 2015; published 12 October 2015)

We study the probability that two directed polymers in the same random potential do not intersect. We use the replica method to map the problem onto the attractive Lieb-Liniger model with generalized statistics between particles. Employing both the nested Bethe ansatz and known formula from MacDonald processes, we obtain analytical expressions for the first few moments of this probability and compare them to a numerical simulation of a discrete model at high temperature. From these observations, several large time properties of the noncrossing probabilities are conjectured. Extensions of our formalism to more general observables are discussed.

DOI: 10.1103/PhysRevE.92.040102 PACS number(s): 05.40.−a, 82.35.Lr, 05.10.Gg, 64.70.qj

Introduction. Recently there has been considerable progress in calculating the free energy and its fluctuations for directed polymers or directed paths in random media. This problem arises in a variety of fields, including optimization and glasses [1], vortex lines in superconductors [2], domain walls in magnets [3], disordered conductors [4], Burgers equation in fluid mechanics [5], exploration-exploitation tradeoff in population dynamics and economics [6], and biophysics [7,8]. Moreover, an exact mapping connects the Directed Polymer (DP) in $1+d$ dimension to the Kardar-Parisi-Zhang (KPZ) equation [9] in dimension d, which, in $d=1$, is at the center of an amazingly rich universality class, including discrete growth and particle transport models, with surprising connections in mathematics to random permutations and random matrices.

Two very different methods led to exact solutions: one based on the limit of discrete lattices, e.g., particle models such as q-TASEP (q deformed totally asymmetric simple esclusion process), often yielding rigorous results [10–14]; and the other one based on replica, a standard approach in the physics of disordered systems [15], and the mapping to a continuum quantum integrable system, solvable by Bethe ansatz [16–19]. The calculation of the nth moment of the DP partition sum is reduced to the time evolution of a n-particle quantum state, determined by the initial conditions. The evolution is performed with the attractive Lieb-Liniger Hamiltonian, whose spectrum is exactly computable [20,21]. The derivation based on the replica Bethe ansatz (RBA) involves some guessing and has often anticipated rigorous results from the math community. For instance, for the DP with two fixed endpoints, corresponding to the *droplet initial condition* in the KPZ equation, both approaches obtain the free energy as a Fredholm determinant, showing convergence at large time to the Tracy-Widom distribution for the largest eigenvalue of a random matrix [11,12,16,17,22].

An outstanding challenge is to extend these methods and results to collections of directed paths with hard-core repulsion, a difficult problem involving both interaction and disorder in a nonperturbative way. It arises in the above examples, e.g., populations competition, steps in vicinal surfaces, or the vortex glass in two-dimensional (2D) superconductors [23]. There was progress in that direction in the context of vortex arrays [24], within the multilayer PNG growth model [25], and the semidiscrete DP hierarchies [13,26], with emerging connections to the spectrum of random matrices. Within the RBA method, in almost all cases up to now, only the $1d$ Bose gas was considered, i.e., with initial conditions corresponding to a fully symmetric quantum state. Here we consider infinite hard-core repulsion, modeled by a noncrossing condition, which requires more general initial conditions.

The aim of this Rapid Communication is to study continuum DP observables for noncrossing paths. We develop the more general nested replica Bethe ansatz (NRBA) and connect it to another recently developed method [13]. Here, as a first step, we focus on the calculation of *crossing probabilities*, but we expect the potential outcome of the method to be broader.

We introduce the partition function of a directed polymer with fixed endpoints

$$Z_\eta(x;y|t) \equiv \int_{x(0)=x}^{x(t)=y} Dx\, e^{-\int_0^t d\tau [\frac{1}{4}(\frac{dx}{d\tau})^2 - \sqrt{2\bar{c}}\,\eta(x(\tau),\tau)]} \tag{1}$$

in a random potential with white-noise correlations $\overline{\eta(x,t)\eta(x',t')} = \delta(x-x')\delta(t-t')$. Then, the probability that two polymers with fixed endpoints do not cross in a given realization η of the potential is expressed as

$$p_\eta(x_1,x_2;y_1,y_2|t) \equiv 1 - \frac{Z_\eta(x_2;y_1|t)Z_\eta(x_1;y_2|t)}{Z_\eta(x_1;y_1|t)Z_\eta(x_2;y_2|t)} \tag{2}$$

since all paths with at least one intersection can be obtained from paths with y_1,y_2 exchanged [27] (see Fig. 1). For simplicity, we consider the random variable defined by the limit of near-coinciding endpoints

$$p_\eta(t) \equiv \lim_{\epsilon\to 0} \frac{p_\eta(-\epsilon,\epsilon;-\epsilon,\epsilon|t)}{4\epsilon^2} = \partial_x \partial_y \ln Z_\eta(x;y|t)|_{\substack{x=0\\y=0}} \tag{3}$$

where the last equality, derived from relation (2), belongs to a larger set of relations between noncrossing probabilities and the single path free energy [28,29]. We now present a technique to calculate all the moments of $p_\eta(t)$ at arbitrary time t with explicit results for the first few.

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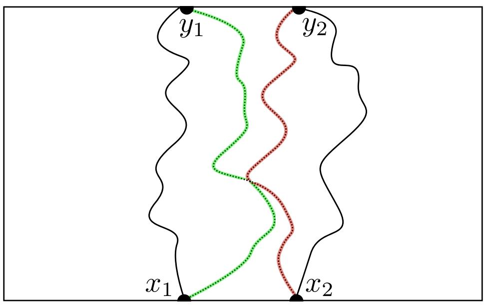

FIG. 1. (Color online) Paths with fixed endpoints. By exchanging the paths after the last intersection, one builds a mapping between crossing paths and paths with the final endpoints exchanged.

Replica trick and nested Bethe ansatz. The average of products $\mathcal{Z}_n = \overline{Z_\eta(x_1;y_1|t)\dots Z_\eta(x_n;y_n|t)}$ satisfies [30]

$$\mathcal{Z}_n(\mathbf{x};\mathbf{y}|t) = \langle x_1 \dots x_n | e^{-tH_n} | y_1 \dots y_n \rangle \tag{4}$$

for any integer n, in quantum mechanical notations, where bold symbols are shorthand for ordered sets of variables and the Lieb-Liniger Hamiltonian reads

$$H_n \equiv -\sum_{i=1}^{n} \partial_{x_i}^2 + 2c \sum_{1\leqslant i<j\leqslant n} \delta(x_i - x_j) \tag{5}$$

with $c = -\bar{c} < 0$. To use the replica trick we introduce

$$\Theta_{n,m}(t) \equiv \lim_{\epsilon\to 0} \overline{[(2\epsilon)^{-2} Z_\eta^{(2)}(\epsilon)]^m [Z_\eta(0;0|t)]^{n-2m}}, \tag{6}$$

where we set $Z_\eta^{(2)}(\epsilon) \equiv Z_\eta(\epsilon;\epsilon|t) Z_\eta(-\epsilon;-\epsilon|t) - Z_\eta(-\epsilon;\epsilon|t) Z_\eta(\epsilon;-\epsilon|t)$, so that $\overline{p_\eta(t)^m} = \Theta_{0,m}(t)$. The advantage of this expression is that for integers n,m with $n \geqslant 2m$, it can be expressed in terms of relation (4):

$$\begin{aligned}\Theta_{n,m}(t) &= \lim_{\epsilon\to 0} (2\epsilon)^{-2m} \langle \Psi_m(\epsilon)| e^{-tH_n} |\Psi_m(\epsilon)\rangle \\ &= \sum_\mu \frac{|\mathcal{D}_m \psi_\mu(\mathbf{0})|^2}{||\mu||^2} e^{-tE_\mu},\end{aligned} \tag{7}$$

where $|\Psi_m(\epsilon)\rangle = 2^{-m/2} (\otimes_{j=1}^m |\epsilon,-\epsilon\rangle - |-\epsilon,\epsilon\rangle) \otimes |0\dots 0\rangle$ and a complete set of eigenstates $|\mu\rangle$ of H_n of energies E_μ has been inserted with $\psi_\mu(\mathbf{x}) \equiv \langle \mathbf{x}|\mu\rangle$. Here $\mathcal{D}_m$ is a differential operator obtained from the limit $\epsilon \to 0$ of $(2\epsilon)^{-m} \langle \Psi_m(\epsilon)|\mu\rangle$, e.g., $\mathcal{D}_1 = 2^{-1/2}(\partial_{x_1} - \partial_{x_2})|_{\mathbf{x}=0}$. Since H_n is integrable by Bethe ansatz, the eigenstates, with eigenvalues $E_\mu = \sum_{j=1}^n \mu_j^2$, take the form

$$\psi_\mu(\mathbf{x}) = \sum_{P,Q\in\mathcal{S}_n} \vartheta_Q(\mathbf{x}) A_Q^P \exp\left[i \sum_{j=1}^n x_{Q_j} \mu_{P_j} \right], \tag{8}$$

where $\{\mu_1,\dots,\mu_n\}$ is a set of rapidities, $\mathcal{S}_n$ is the set of n permutations, and $\vartheta_Q(\mathbf{x})$ is the indicator of the sector $x_{Q_1} \leqslant x_{Q_2} \leqslant \dots \leqslant x_{Q_n}$. However, $|\Psi_m(\epsilon)\rangle$ is not a symmetric state under the exchange of the coordinates; thus the quantum dynamics described by relation (5) does not belong to the bosonic sector. Nonetheless, it is still possible to explicitly determine the eigenstates [31] corresponding to different representations of the symmetric group. It is enough to choose the vectors A_Q^P, for all fixed permutation P, inside an irreducible representation of S_n. The relevant case for us is the representation corresponding to a two-rows Young diagram $\xi = (n-m,m)$, where we denote a diagram as the decreasing sequence of row lengths [32,33]. For instance, for $n = 8$ and $m = 3$ we have

(5,3) ≡	1	3	5	7	9
	2	4	6		

(9)

and the filling indicates antisymmetric wave functions under the exchange of coordinates $x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_4$, $x_5 \leftrightarrow x_6$, which are in the symmetry class selected by the action of $\mathcal{D}_{m=3}$. These representations can be built explicitly as the Hilbert space of an integrable spin-1/2 chain with n sites restricted to the sector with m down spins. Then the eigenstates of H_n on a ring of length L are obtained by diagonalizing simultaneously the spin model. This leads to the so-called nested-Bethe-ansatz (NBA) equations

$$\prod_{\substack{b=1\\ b\neq a}}^{m} \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^{n} \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2}, \tag{10a}$$

$$\prod_{\substack{k=1\\ k\neq j}}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^{m} \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}, \tag{10b}$$

where $\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$ and same for the λ's, the auxiliary rapidities on the spin chain that impose the appropriate symmetry to the wave function. Solutions of Eqs. (10) provide the eigenstates of relation (5) in the appropriate symmetry class and the wave functions are obtained setting $A_Q^P = A_{\text{bos}}^P \langle\langle \Psi_Q | \omega(P\boldsymbol{\mu})\rangle\rangle$ with

$$|\omega(\boldsymbol{\mu})\rangle\rangle = \sum_{a_1,\dots,a_m=1}^{n} \alpha(\mathbf{a}|\boldsymbol{\mu}) \sigma_-^{a_1} \dots \sigma_-^{a_m} |+\rangle\rangle. \tag{11}$$

Here, $(P\mu)_j \equiv \mu_{P_j}$ and $|\dots\rangle\rangle$ indicates states in the auxiliary spin space, σ_-^a is the lowering spin operator at site a, $a = 1,\dots,m$, acting on the reference state $|+\rangle\rangle = |\uparrow \dots \uparrow\rangle\rangle$. The vector of states $|\Psi_Q\rangle\rangle$ is fixed by the filling of ξ and performs the unitary mapping between the spin-chain representation and a particular representation of shape $(n-m,m)$, such that the exchange of two spins is mapped into the exchange of two particles. Here $A_{\text{bos}}^P = \Omega_{\boldsymbol{\mu}}^0 / \Omega_{P\boldsymbol{\mu}}$ accounts for the bosonic phase scattering with $\Omega_{\boldsymbol{\mu}} \equiv \prod_{j<l} f(\mu_{lj})$, $\Omega_{\boldsymbol{\mu}}^0 = \sqrt{\Omega_{\boldsymbol{\mu}} \Omega_{-\boldsymbol{\mu}}}$ and $f(u) \equiv u/(u - ic)$, while

$$\alpha(\mathbf{a}|\boldsymbol{\mu}) = \text{sym}_{\boldsymbol{\lambda}} \left[\prod_{k<l} \left(1 + \frac{ic\,\text{sgn}(a_{lk})}{\lambda_{lk}} \right) \prod_{l=1}^{m} \kappa_{a_l}(\lambda_l|\boldsymbol{\mu}) \right],$$

$$\kappa_a(u|\boldsymbol{\mu}) = \frac{ic}{u - \mu_a - ic/2} \prod_{b=a}^{n} \frac{u - \mu_b - ic/2}{u - \mu_b + ic/2},$$

and $\text{sym}_{\boldsymbol{\lambda}}[W(\boldsymbol{\lambda})] = \sum_R W(R\boldsymbol{\lambda})/m!$ is the symmetrization of $W(\boldsymbol{\lambda})$ over the variables $\boldsymbol{\lambda}$.

Average of $p_\eta(t)$. We now consider the case $m = 1$ which selects the subspace of wave functions $\Psi_{\boldsymbol{\mu}}(x_1 = x_2) = 0$. Then, the wave function in Eq. (8) remains continuous, even after the action of $\mathcal{D}_1$, and we can average over different orderings of the coordinates $\mathbf{x}$. Hence $\mathcal{D}_1\psi_\mu(\mathbf{0}) = \frac{1}{n!}\sum_{P,Q} d_1(Q^{-1}P\boldsymbol{\mu})A_Q^P$, where $d_m(\boldsymbol{\mu}) = (-i)^m\mathcal{D}_m e^{i\boldsymbol{\mu}\cdot\mathbf{x}}|_{\mathbf{x}=0}$. We then obtain [34]

$$\frac{1}{n!}\sum_Q d_1(Q^{-1}\boldsymbol{\mu})\,|\Psi_Q\rangle\rangle = \sqrt{Z}\sum_{a=1}^n(\mu_a-\hat{\mu})\sigma_-^a\,|+\rangle\rangle \quad (12)$$

with $Z=[(n-1)n!]^{-1}$ ensuring normalization and $\hat{\mu} = \sum_b \mu_b/n$. It leads to $|\mathcal{D}_1\psi_\mu(\mathbf{0})|^2 = n(n-2)!|\Omega_{\boldsymbol{\mu}}|^2|\mathcal{F}(\lambda,\boldsymbol{\mu})|^2$, where $\mathcal{F}(\lambda,\boldsymbol{\mu}) = \mathrm{sym}_{\boldsymbol{\mu}}\,[\Omega_{\boldsymbol{\mu}}^{-1}\sum_a(\mu_a-\hat{\mu})\kappa_a(\lambda|\boldsymbol{\mu})]$ and we note $\lambda_{a=1}=\lambda$. The sum over all solutions of Eqs. (10) must then be performed according to Eq. (7), a formidable task in general. However, Eq. (10b) simplifies dramatically when $L\to\infty$. For $\bar{c}>0$, the n rapidities $\mu_1,\ldots,\mu_n$ are organized in n_s bound states, each composed by $m_j \geqslant 1$ particles, with $\sum_{j=1}^{n_s} m_j = n$. The rapidities inside a bound state follow a regular pattern in the complex plane $\mu^{j,a} = k_j + \frac{i\bar{c}}{2}(m_j+1-2a) + i\delta^{j,a}$, named string. Here $a=1,\ldots,m_j$ labels the rapidities inside the string and $\delta^{j,a}$ are exponentially small for large L. A study of Eqs. (10) reveals that, at variance with the bosonic case, not all string configurations are actually allowed, which is consistently with the symmetry of the wave function [28], see also Refs. [35–37]. For those allowed, using Refs. [22,38], we obtain their norm Ref. [39] as

$$\frac{||\psi_\mu||^2}{(\Omega_{\boldsymbol{\mu}}^0)^2} = \frac{(L\bar{c})^{n_s}\prod_j m_j^2}{\bar{c}^n\,\Phi(\mathbf{k},\mathbf{m})}\sum_{l=1}^n\frac{4c^2}{c^2+4(\mu_l-\lambda)^2}\,, \quad (13)$$

$$\Phi(\mathbf{k},\mathbf{m}) = \prod_{1\leqslant j<j'\leqslant n_s}\frac{(k_j-k_{j'})^2+c^2(m_j-m_{j'})^2/4}{(k_j-k_{j'})^2+c^2(m_j+m_{j'})^2/4}\,. \quad (14)$$

For each configuration of rapidities following the string ansatz, a multiplet of eigenstates is given by the set $\{\lambda^{(1)},\ldots,\lambda^{(n-1)}\}$ of solutions of Eq. (10a), i.e., $Q(\lambda)\equiv P_+(\lambda)/P_-(\lambda)=1$, where $P_\pm(\lambda)=\prod_i(\lambda-\mu_i\pm ic/2)$. These values cannot be determined analytically for general n; however, the sum over them can be performed using residue theorem

$$\sum_i w(\lambda^{(i)}) = \oint_{\mathcal{C}}\frac{dz}{2\pi i}w(z)\frac{Q'(z)}{Q(z)-1},$$

where $w(z)$ is any analytic function inside the contour $\mathcal{C}$, which encircles all the solutions $\lambda^{(i)}$ and no other singularity of the integrand. Equivalently, the integral can be computed taking the poles outside the contour, which in the case $w(z) = \frac{|\mathcal{F}(\lambda^{(i)},\boldsymbol{\mu})|^2}{||\psi_\mu||^2}$ are given by $z_k=\mu_k-ic/2$ [39]. The sum can then be performed analytically. Moreover, for $L\to\infty$, string momenta become free and we can replace $\sum_{k_j}\to m_jL\int\frac{dk_j}{2\pi}$, which leads to

$$\Theta_{n,1}(t) = \sum_{n_s=1}^n\frac{n!\bar{c}^n}{n_s!(2\pi\bar{c})^{n_s}}\sum_{(m_1,..m_{n_s})_n}\times\prod_{j=1}^{n_s}\int_{-\infty}^{+\infty}\frac{dk_j e^{-A_2 t}}{m_j}\Phi(\mathbf{k},\mathbf{m})\Lambda_{n,1}(\mathbf{k},\mathbf{m}) \quad (15)$$

with $\Lambda_{n,1}=[n(n-1)]^{-1}h_2$. Here, $(m_1,\ldots,m_{n_s})_n$ indicates sum over all integers $m\geqslant 1$ whose sum equals n and we defined $h_p=\sum_{j<l}(\mu_j-\mu_l)^p-(i\bar{c})^p(j-l)^p$. The rapidities μ_j are written as a function of string sizes and momenta according to the string ansatz, so that $\Lambda_{n,1}$ vanishes on the n strings. The conserved charges of the Lieb-Liniger model have been introduced as $A_p=\sum_{j=1}^n\mu_j^p$, A_2 being the energy. A crucial property of Eq. (15) is that by replacing $\Lambda_{n,1}\to\Lambda_{n,0}\equiv 1$, one recovers the formula for $\mathcal{Z}_n(t)\equiv\mathcal{Z}_n(\mathbf{x}=\mathbf{0};\mathbf{0}|t) = \Theta_{n,0}(t)$ as given in Ref. [16]. Therefore, rewriting $\Lambda_{n,1}$ in terms of the conserved charges and using the statistical tilt symmetry (STS) (see, e.g., the appendix of Ref. [18]), we obtain

$$\Theta_{n,1}(t) = \frac{1}{n-1}\left[\frac{n(n^2-1)\bar{c}^2}{12}-\partial_t-\frac{1}{2t}\right]\mathcal{Z}_n(t)\,. \quad (16)$$

This expression is exact for $n\geqslant 2$ and allows the analytical continuation $n\to 0$. In particular, we obtain

$$\overline{p_\eta(t)} = \lim_{n\to 0}\Theta_{n,1}(t) = \frac{1}{2t}\,. \quad (17)$$

This is in fact the exact result for $p_\eta(t)$ without disorder, i.e., $\eta(x,t)=0$. This remarkable conclusion can also be obtained by averaging relation (3) and recalling that the dependence of the *average* free energy of a path with respect to its endpoints is entirely fixed by the STS, namely $\overline{\ln Z_\eta(x;y|t)} = h(t)-(x-y)^2/(4t)$, where

$$h(t) = \overline{\ln Z_\eta(0;0|t)} \quad (18)$$

is our averaged free energy (and average KPZ height).

Alternative derivation. A different approach was recently proposed in Ref. [13] (remark 5.25) where nonintersecting paths were also studied. There, it was proposed a multicontour-integral formula is associated to a partition of n. We identify the partition with a Young diagram and for the two-row case of our interest, it can be put in the form

$$\Theta_{n,m}(t) = \frac{1}{2^m}\int\frac{dz_1}{2\pi}\cdots\int\frac{dz_n}{2\pi}e^{-t\sum_{k=1}^n z_k^2}\times\left(\prod_{1\leqslant k<j\leqslant n}f(z_{kj})\right)\left(\prod_{q=1}^m h(z_{2q-1,2q})\right), \quad (19)$$

where $z_{kj}=z_k-z_j$, $h(u)=u(u-ic)$ and the integration contours are parallel to the real axis with an imaginary part C_j for z_j satisfying $C_{j+1}>C_j+\bar{c}$. Shifting back all the contours to the real axis, we encounter many poles whose residues reduce to integrals with a smaller number of integration variables. This expansion can then be organized to reproduce the one based on strings in Eq. (15), with $\Lambda_{n,1}$ replaced by [28]

$$\Lambda_{n,m}(\mathbf{k},\mathbf{m}) = \frac{1}{2^m}\,\mathrm{sym}_{\boldsymbol{\mu}}\left[\frac{\prod_{q=1}^m h(\mu_{2q-1,2q})}{\prod_{1\leqslant j<k\leqslant n}f(\mu_{kj})}\right] \quad (20)$$

and again the $\boldsymbol{\mu}$ given by the string ansatz. Interestingly, $\Lambda_{n,m}$ is always a polynomial in the μ's of degree $2m$ as can be seen considering the residue at coinciding points. Moreover, Eq. (20) agrees with the result obtained from the NBA for $m=1$, which gives a completely independent check to the proposition in Eq. (19). For $m>1$, the calculation from NBA becomes more involved but we will continue by assuming that Eq. (20) retains its validity.

Higher moments of $p_\eta(t)$. We now focus on $m = 2$. Upon symmetrization in Eq. (20) one obtains

$$\Lambda_{n,2}(\boldsymbol{\mu}) = \frac{h_2^2 - (n-1)h_4 - n(n-1)^2\bar{c}^2 h_2}{n(n-1)(n-2)(n-3)}. \tag{21}$$

After tedious calculations, it can be rewritten in terms of the conserved charges $\Lambda_{n,2}(\{A_p\})$ [28]. In contrast with the $m = 1$ case, higher charges, up to $p = 4$, are involved. It is therefore useful to formally generalize the partition function to $\mathcal{Z}_n^{\boldsymbol{\beta}}(t)$ which is obtained from the expression of $\mathcal{Z}_n(t)$ replacing the imaginary time evolution $e^{-A_2 t}$ with the more general $e^{-A_2 t + \sum_{p\geqslant 1}\beta_p A_p}$. This is the partition function of a generalized Gibbs ensemble (GGE) [40], which we show can be related to a Fredholm determinant [28]. Here, we use it as a generating function: $\Theta_{n,2}(t) = \Lambda_{n,2}(\{\partial_p\})\mathcal{Z}_n^{\boldsymbol{\beta}}(t)$, formally replacing $A_p \to \partial_p \equiv \partial_{\beta_p}$ and setting $\beta_p \to 0$ at the end. Deriving extended STS identities from the invariance $\mu_j \to \mu_j + k$ in $\mathcal{Z}_n^{\boldsymbol{\beta}}(t)$, for arbitrary k, we are able to re-express it only from the energy A_2, leading to

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right)h(t)\,. \tag{22}$$

Hence the second moment is determined at all times from the average free energy $h(t)$ (18). We did not find a direct derivation of this remarkable result, and it may be a consequence of integrability. At large time $h(t) \simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2}(\bar{c}^2 t)^{1/3}$ [11,12,16,17], so that

$$\overline{p_\eta(t)^2} \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}} \tag{23}$$

with $\overline{\chi_2} = -1.77(1)$ the mean of the Tracy-Widom GUE distribution [41] (see Ref. [42] for numerical details). Repeating this procedure for $m = 3$ we can again use $\mathcal{Z}_n^{\boldsymbol{\beta}}(t)$. Now higher charges are involved and the result is expressed as derivatives of a Fredholm determinant [28]. It simplifies at large time, leading to

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}. \tag{24}$$

It is natural to conjecture the leading decay $\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$ for any integer $m > 1$. However, the knowledge of moments at long times is not sufficient to reconstruct the full distribution of p: in view of Ref. [26], we further surmise that $p_\eta(t)$ tends to zero (sub)exponentially at large t for all but a small fraction $\sim 1/(\bar{c}^2 t)$ of environments where typically $p_\eta(t) \sim \bar{c}^2$. This is consistent with the conjecture [28] $\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3}$ where $a = \overline{\chi_2} - \overline{\chi_2'}$ is the average gap between the first (χ_2) and second (χ_2') GUE (scaled) largest eigenvalues, with $a \approx 1.9043$ [43] (note that $a \approx 1.49134$ for the hard wall problem [44]).

Comparison with numerics. To check our results, we study a discrete directed polymer on a square lattice [16], defined according to the recursion (with integer time $\hat{t}$ running along the diagonal)

$$Z_{\hat{x},\hat{t}+1} = \left(Z_{\hat{x}-\frac{1}{2},t} + Z_{\hat{x}+\frac{1}{2},\hat{t}}\right)e^{-\beta V_{\hat{x},\hat{t}+1}} \tag{25}$$

with $V_{\hat{x},\hat{t}}$ sampled from the standard normal distribution. In the high temperature limit $\beta \ll 1$, it maps into the continuous DP (1) at $\bar{c} = 1$ with $x = 4\hat{x}\beta^2$ and $t = 2\hat{t}\beta^4$ [16]. We consider two polymers with initial conditions $Z^{\pm}_{\hat{x},\hat{t}=1} = \delta_{\hat{x},\pm 1/2}$ and ending at time $\hat{t}$ at $\hat{x} = \pm 1/2$. For each realization of the $V_{\hat{x},\hat{t}}$, the noncrossing probability $\hat{p}$ on the lattice is efficiently computed using the image method [27]. By comparison with relation (3), we deduce $\hat{p} \simeq 16p\beta^4$, for $\beta \to 0$, due to the rescaling of the factor $\epsilon = 4\beta^2$. The numerical results and the analytical predictions [Eqs. (17,23)] are shown in Fig. 2. For the first moment $\overline{p_\eta(t)}$ the agreement is excellent even at finite temperature, presumably due to robustness of Eq. (17) at any time. The numerical check of Eq. (23) is more delicate: Indeed, the large-time behavior of the second moment depends strongly on temperature and approaches our prediction only for $\beta \ll 1$; see Fig. 2 (right). However, the leading decay is found consistent with t^{-1} down to zero temperature, where the polymer paths do not fluctuate thermally and for any $m > 0$: $\overline{\hat{p}^m} = \overline{\hat{p}}$. In order to interpolate between zero and high temperatures, we conjecture the large time behavior of the moments on the lattice $\overline{\hat{p}^m} \simeq c_m(\beta)/\hat{t}$, with $c_m(\beta) \simeq$

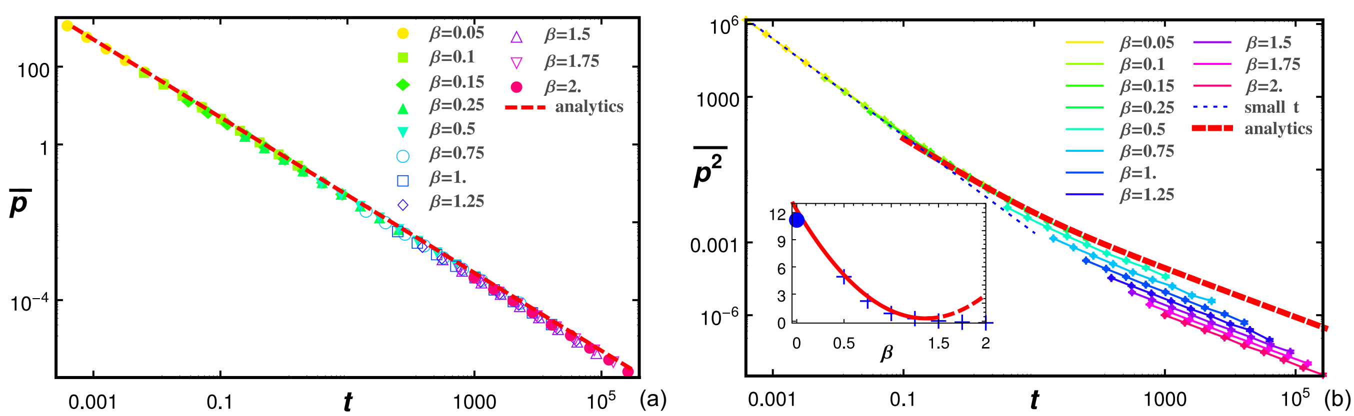

FIG. 2. (Color online) The first (a) and second moments (b) of p_η in the continuum limit are shown vs t for several value of β. Numerical simulations with at least 2×10^5 realizations and time up to $\hat{t} = 8192$. The value of p_η introduced in Eq. (3) is obtained by $p_\eta(t) = \hat{p}/(16\beta^4)$, with $\hat{p}$ being the probability on the lattice, and the time is scaled to t by $\hat{t} = t/(2\beta^4)$. Dashed red (gray) line: analytical predictions (17) and (23). In panel (b), the small t expansion $\overline{p_\eta^2} \simeq 4t^{-2}$ is shown with a dashed blue (gray) line. Inset in panel (b): plot of $\overline{\hat{p}^2}\,\hat{t}/\beta^4 \simeq \beta^{-4}c_2(\beta)$ vs β and extrapolation at small $\beta = 0$ from the best fit with a quadratic function in β at fixed $t = 10^3$. It shows a finite limit consistent with our prediction $\simeq 32/3 = 10.6(7)$ (blue [gray] circle).

$\gamma_m 2^{4m-1}\beta^{4(m-1)}$ at high temperatures and $c_m(\beta\to\infty)=c_\infty$ is a constant that we expect to be nonuniversal [28]. This agrees with the intuitive picture of the zero-temperature deterministic path, weakly perturbed by thermal fluctuations. Checking the subleading terms in Eq. (23) would require much more intensive numerics.

Conclusions. We presented a general formalism to calculate the statistics of N mutually avoiding directed polymers in a random potential, with explicit results for $N=2$. Multi-polymer observables are reduced to a compact form in terms of conserved quantities of the Lieb-Liniger Hamiltonian and expressed at all times by derivatives of a Fredholm determinant, i.e., the GGE partition function. As a simplest example we obtained the lowest moments of noncrossing probabilities, with an exact relation between the variance and the free energy, a nontrivial scaling $t^{-5/3}$ for the subleading part and a prediction of leading behavior for all moments. The full distribution of the noncrossing probability is under current investigation. Going beyond the infinite hard-core repulsion remains for the moment elusive, but we are confident that further developments of the present method and full exploitation of its integrable structure will allow further progress in the elusive interplay between disorder and interactions.

Acknowledgments. We thank A. Borodin, I. Corwin, and A. Rosso for fruitful discussions. This work is supported by Investissements d'Avenir LabEx PALM (ANR-10-LABX-0039-PALM) and by PSL Grant No. ANR-10-IDEX-0001-02-PSL.

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