Wave-number spectrum of dissipative drift waves and a transition scale

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We study the steady state spectrum of the Hasegawa-Wakatani (HW) equations that describe drift wave turbulence. Beyond a critical scale k_c, which appears as a balance between the nonlinear time and the parallel conduction time, the adiabatic electron response breaks down nonlinearly and an internal energy density spectrum of the form $F(k_\perp) \propto k_\perp^{-3}$, associated with the background gradient, is established. More generally a dual power law spectrum, approximately of the form $F(k_\perp) \propto k_\perp^{-3}(k_c^{-2} + k_\perp^{-2})$ is obtained, which captures this transition. Using dimensional analysis, an expression of the form $k_c \propto C/\kappa$ is derived for the transition scale, where C and κ are normalized parameters of the HW equations signifying the electron adiabaticity and the density gradient, respectively. The results are numerically confirmed using a shell model developed and used for the Hasegawa-Wakatani system.

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I. INTRODUCTION

The standard Hasegawa-Wakatani (HW) equations [1] were originally derived as a reduced nonlinear model to study key physical aspects of dissipative drift wave turbulence. In the "adiabatic limit," the equations reduce to the Charney-Hasegawa-Mima (CHM) equation [2,3], which is used in the description of drift waves in magnetized plasmas and Rossby waves in the atmosphere [4,5], while in the "hydrodynamic limit" it reduces to the two-dimensional Navier-Stokes equation. The equation can be used to describe various physical phenomena from geophysical fluids dynamics to basic two-dimensional fluid turbulence in these limiting forms. Here we study the wave-number spectra of the HW equations in various limiting cases.

The coupled equations of the HW model, governing the normalized electric potential $\Phi = e\Phi/T_e$ and density $n = n/n_0$ are

$$\left(\frac{\partial}{\partial t} - \nabla\Phi \times \hat{z}\cdot\nabla\right)\nabla^2\Phi - C(\Phi - n) = D_\Phi, \quad (1a)$$

$$\left(\frac{\partial}{\partial t} - \nabla\Phi \times \hat{z}\cdot\nabla\right)n + \kappa\frac{\partial\Phi}{\partial y} - C(\Phi - n) = D_n, \quad (1b)$$

where $C = (T_e k_\parallel^2)/(e^2 n_0 \eta \omega_{ci})$ is a measure of parallel electron conduction rate (sometimes called the adiabaticity parameter), $\kappa = -\rho_s d(\ln n_0)/dx$ is the normalized density gradient, and $D_{\Phi,n}$ are the dissipation of vorticity and density, respectively. Independent variables t and x are normalized with the ion cyclotron frequency $\omega_{ci} = eB/m_i$ and the ion sound gyroradius $\rho_s = \sqrt{T_e/m_i}\,\omega_{ci}^{-1}$. B, e, m_i, T_e, and η are the magnetic field, electron charge, ion mass, electron temperature, and the plasma resistivity, respectively. The system consisting of Eqs. (1a) and (1b) can be obtained from equations of plasma vorticity and electron continuity, using the parallel Ohm's law to eliminate the parallel current. Note that Eqs. (1a) and (1b) together imply potential vorticity (PV) conservation, where the PV for the system can be defined as $q \approx \nabla^2\Phi - \ln n$. Note also that a modification for the vorticity equation is actually necessary in order to account for zonal flows [6], and their interaction with turbulence [7–10]. We do not consider this variation here, since our motivation is to study the wave number spectrum implied by the Hasegawa-Wakatani system as is.

The limit $C \ll 1$ corresponds to the hydrodynamic regime since $\nabla^2\Phi$ evolution reduces to the two-dimensional Navier-Stokes equation, and the density n is advected as a passive scalar. In general, the turbulence arising from the HW equations is self-consistently driven by the instability and including the dissipation $D_{\Phi,n}$ is necessary in order to obtain a steady state spectrum. In the limit of $C \gg 1$ where the electron response becomes adiabatic, the equations gradually approach the CHM equation with $n \approx \Phi$, and the linear instability disappears. In this limit one has to include forcing as well as dissipation in order to obtain a steady state spectrum.

For $C \ll 1$ the the spectral kinetic energy spectrum $E(k)$ is the well known Kraichnan-Kolmogorov spectra [11,12], i.e., $E(k) \sim k^{-5/3}$ and $E(k) \sim k^{-3}$ for the inverse energy and forward enstrophy cascades, respectively. Note that we use k instead of $k_\perp$ consistently in these expressions and in the remainder of this paper, since $k_\parallel$ is simply absorbed into C and the considered system of equations is two dimensional. The spectral density of internal energy $F(k)$ evolves as a passive scalar with the power law $F(k) \sim k^{-5/3}$ corresponding to the Obukhov-Corrsin spectrum [13,14] and $F(k) \sim k^{-1}$ corresponding to the Batchelor spectrum [15]. For $C \gg 1$ the energy injection moves to smaller scales and the kinetic energy spectrum becomes that of HM $E(k) \sim k^{-3}$ or $F(k) \sim k^{-5}$ for the forward enstrophy cascade roughly everywhere [2]. However, using dimensional analysis for the power balance between the linear and nonlinear terms, we find that for $C \gg 1$ there is a shift in the balance for a scale $k \sim k_c$, where $k_c \sim C/\kappa$. This results in a new spectrum of density fluctuations for the HW in the $C \gg 1$ limit. The spectrum is associated with the background gradient and takes the form $F(k) \propto k^{-3}$ for $k > k_c$. Table I presents a summary of these results in addition to the particle flux spectra $\Gamma(k)$ that will be discussed further.

TABLE I. Summary of the various power law spectra for the Hasegawa-Wakatani system. Here, k_i indicates energy injection and k_c indicates the critical scale where the nonlinear turnover rate balances parallel electron conduction.

	Kinetic energy		
	$k < k_i$ $E(k) \propto k^{-5/3}$	$k > k_i$ $E(k) \propto k^{-3}$	
	Internal energy		
$C \ll 1$	$k < k_i$ $F(k) \propto k^{-5/3}$	$k > k_i$ $F(k) \propto k^{-1}$	
$C \gg 1$	$k < k_i$ $F(k) \propto k^{-11/3}$	$k > k_i$ $F(k) \propto k^{-5}$	$k > k_c$ $F(k) \propto \kappa^2 k^{-3}$
	Particle flux		
$C \ll 1$	$k < k_i$ $\Gamma(k) \propto \kappa k^{-7/3}$		$k > k_i$ $\Gamma(k) \propto \kappa k^{-3}$
$C \gg 1$	$k < k_i$ $\Gamma(k) \propto \frac{\kappa k^{1/3}}{(1+k^2)}$		$k > k_i$ $\Gamma(k) \propto \frac{\kappa k^{-1}}{(1+k^2)}$

II. DERIVATION OF SPECTRA AND TRANSITION SCALE

To derive the power law spectra in Table I and the critical scale k_c, we start with the energy budget equation for the density fluctuations

$$\partial_t F(k) + \frac{\partial}{\partial k}\Pi(k) + 2\kappa\Gamma(k) - 2C\left[\frac{H(k)}{k} - F(k)\right] = \mathcal{E}(k). \tag{2}$$

Equation (2) results from multiplying the Fourier transform of Eq. (1b) by n, integrating over spectral shells, and representing it in terms of $E(k) = \int |\Phi_{\mathbf{k}}|^2 k^3 d\alpha_k$, the spectral kinetic energy density; $F(k) = \int |n_{\mathbf{k}}|^2 k d\alpha_k$, the spectral internal energy density; $H(k) = \int \mathrm{Re}[\Phi_{\mathbf{k}}^* n_{\mathbf{k}}] k^2 d\alpha_k$, the spectral density of the real part of the conserved cross-helicity term, $\Gamma(k) = \int \sin\alpha_k \mathrm{Im}[\Phi_{\mathbf{k}}^* n_{\mathbf{k}}] k^2 d\alpha_k$, the spectral particle flux density; and $\mathcal{E}(k)$, the rate of dissipation. The nonlinear term is modeled as a flux in k space, $\frac{\partial}{\partial k}\Pi(k)$, where $\Pi(k) \propto E(k)^{1/2} F(k) k^{5/2}$ dimensionally. For $C \ll 1$, the fourth term, proportional to C is negligible, we get the usual $F(k) \sim k^{-5/3}, k^{-1}$ which are the Obukhov-Corrsin and Batchelor spectra for the passive scalar, respectively. On the other hand, for $C \gg 1$ the expected spectrum for the density is $n_k^2 \approx \Phi_k^2$, which gives $F(k) \propto k^{-11/3}$ and $F(k) \propto k^{-5}$. Recall that $E_K(k)$ and $E_I(k)$, which are usually defined for the Hasegawa-Mima system as the kinetic and internal energies, are the limiting cases of $E(k)$ and $F(k)$, respectively. However, the existence of κ modifies the balance, and the fourth term in Eq. (2) becomes negligible relative to the second and third terms for $k > k_c$. The steady state solution for this case is

$$F(k) \propto \kappa^2 k^{-3},$$

which is the associated "anisotropic" spectrum in the presence of a background gradient as first discussed (to our knowledge) in Ref. [16] and later in Ref. [17]. Note that this is not a "constant flux" solution for the fluctuations in the classical Kolmogorov sense and is rather linked to the mixing length spectrum [18,19]. Here we argue that the transition from the $F(k) \propto k^{-5}$ to $F(k) \propto \kappa^2 k^{-3}$ take place at the critical scale $k_c \sim C/\kappa$.

Notice that in general, Eq. (2) should be supplemented by an equation for the kinetic energy and the real and imaginary parts of the cross correlation. Even considering a completely anisotropic spectrum [i.e., $|\Phi_{\mathbf{k}}|^2 \equiv \Psi(k,\alpha_k)$] for any of these quantities (including particle flux), one can define its angle average [such as $E(k)$ for spectral energy] and consider its evolution since the local nonlinear transfer can still be modeled as a flux in the k space (denoting scale rather than a directional wave number). Using the Fourier transforms of Eqs. (1a) and (1b), a *simplified* equation for the anisotropic correlation $\langle \Phi_{\mathbf{k}} n_{\mathbf{k}}^* \rangle$ can be obtained:

$$\tau_k^{-1}\langle \Phi_{\mathbf{k}}^* n_{\mathbf{k}} \rangle - \frac{C}{k^2}(\langle n_{\mathbf{k}} \rangle^2 + k^2 \langle \Phi_{\mathbf{k}} \rangle^2) + i\kappa k_y \langle \Phi_{\mathbf{k}} \rangle^2 \approx 0 \tag{3}$$

whose real and imaginary parts can be used to compute $H(k)$ and $\Gamma(k)$ in Eq. (2). Notice that τ_k can be linear [i.e., $\tau_k^{-1} \sim C(1+k^2)/k^2$], or nonlinear (i.e., $\tau_k \sim \tau_k^{n\ell}$) depending on the value of C. While for an intermediate C, one would have to solve the full linear propagator [20], the limiting cases are dominated by these simpler τ_k. In any case the solution may be written as

$$\Gamma(k) \approx -\tau_k \int \sin^2\alpha_k (\kappa \langle \Phi_{\mathbf{k}} \rangle^2 k^2) k d\alpha_k \tag{4}$$

which gives

$$\frac{\partial}{\partial k}\Pi(k) \approx \kappa^2 E(k)^{1/2} k^{-3/2}$$

for $C \ll 1$ [where we used $\tau_k \sim \tau_k^{n\ell} \sim [u(k)k]^{-1} \sim E(k)^{-1/2} k^{-3/2}$]. For $\kappa \ll 1$, we get the usual Obukhov-Corrsin and Batchelor spectra, while for substantial κ, we get the case discussed in Ref. [17] for the passive scalar with background gradient. Notice that the particle flux spectrum implied by $\Gamma(k) \approx -\kappa E(k)^{1/2} k^{-3/2}$ for the Obukhov-Corrsin case is consistent with the three-dimensional results in Refs. [21] and [22] in contrast to the results in Ref. [23].

In the opposite limit $C \gg 1$, we can first compute $H(k)$ by taking the real part of Eq. (3), $H(k) = \frac{k(E(k)+F(k))}{(1+k^2)}$, and using $\tau_k^{-1} \sim C(1+k^2)/k^2$ in Eq. (4) we get

$$\frac{\partial}{\partial k}\Pi(k) = 2C\left\{\frac{\kappa^2}{C^2}\frac{k^2}{(1+k^2)}E(k) + \left[\frac{E(k) - F(k)k^2}{(1+k^2)}\right]\right\}.$$

Now considering $E(k) \propto k^{-3}$, and $F(k) \propto k^{-5}$ at the low k part of the spectrum [i.e., the scalar spectrum $F(k)$ being determined by a balance between $E(k)$ and $F(k)k^2$], as k slowly increases, one will reach a point where $(\kappa^2/C^2)E(k) \propto k_c^2 k^{-3}$ take over the other two terms which go as k^{-5}. This point is defined as the critical scale k_c, discussed in our paper. Noting that at this point the nonlinear flux also becomes nonzero [i.e., $\Phi_k \neq n_k \Rightarrow \frac{\partial}{\partial k}\Pi(k) \neq 0$], it is in fact a scale at which a linear term is balanced by a nonlinear term. In other words, beyond this scale one can drop the $E(k)$ as opposed to $E(k)k^2/k_c^2$ further assuming that $k_c \gg 1$ (so that $k^2 \gg 1$ as well) we get

$$C\frac{\partial}{\partial k}\Pi(k) \approx 2\left\{\frac{\kappa^2}{C^2}E(k) - F(k)\right\}.$$

Using $\Pi(k) \propto E(k)^{1/2}F(k)k^{5/2}$ and $E(k) \propto \beta^{2/3}k^{-3}$, where β is the enstrophy dissipation, we get $F(k) \propto \kappa^2 k^{-3}$ as the solution.

The two solutions can be written approximately in the form $F(k) \propto \beta^{2/3}k^{-3}(k_c^{-2} + k^{-2})$, where the limiting forms are captured nicely. Obviously, however, the form of the transition cannot be described accurately by this expression.

III. VERIFICATION USING A SHELL MODEL

In order to verify the prediction for the power spectra, capture the details of the transition, and demonstrate the parametric dependency that we derive for k_c, a shell model is introduced for the HW equations [24]. Note that direct numerical simulations of the Hasegawa-Wakatani system become expensive relatively rapidly with increasing value of C [25,26], and it becomes very difficult to have sufficient resolution in order to resolve a dual power law spectrum, let alone to see the scaling of the transition point of such a spectrum. Note also that the transition may appear at *unphysical scales*, much smaller than the ion Larmor radius, for which the use of a Hasegawa-Wakatani equation may not be justified. However, since we consider a pure Hasegawa-Wakatani system in slab geometry (instead of a modified one in a sheared torus), and consider homogeneous isotropic turbulence (apart from the background inhomogeneities which appear as constant coefficients), we do not claim the model is physically relevant for tokamaks. Our aim here is to study the *nonlinear* implications of this model *as is*.

In order to *derive* a shell model, the HW equations for Φ and n in Fourier space are rewritten in terms of the shell variable $\Phi_n = [\frac{1}{k_n^2}\int_{k_n}^{k_{n+1}} E(k)dk]^{1/2}$ and $n_n = [\int_{k_n}^{k_{n+1}} F(k)dk]^{1/2}$ where the spacing between the discrete shells follows $k_n = k_0 g^n$ where $g > 1$ and k_0 is the wavelength of the largest scale. Following the Gledzer-Okhitani-Yamada [27] prescription for a finite representation of the nonlinear term, the nth shell is allowed to interact nonlinearly only with the neighboring shells $n+1, n+2, n-1$, and $n-2$. Using the conservation of kinetic energy $E = \sum|\Phi_n|^2 k_n^2$, internal energy $N = \sum n_n^2$, enstrophy $W = \sum|\Phi_n|^2 k_n^4$, and effective helicity $H = \sum k_n^2 \Phi_n n_n$ [28], the HW Eqs. (1) are written as a set of ordinary differential equations in the shell variables:

$$\begin{aligned}\frac{d\Phi_n}{dt} &= \alpha k_n^2(g^2-1)\left[\frac{1}{g^7}\Phi_{n-1}^*\Phi_{n-2}^* \right.\\ &\quad \left. - \frac{g^2+1}{g^3}\Phi_{n-1}^*\Phi_{n+1}^* + g^3\Phi_{n+1}^*\Phi_{n+2}^*\right]\\ &\quad + \frac{C(\Phi_n - n_n)}{k_n^2} - \left(\nu_\Phi k_n^{-6} + \nu'_\Phi k_n^4\right)\Phi_n, \qquad (5a)\end{aligned}$$

$$\begin{aligned}\frac{dn_n}{dt} &= \alpha k_n^2\left[\frac{1}{g^3}(\Phi_{n-2}^* n_{n-1}^* - \Phi_{n-1}^* n_{n-2}^*) \right.\\ &\quad - \frac{1}{g}(\Phi_{n-1}^* n_{n+1}^* - \Phi_{n+1}^* n_{n-1}^*)\\ &\quad \left. + g(\Phi_{n+1}^* n_{n+2}^* - \Phi_{n+2}^* n_{n+1}^*)\right]\\ &\quad + C(\Phi_n - n_n) + i\kappa k_n\Phi_n - \left(\nu k_n^{-6} + \nu' k_n^4\right)n_n. \quad (5b)\end{aligned}$$

Choosing $N = 60$ shells, we look for the steady state solution of HW Eq. (5) for different choices of the parameters C and κ. The power spectra for the shell variables Φ_n, n_n in this steady state are related to spectral energy densities as $E(k_n) \sim \langle\Phi_n^2\rangle k_n$ and $F(k_n) \sim \langle n_n^2\rangle k_n^{-1}$, where $\langle n_n^2\rangle$ and $\langle\Phi_n^2\rangle$ are averaged over 100 steady state solutions for Eq. (5). Similarly $H(k_n) \sim \mathrm{Re}\langle\Phi_n^* n_n\rangle$ and $\Gamma(k_n) \sim \mathrm{Im}\langle\Phi_n^* n_n\rangle$. We drop the n subscript from here on keeping in mind that the results presented are those from the shell model.

There is no need to force the system in the case of $C \ll 1$ since the HW equations have a self-consistent drive for low C. For $C \gg 1$, the self-consistent instability is very weak, therefore we apply a small forcing on two shells at the scale of the instability to achieve the steady state spectrum at a reasonable computational time. We choose a magnitude for the forcing that is small enough such that the power law of the spectrum is independent of it. In both limits of C, dissipation factors $\nu = 10^{-18}$ and $\nu' = 10^{-24}$ are used to achieve a steady state turbulence cascade.

Figure 1 is a log-log plot of $E(k)$, $F(k)$, and $\Gamma(k)$ in both limits $C \ll 1$ ($C = 0.001$) to the left and $C \gg 1$ ($C = 4$) for $\kappa = 0.1$ to the right. Note that for $C \ll 1$ the shell model does not capture exactly the Kraichnan-Kolmogorov dual cascade for the potential, $E(k) \sim k^{-5/3}$, k^{-3}. This is due to the equipartition theorem [29] where the steady state solution to the shell model is found at a local minimum of k^{-1} for the inverse cascade. The Obukhov-Corrsin and Batchelor passive scalar spectra for the density, $F(k) \sim k^{-5/3}$, k^{-1} are therefore modified according to the modification in $E(k)$. Following the dimensional argument for $F(k)$ given $E(k) \sim k^{-1}$ results in $F(k) \sim k^{-2}$ instead. Furthermore, since the shell model does not capture the phase information, the resulting particle flux spectrum is simply $\Gamma(k) \sim \sqrt{E(k)F(k)}$. Indeed the resulting power spectra for the particle flux in the shell model simulation are $\Gamma(k) \sim k^{-3/2}$, k^{-2}. The shell model results for $C \gg 1$, however, agree with the analytic predicted. We get the adiabatic spectrum for $E(k) \sim k^{-3}$ and demonstrate the existence of k_c where $F(k) \sim k^{-5}$ for $k < k_c$ and $F(k) \sim k^{-3}$ for $k > k_C$. For the adiabatic case, $\Gamma(k) \sim k^{-5/3}$, k^{-3} also as theoretically predicted. Since we are interested in investigating the new transition scale for the internal energy and the predicted scaling in the case of $C \gg 1$, we use the shell model which is shown to reliably capture the internal energy cascade and depict k_c. We numerically compute the value of k_c as a function of C and κ to confirm the relation we derived for the scale $k_c \propto C/\kappa$ at which the bifurcation occurs.

One interesting observation is that the internal energy spectrum in the high-k limit, $k > k_c$, gradually transitions from the passive scalar $F(k) \sim k^{-1}$ in the hydrodynamic regime to that associated with a background gradient $F(k) \sim k^{-3}$ in the adiabatic regime as C is varied. Since our analytical derivations were constrained to the asymptotic limits of C, we use the shell model to study the transition of the spectra of $F(k)$ from k^{-1} for $C \ll 1$ to k^{-3} for $C \gg 1$ for $k > k_c$. We plot in Fig. 2 the spectra and the power χ of the internal energy spectrum $F(k) \sim k^\chi$ where $k > k_c$, $\kappa = 10$, and values of C vary from 1 to 60 at increments of five.

We find that the transition that the internal energy power spectrum $F(k)$ undergoes, has a universal form as a function of C which is similar for all κ but spans increasingly larger

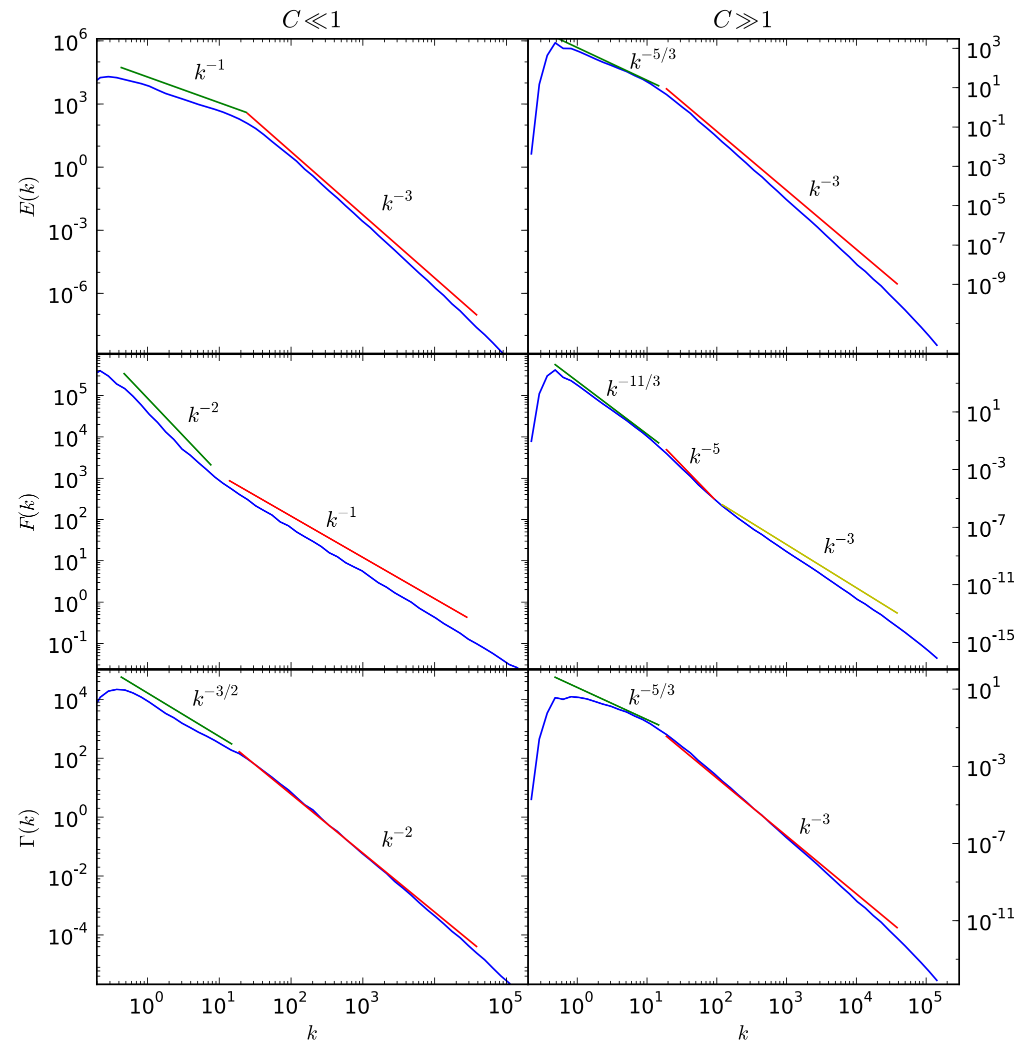

FIG. 1. (Color online) Log-log plot of the power spectra of $E(k)$ (top), $F(k)$ (middle), and $\Gamma(k)$ (bottom) where $\kappa = 1.0$ and $C \ll 1$ ($C = 0.01$) on the left and $C \gg 1$ ($C = 100$) on the right, using the shell model.

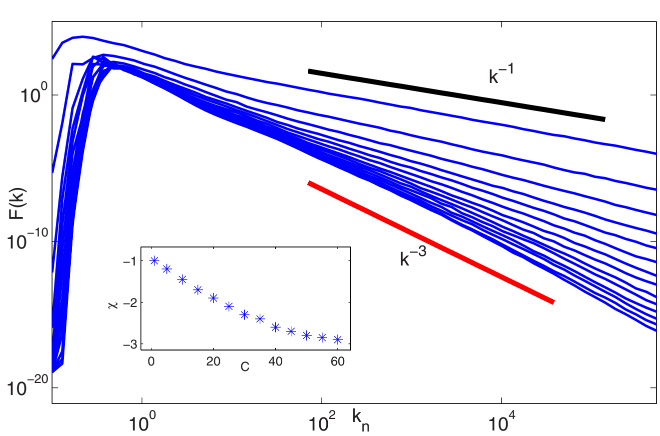

FIG. 2. (Color online) Log-log plot of the high-k spectra of the internal energy $F(k) \sim k^{\chi}$ for $\kappa = 10$ and different values of $0.1 < C < 60$. The value of χ is plotted in the smaller figure on the bottom right. The transition is steeper closer to the limit $C \ll 1$ and becomes more gradual for larger C.

regions in C for larger κ. We denote the value of C for a given κ for which the transition for $k > k_c$ occurs from the hydrodynamic regime, $F(k) \sim k^{-1}$, to the intermediate regime as $C_{t1}(\kappa)$ and from the intermediate regime to the adiabatic regime, $F(k) \sim k^{-3}$, as $C_{t2}(\kappa)$. We find that $C_{t1,t2}(\kappa)$ are straight lines in (C,κ) parameter space with slopes approximately 1.6 and 0.12. The behavior of the transition of χ where $F(k) \sim k^{\chi}$ for different κ is a universal function of C between $C_{t1}(\kappa)$ and $C_{t2}(\kappa)$ with $\chi \approx -1$ at $C_{t1}(\kappa)$ and $\chi \approx -3$ at $C_{t2}(\kappa)$ for a given κ. We plot in Fig. 3 the value of χ, such that $F(k) \sim k^{\chi}$, in the parameter space (C,κ).

These results are validated qualitatively against an open source plasma fluid solver called BOUT++ [30] that has the Hasegawa-Wakatani model as an example. BOUT++ captures the transition to k^{-3} in $F(k)$ and the results are found to agree with the isotropic shell model (except that the details had to be modified to capture the differences in the way the system was driven).

We have also developed an anisotropic version of the shell model which separates three directions x, y, and $x = y$ directions as introduced in Ref. [31]. The results also agree

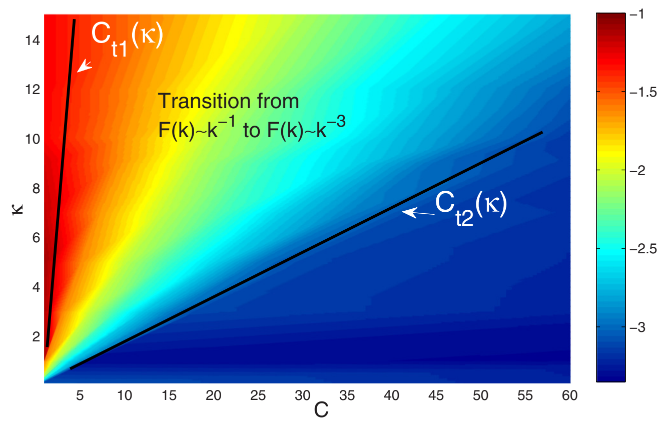

FIG. 3. (Color online) Scan of the power χ in parameter space (κ,C) of the internal energy spectrum $F(k) \sim k^{\chi}$. For a given κ the value of the power χ in the power spectrum transitions from a value $\chi \approx -1$ at C_{1t} to $\chi \approx -3$ at C_{t2} in the universal form as presented in Fig. 2 for $\kappa = 10$. The lines $C_{t1}(\kappa)$ and $C_{t2}(\kappa)$ separate the intermediate regime in the parameter space from the hydrodynamic (red) and adiabatic regime (blue), respectively.

rather well with the isotropic model. The transition takes place for the nonzonal directions only; however, taking the integral over angles in phase space, since the scale $k^{-3} \gg k^{-5}$ the resulting spectrum agrees with the isotropic shell model. It could be argued that $k_c(\alpha_k) \propto C/\kappa\ \sin\alpha_k$ which is consistent with an anisotropic k_c computed from Elperin's results in Ref. [17]. This would imply that the k_c becomes infinite for $\alpha_k = 0$, explaining therefore that the transition does not take place for the zonal direction. However, this is difficult to verify numerically.

IV. CONCLUSION

Based on dimensional analysis of the Hasegawa-Wakatani equations, a dual power law spectrum for the density fluctuations of the form $F(k_\perp) \propto \beta^{2/3}k_\perp^{-3}(k_c^{-2} + k_\perp^{-2})$ has been predicted with $k_c \propto C/\kappa$. A shell model has been developed for the HW equations and validated by reproducing the expected spectra of $E(k)$ and $F(k)$ in the hydrodynamic regime $C \ll 1$ and the adiabatic regime $C \gg 1$ (see Table I for a summary). This model was then used to verify the parametric dependence of the predicted critical scale $k_c \propto C/\kappa$ numerically. The transition of the density spectrum from $F(k) \sim k^{-1}$ for $C \ll 1$ to $F(k) \sim k^{-3}$ for $C \gg 1$ for high k is found in the simulations to be of a universal form scalable with κ. Studying the intermediate regime numerically reveals a universality in the transition from the hydrodynamic limit ($C \ll 1$) to the adiabatic limit ($C \gg 1$). Further study should be made to incorporate zonal flows that arise from turbulence in an anisotropic shell model [31].

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