Erratum: Message passing and moment closure for susceptible-infected-recovered epidemics on finite networks [Phys. Rev. E 89, 022808 (2014)]

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We report an error in Eq. (16) of the original paper. This equation is not correct because the right-hand side does not fully account for the information that $i \leftarrow B$. However, Eq. (16) is only used to prove inequality (17), which still holds. Here, we provide an alternative proof of inequality (17) using "associated" random variables. Our argument is very similar to and inspired by Donnelly's proof for the existence of "positive correlations" in stochastic susceptible-infected-recovered (SIR) models (see Ref. [1], Sec. 3).

We first summarize the method of proof. We shall define random variables $f_{i,B}$, where $i \in V$ and $B \subset N_i$, all on the same sample space, such that $f_{i,B} = \mathbb{1}(i \leftarrow B)$, where $\mathbb{1}$ denotes the indicator function. We then show that $f_{i,B}$ is a nondecreasing function of the set of random variables X, where it is the same set of random variables for every choice of $i \in V$ and $B \subset N_i$. Finally, we show that the set X of random variables is "associated," as defined by Esary *et al.* [2]. By the definition of "association," these facts imply that

$$\mathrm{Cov}[f_{i,B}, f_{i,C}] = E[f_{i,B} f_{i,C}] - E[f_{i,B}]E[f_{i,C}] = P(i \leftarrow B, i \leftarrow C) - P(i \leftarrow B)P(i \leftarrow C) \geqslant 0$$

(see definition 1.1 in Esary *et al.* [2]), which establishes inequality (17).

We define the random weighted graph D' to be generated in the same way as in the original paper, except that we omit steps 3 and 4. We now define the following set of random variables,

$$X \equiv \cup_{i \in V, j \in N_i} \{I_i(0)^c, R_i(0), \tau_i^*, \omega_{ji}\},$$

where $I_i(0)^c$ is the indicator variable for the complement of the event that i is initially infected, $R_i(0)$ is the indicator variable for the event that i is initially recovered, and $\tau_i^* = -\tau_i$ (with τ_i and ω_{ji} as in the original paper).

We now define $f_{i,B}$ such that $f_{i,B} = 0$ if there exists a path (in D') from some $k \in V$ to i, with a member of B as the penultimate individual, where (1) $I_k(0)^c = 0$, (2) for every individual j in the path $R_j(0) = 0$, (3) for each ω_{ji} in the path $|\tau_i^*| > \omega_{ji}$, (4) its total weighting (summing the ω_{ji} along the path) is less than t, otherwise $f_{i,B} = 1$. Clearly then, $f_{i,B}$ is a function of the random variables X and is nondecreasing with respect to each.

Esary *et al.* [2] (see properties P_2 and P_3) proved that (a) if two sets of associated random variables are independent of one another, then their union is a set of associated random variables, and (b) the set consisting of a single random variable is associated. Hence, we need to show that $I_i(0)^c$ and $R_i(0)$ are associated for any given $i \in V$, since then X is formed from a union of sets of associated variables where there is independence between the sets (it is assumed in the original paper that all contact and recovery processes are mutually independent, and independent from the initial conditions).

We know that $I_i(0)^c$ and $R_i(0)$ are associated from Theorem 4.3 of Ref. [2], since

$$P(R_i(0) = 1 \mid I_i(0)^c = 1) \geqslant P(R_i(0) = 1 \mid I_i(0)^c = 0),$$

the right-hand side being zero.

More generally, when there is statistical dependence between arbitrary members of X, inequality (17) holds if the set X is associated [$f_{iB}(X)$ remains nondecreasing and equal to $\mathbb{1}(i \leftarrow B)$]. This allows for the possibility of suitably correlated contact or infectious periods and suitably correlated initial conditions. For example, inequality (17) holds in the case where, grouping together the random variables relating to the "behavior" of a single individual, the sets

$$X_i \equiv \{\tau_i^*, \cup_{j \in N_i} \omega_{ji}\}$$

are mutually independent for all $i \in V$ while, for each $i \in V$, the variables in X_i are associated, and where the set of initial-condition variables

$$\cup_{i \in V} \{I_i(0)^c, R_i(0)\}$$

is associated and independent from $\cup_{i \in V} X_i$.

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[1] P. Donnelly, The correlation structure of epidemic models, Math. Biosci. **117**, 49 (1993).
[2] J. D. Esary, F. Proschan, and D. W. Walkup, Association of random variables, with applications, Ann. Math. Stat. **38**, 1466 (1967).