Analytical solitonlike solutions and the dynamics of ultracold Fermi gases in a time-dependent three-dimensional harmonic potential

Ying Wang, Guosen Luo, and Yu Zhou*
School of Mathematics and Physics, Jiangsu University of Science and Technology, Jiangsu 212003, China

Chao Hang
State Key Laboratory of Precision Spectroscopy and Department of Physics, East China Normal University, Shanghai 200062, China and NYU-ECNU Joint Institute of Physics at NYU-Shanghai, Shanghai 200062, China

(Received 11 May 2015; published 16 September 2015)

We present a theoretical study of solitonlike solutions and their dynamics of ultracold superfluid Fermi gases trapped in a time-dependent three-dimensional (3D) harmonic potential with gain or loss. The 3D analytical solitonlike solutions are obtained without introducing any additional integrability constraints used elsewhere. The propagation of both bright- and dark-soliton-like solutions is investigated. We show that the amplitudes of dark-soliton-like solutions exhibit periodic oscillation, whereas those of the bright-soliton-like ones do not show such behavior. Moreover, we highlight that the oscillation periods of dark-soliton-like solutions predicted by our approach are matched very well with those observed in a recent experiment carried out by Yefsah *et al.* [T. Yefsah, A. T. Sommer, M. J. H. Ku, L. W. Cheuk, W. Ji, W. S. Bakr, and M. W. Zwierlein, Nature (London) **499**, 426 (2013)] in both Bose-Einstein condensation and unitarity regimes.

DOI: 10.1103/PhysRevE.92.032910 PACS number(s): 05.45.−a, 03.75.Lm, 47.37.+q

I. INTRODUCTION

Self-stabilization against dispersion via a nonlinear interaction can induce a class of fascinating shape-preserving wave phenomena, termed solitons, in nonlinear media. They are characterized as localized solitary wave packets that sustain the ability of surviving mutual collisions without changing shapes and velocities due to the mathematical integrability of the underlying equation of motion [1]. Solitons have nowadays been discovered in a wide range of physical research fields from particles to astrophysics, as well as in acoustics and optics [2]. They might be either bright or dark, relying on the details of effective nonlinearity and boundary conditions, where the bright soliton is a peak in the intensity while the dark one is a notch with a typical phase step across it [3]. Although there were lots of investigations on solitons in the classical frame, the creation of nonspreading matter-wave packets in Bose-Einstein condensates (BECs) and degenerate Fermi gases in laboratories witnesses the renaissance of soliton studies in the quantum scenario [4–9]. In these quantum coherent systems, solitons display the nonlinear behavior of collective modes which reveals the rich physics of ultracold atoms and displays well-controlled characteristics [10]. In the last few years, both bright and dark solitons were observed in BECs with attractive and repulsive interactions, respectively, in a few pioneering experiments [11]. Their dynamics such as reflections [12], collisions [13], and even solitonic vortices [14,15] in ultracold atoms have also been recently reported.

On the other hand, the successful experimental realization of the superfluid from the Bardeen-Cooper-Schrieffer (BCS) state to the BEC in ultracold, trapped Fermi gases initiated considerable efforts in the study of BCS-BEC crossover [16]. In contrast to the intensive work on linear collective excitations, studies on the nonlinear behavior of collective modes in superfluid Fermi gases are still rare. From the mathematical viewpoint, this is because the Bogoliubov–de Gennes (BdG) equation, which describes the motion of superfluid Fermi gases, is much more difficult to solve analytically than the Gross-Pitaevskii equation (GPE), which describes the motion of BECs. It is known that many properties from the BCS state to the BEC limit across unitarity evolve smoothly [17]; recently, an order-parameter (macroscopic wave function) equation controlling the dynamics of trapped superfluid Fermi gases at zero temperature was suggested, which has the form of the nonlinear Schrödinger (NLS) equation with polytropic nonlinearity [18–20]. This equation, which is called the generalized Gross-Pitaevskii equation (GGPE) [21–29], can be used to study the collective behaviors, such as solitonlike dynamics, in the BEC and the unitarity regime for the positive atom-atom scattering length.

We also note that dimensional reduction of the system was usually used in previous works to avoid complex and unstable solutions. It is known that for BECs with one-dimensional setup, an exact soliton solution can be reached [30,31]. Such an exact soliton solution comes from the integrability of the model, and its amplitude and velocity are directly related to conserved constants of the motion. However, in real experiments, the ideal homogeneous one-dimensional (1D) setup is impossible. A quasi-1D layout using tight harmonic trapping confinement in radial directions is a useful solution, which intrinsically breaks the integrability and prevents exact soliton solutions. Nevertheless, in such case a solitonlike solution can be reached if the border of the system does not affect the wave propagation over relevant experimental timescales, which retains many similar properties with the exact soliton solution [32–37].

In this work, we present the study of solitonlike solutions and their dynamics in a superfluid Fermi gas trapped in a time-dependent three-dimensional (3D) harmonic potential with gain or loss. We note that analytical 3D solitons in a trapped BEC have been examined by a few studies [38,39],

*yzhou@just.edu.cn

recently. However, no such studies have been carried out in the context of superfluid Fermi gases as far as we know. To obtain the analytical solitonlike solutions, we solve the GGPE based on the self-similar approach combined with scale and phase transformations, ruling out any integrable constraints used elsewhere (e.g., in Refs. [40–45]). We study the propagation of both bright- and dark-soliton-like solutions, as well as their dynamics, and find that the amplitude of dark-soliton-like solutions exhibits periodic oscillation, whereas the amplitude of bright-soliton-like solutions shows no periodic behavior. In addition, we highlight that the oscillation periods of dark-soliton-like solutions predicted by our theory are matched very well with those observed in a recent experiment [46], in both BEC and unitarity limits.

This paper is arranged as follows. The next section gives the self-similar approach for obtaining the analytical 3D solitonlike solutions of the GGPE. In Sec. III, we discuss the propagation of dark- and bright-soliton-like solutions. Their dynamics are also investigated in detail. We find that our results are matched very well with the experimental observations. In the last section a summary and discussion of main results obtained in this work are given.

II. SELF-SIMILAR SOLUTIONS OF THE GENERALIZED GROSS-PITAEVSKII EQUATION

We consider a superfluid Fermi gas, in which fermionic atoms (i.e., ^{6}Li or ^{40}K) have two different internal states with equal atomic number. In the ground state all atoms are paired with the condensed atom pair density, denoted as n_s. By means of the Feshbach resonance technique the BCS-BEC crossover can be realized easily by tuning an applied magnetic field [16,17] and, hence, changing the s-wave scattering length a_s. Usually, a dimensionless interaction parameter $\eta \equiv 1/k_F a_s$ (where k_F is the Fermi wave number) is introduced to distinguish different superfluid regimes, i.e., the BCS regime ($\eta \leqslant -1$), the BEC regime ($\eta \geqslant 1$), and the BEC-BCS crossover regime ($-1 < \eta < 1$). The special case $\eta = -\infty$ ($\eta = +\infty$) is called the BCS (BEC) limit, while $\eta = 0$ is called the unitarity limit. The smooth transition from BCS to BEC means that the dynamical behavior of superfluid Fermi gases in various superfluid regimes can be investigated in a unified way [16].

Recently, an order-parameter (macroscopic wave function) equation governing the dynamics of trapped superfluid Fermi gases near the BEC and unitarity regime at zero temperature was suggested, which is referred to as the GGPE [21–29]. The dimensionless form of this equation is

$$i\psi_t + \rho_0(t)\Delta\psi + g_0(t)|\psi|^{2\gamma}\psi + \sum_j k_{0j}(t)x_j^2\psi = i\Gamma_0(t)\psi, \tag{1}$$

where $j = x,y,z$ and Δ is the Laplacian operator in the Cartesian coordinates. The time-dependent functions $\rho_0(t)$, $g_0(t)$, $k_{0j}(t)$, and $\Gamma_0(t)$ characterize the quantum pressure (which provides the necessary dispersion for the formation of a soliton), nonlinearity (induced from interatomic interactions), trapping potential (usually harmonic), and gain and loss, respectively. In obtaining Eq. (1) we have utilized the scale transformation $\boldsymbol{r}\sqrt{2m/\hbar} \to \boldsymbol{r}$ with m being the atomic mass. The polytropic power index γ is a function with respect to the interatom interaction (s-wave scattering length) [21–29], varying continuously from $\gamma = 1$ for the BEC limit (corresponding to $\eta = +\infty$) to $\gamma = 2/3$ for the unitarity limits (corresponding to $\eta = 0$, respectively) [47–50].

In general, the procedure for obtaining analytical solutions of Eq. (1) needs extra integrable constraints; however, in the following, we show a path for obtaining exact solutions without using any integrable constraints between coefficients of the equation. First, we scale the spatial and temporal coordinates as $\boldsymbol{r}' = \sigma(t')\boldsymbol{r}$ and $t' = \int \rho_0(t)\sigma^2(t)dt$, with $\sigma(t)$ being a time-dependent function to be determined in the ensuing steps. The original wave function is transformed as

$$\psi(\boldsymbol{r},t) = \sigma^{3/2}(t')\exp\left[\frac{\sigma_{t'}(t')}{\sigma^3(t')}\sum_j x_j'^2\right]\varphi(\boldsymbol{r}',t'). \tag{2}$$

Changing notation from $(\boldsymbol{r}',t')$ to $(\boldsymbol{r},t)$ and using Eq. (2), the transformed GGPE is of the form

$$i\varphi_t + \Delta\varphi + g(t)|\varphi|^{2\gamma}\varphi + \sum_j k_j(t)x_j^2\varphi = i\Gamma(t)\varphi, \tag{3}$$

where $g(t) = g_0(t)\sigma^{3\gamma-2}(t)/\rho_0(t)$, $\Gamma(t) = \Gamma_0(t)/[\rho_0(t)\sigma^2]$, and $k_i(t) = \{4k_{0j}(t) + \sigma_t^2(t)/\sigma^2(t) - [\sigma_t(t)/\sigma(t)]_t\}/[4\rho_0(t)\sigma^4(t)]$.

The self-similar projective equation of the 3D GGPE (3) is

$$iu_\tau + \varepsilon u_{\varsigma\varsigma} + \delta|u|^{2\gamma}u = 0, \tag{4}$$

which admits bright- and dark-soliton-like solutions of the form [20,21]

$$u_+(\varsigma,\tau) = C_+\text{sech}^{1/\gamma}B_+(\varsigma + c\tau)\exp[i(d_1\varsigma + d_2\tau)],\quad \delta > 0, \tag{5a}$$

$$u_-(\varsigma,\tau) = C_-\tanh^{1/\gamma}B_-(\varsigma + c\tau)\exp[i(b_1\varsigma + b_2\tau)],\quad \delta < 0, \tag{5b}$$

for attractive ($\delta > 0$) and repulsive ($\delta < 0$) interactions, respectively. Here $C_\pm$, $B_\pm$, c, $b_{1,2}$, and $d_{1,2}$ are free parameters determining the amplitude, width, central localization, group velocity, and phase of the solitons, respectively. Now we introduce the similarity ansatz for φ as

$$\varphi(\boldsymbol{r},t) = A(t)u[\varsigma(\boldsymbol{r},t),\tau(t)]\exp[ia(\boldsymbol{r},t)], \tag{6}$$

where $u(\varsigma,\tau)$ is the solution of Eq. (4); $A(t)$, $\varsigma(\boldsymbol{r},t)$, $\tau(t)$, and $a(\boldsymbol{r},t)$ are undetermined functions which are differentiable with respect to the spatial and temporal coordinates. Substituting the solution (6) into GGPE (3) and putting the resultant equation into the same form as Eq. (4), we obtain the following relations:

$$2gA^{2\gamma} - \delta\tau_t = 0, \tag{7a}$$

$$\sum_j \varsigma_{jj} = 0, \tag{7b}$$

$$\sum_j \varsigma_j^2 - \varepsilon\tau_t = 0, \tag{7c}$$

$$\varsigma_t + 2\sum_j \varsigma_j a_j = 0, \tag{7d}$$

$$2A_t - 2\Gamma A + 2A\sum_j a_{jj} = 0, \tag{7e}$$

$$a_t + a_x^2 + a_y^2 + a_z^2 - \sum_j k_j x_j^2 = 0, \tag{7f}$$

which result in

$$A(t) = \sqrt{\frac{3\delta}{\varepsilon g}}G, \tag{8a}$$

$$\varsigma(\boldsymbol{r},t) = -6C_1\int G^2 dt + G\sum_j x_j + C_2, \tag{8b}$$

$$\tau(t) = \frac{3}{\varepsilon}\int G^2 dt + C_3, \tag{8c}$$

$$a(\boldsymbol{r},t) = \frac{g_t + 2\Gamma g}{4g}\sum_j x_j^2 + C_1 G\sum_j x_j - 3C_1^2 \times \int G^2 dt + C_4, \tag{8d}$$

with $j = x,y,z$, $G = \exp[-\int(g_t/g + 2\Gamma)dt]$, and $C_{1,2,3,4}$ being the integral constants. Then, the general consistence condition reads

$$4\Gamma^2 + 4\frac{g_t}{g}\Gamma + 2\Gamma_t + \frac{g_{tt}}{g} - \frac{4}{3}\sum_j k_j(t) = 0. \tag{9}$$

Since the functions $\Gamma(t)$, $g(t)$, and $k_j(t)$ are all dependent on $\sigma(t)$, the condition (9) actually imposes a requirement of $\sigma(t)$, while the functions $\rho_0(t)$, $g_0(t)$, $k_{0i}(t)$, and $\Gamma_0(t)$ are allowed to change freely. The ansatz (6) together with the condition (9) present the exact 3D solitonlike solutions of GGPE (3).

The ansatz (6) leads to the final solutions of the 3D GGPE in the form of bright or dark solitons indicated by Eqs. (5). However, the obtained set of analytical solutions is not the only possible solitonlike solution of the 3D GGPE. In principle, one can write the ansatz in the following general form:

$$\varphi(\boldsymbol{r},t) = A(t)H(u[\varsigma(\boldsymbol{r},t),\tau(t)])\exp[ia(\boldsymbol{r},t)]. \tag{10}$$

Here $H(u)$ is a function of $u(\varsigma,\tau)$, which can be solved by a certain 1D nonlinear partial differential equation. In this work, we are interested in the solutions which have the simplest form [i.e., the ansatz (6)] and possess the most evident physical meaning as shown below.

III. DARK AND BRIGHT SOLITONS IN DEGENERATE FERMI GASES

A. Dark-soliton-like solutions

Now we consider a practical case in which k_{0j} ($j = x,y,z$) are all time independent and satisfy the condition $k_{0x} = k_{0y} = k_\perp > k_{0z}$ with $\rho_0(t) = 1$. In this case the ansatz (6) allows a dark-soliton-like solution indicated by solution (5b), reading

$$|\psi(\boldsymbol{r},t)| = D_{3\mathrm{D}}\sigma(t)^{3/2}A(t)\tanh^{1/\gamma} \times \left[\sigma(t)p(t)\sum_j k_{cj}x_j + q(t)\right], \tag{11}$$

where $D_{3\mathrm{D}}$ is a constant, $p(t) = \exp(-2\Gamma_0 t/\rho_0)$, and $q(t) = \int \exp(-2\Gamma_0/\rho_0 t)\sigma^2(t)dt$ according to Eqs. (8). We are interested in this case which can be compared with the one observed in the experiment carried out by the MIT group [46]. To this end, we take $a(\boldsymbol{r},t) = s(t)\sum_j x_j^2 + s_1(t)\sum_j x_j + s_2(t)$ according to Eqs. (8), where $s_1(t)$ and $s_2(t)$ are determined from the initial condition and can be set to zero in the laboratory reference frame. In addition, we have $p(t) \propto A^{2/3}(t)$ and $g \propto \sigma^{3\gamma-2}(t)$ with experimental proportional constants. Setting $p(t) \propto \sigma^\nu(t)$, with ν being a constant determined by the normalization at a certain time (e.g., $t = 0$), we reach the expression $\delta = \sigma^{\beta(\gamma)}(t)$.

From Eqs. (7c) and (7d), we get the relation $\sigma^{1+\alpha(\gamma)}(t) = A^{2/3}$. Using this relation, we obtain

$$s(t) = -\frac{1-\alpha(\gamma)}{4}\left(\frac{\sigma_{t'}(t)}{\sigma(t)}\right)_{t'}. \tag{12}$$

Noticing that $dt'/dt = \sigma^2(t)$ and substituting Eq. (12) into Eq. (7f), we further obtain the equation for $\sigma(t)$ as

$$k_{cj}\left\{[\alpha^2(\gamma) - \alpha(\gamma)]\left(\frac{\sigma_t(t)}{\sigma(t)}\right)^2 - \alpha(\gamma)\frac{\sigma_{tt}(t)}{\sigma(t)}\right\} + k_{0j}\sigma^2(t) = 0, \tag{13}$$

which requires that $k_{cx}/k_{0x} = k_{cy}/k_{0y} = k_{cz}/k_{0z}$ equals a constant with the constraint $k_{cx} + k_{cy} + k_{cz} = 3$ [in fact, one can define $k_{cx} = k_{cy} = 3\lambda/(1+\lambda)$ and $k_{cz} = 3/(1+\lambda)$ with λ being the aspect ratio]. Equation (13) accommodates the periodic solution which makes the density (the square of the amplitude of the wave function) oscillate. Now we apply the parameters used in the MIT experiment to calculate this period. When the aspect ratio $\lambda = 6.1$, in the BEC limit ($\gamma = 1$) with $\alpha(\gamma) = 1$, Eq. (13) has a solution of $\sigma(t) = \sin^\nu \omega t$ with ν turning out to be 1 and $\omega = \sqrt{(4\Sigma k_{0i})/3}$, and we get $T_s = 4.98T_z$. Notice that $T_s = 4.4T_z$ was reported in the experiment in Ref. [46]; thus, the analytical result matches the experimental one very well. In the unitarity limit ($\gamma = 2/3$) with $\alpha(\gamma) = -1$, Eq. (13) has a solution of $\sigma(t) = (a\tan^2\omega t + b)^q$ [the equations for ω,a,b,q are obtained from the equations $2h_1 q + 2h_2(q-1) + 3h_2 = 0$, $k_z a + 8h_1 q^2 a\omega^2 + 8h_2 q(q-1)a\omega^2 + 2h_2 q\omega^2(3b+4a) = 0$, $k_z b + 2h_1 q^2 a\omega^2 + 2h_2 q(q-1)a\omega^2 + h_2 q\omega^2(4b+a) = 0$, and $k_z b + 2h_2 q a\omega^2 = 0$, with $h_1 = \alpha(\gamma)$ and $h_2 = \alpha^2(\gamma) - \alpha(\gamma)$] with $\omega = \sqrt{(\sqrt{6}-1)\Sigma k_{0i}/5}$, and we get $T_s = 13.08T_z$. Notice that $T_s = 14T_z$ was reported in the experiment in Ref. [46]; we have a good match again.

For very strong confinement in the radial direction, we can introduce a parameter $\lambda = k_\perp/k_{0z} \gg 1$ and define $\chi(t) = \sigma^{2\lambda-2}(t)$. It is obvious that $\chi(t) \to 0$ for $|\sigma(t)| < 1$. Considering the modified transformation $(x,y,z) \to \sigma(t)(\sqrt{\chi(t)}x, \sqrt{\chi(t)}y, z)$, Eqs. (7) are modified as follows:

$$2gA^{2\gamma} - \delta\tau_t = 0, \tag{14a}$$

$$\chi(t)(\varsigma_{xx} + \varsigma_{yy}) + \varsigma_{zz} = 0, \tag{14b}$$

$$\chi(t)(\varsigma_x^2 + \varsigma_y^2) + \varsigma_z^2 - \varepsilon\tau_t = 0, \tag{14c}$$

$$\varsigma_t + 2[\chi(t)(\varsigma_x a_x + \varsigma_y a_y) + \varsigma_z a_z] = 0, \tag{14d}$$

$$2A_t - 2\Gamma A + 2A[\chi(t)(a_{xx} + a_{yy}) + a_{zz}] = 0, \tag{14e}$$

$$a_t + \chi(t)(a_x^2 + a_y^2) + a_z^2 - \sum_j k_j x_j^2 = 0. \tag{14f}$$

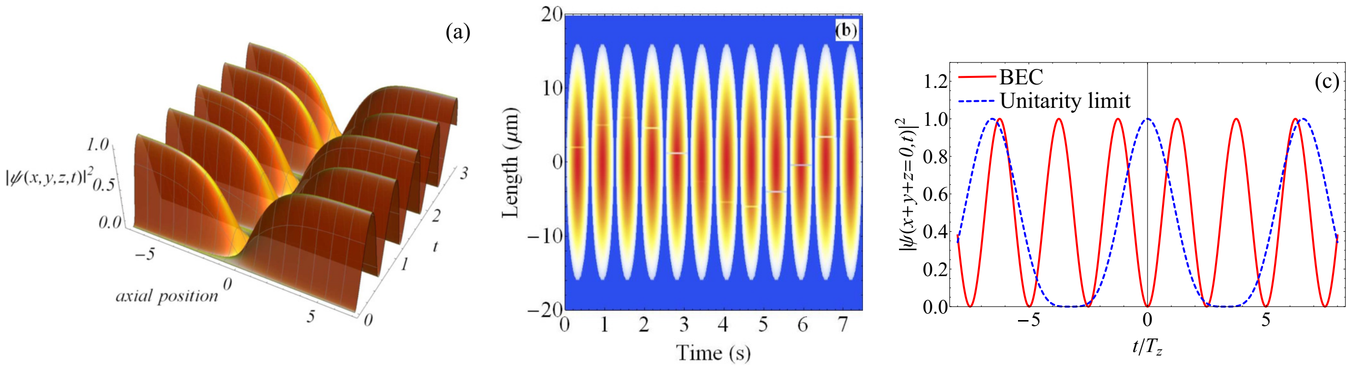

FIG. 1. (Color online) (a) The evolution of $|\psi|^2$ as a function of axial position $(x + y + z)$ and t in the BEC limit ($\gamma = 1$). (b) The oscillation of the soliton position (horizontal notch) with time t. (c) $|\psi|^2$ as a function of t/T_z at axial position $x + y + z = 0$ in the BEC limit where $\gamma = 1$ (solid line) and in the unitarity limit where $\gamma = 2/3$ (dashed line). The oscillation period of the solitonlike solution is $T_s/T_z = 4.98$ in the BEC limit ($\gamma = 1$) and $T_s/T_z = 13.08$ in the unitarity limit ($\gamma = 2/3$). The parameters are taken as $N = 2 \times 10^5$ and $T_z = 93.76$ ms for ^{6}Li atoms.

The equations for $\sigma(t)$ are the same as those given in Eqs. (12) and (13) with $\alpha(1) = 2$ (BEC limit) and $\alpha(2/3) = 1$ (unitarity limit), corresponding to $T_s/T_z = 2.63$ and 10.52 in the BEC and unitarity limits, respectively. In general, $T_s/T_z = 2.63(2/\gamma - 1)^2$.

The analytical results presented here agree very well with the experimental observation on T_s/T_z which demonstrates the validity of polytropic approximation in the mean-field GGPE description and the applicability of the corresponding theoretical approach adopted for the 3D case. Figure 1(a) shows the 3D evolution of a dark-soliton-like solution for the ^{6}Li system with $\gamma = 1$ in the BEC limit while Fig. 1(b) shows the oscillation of soliton position with time t. Figure 1(c) shows the oscillation behavior of the solitonlike solution in time with $\gamma = 1$ (BEC limit) and $\gamma = 2/3$ (unitarity limit). The number of atoms is $N = 2 \times 10^5$ and the trapping period is $T_z = 93.76$ ms. A large difference in oscillation period can be observed between the BEC and the unitarity limits due to the different interaction strengths determined by the different polytropic index γ in the model. Referring to Eq. (3), we have $g = \sigma(t)$ for $\gamma = 1$, whereas $g = 1$ for $\gamma = 2/3$, and trivial interdependence between $\sigma(t)$ and $A(t)$ occurs. This corresponds to the fact that only a minute fraction of atoms from the soliton contribute to the pairing, causing exponentially large effective mass which leads to the large difference in the soliton's oscillation periods (compared with that from BEC limit), as pointed out in Ref. [46].

B. Bright-soliton-like solutions

Now we focus on the case of attractive interaction where $g_0 > 0$ in Eq. (1). From Eqs. (5) and (6), it can be seen that the solution is bright solitonlike. For simplicity and taking into account the realistic experiment, we assign $\rho_0(t)$, $k_0(t)$, $g_0(t)$, and $\Gamma_0(t)$ as constants and get the ordinary differential equation for $\sigma(t)$ from the consistence condition Eq. (9), reading

$$\lambda_1 y_{tt} + \lambda_2 y_t - B_1 y^{1+1/2\theta} - B_2 y^{1+1/\theta} = 0, \qquad (15)$$

where $y(t) = \sigma^{-4\theta}(t)$, $B_1 = k_0$, and $B_2 = \Gamma_0^2/\rho_0$ with λ_1 and λ_2 being constants. The power index θ satisfies $3[\gamma(3\gamma - 1) + 1]\theta^2 - 8(3\gamma + 1)\theta + 16(3\gamma + 1) = 0$.

Referring to Eqs. (5), (6), and (8), the modulus of the wave function could be written as

$$|\psi(\boldsymbol{r},t)| = B_{3D}\sigma(t)^{3/2}e^{-2\Gamma t}\mathrm{sech}^{1/\gamma} \times \left[B_+\sigma(t)e^{-2\Gamma t}\sum_j x_j + b_3 e^{-2\Gamma t} + b_4 t\right], \qquad (16)$$

where B_{3D} and $b_{3,4}$ are constants determined by the normalization condition and the initial condition, and B_+ is the same as in Eqs. (5). One can clearly see that the solution indicated by Eq. (16) is a bright-soliton-like solution with its width evolving as $e^{2\Gamma t}/\sqrt{\sigma(t)}$. Equation (15) is integrable through the intermediate ansatz $y_t = \beta_1 y^{1+1/4\theta} + \beta_2 y$. The analytical solution of Eq. (15) takes the form of $\sigma(t) = 1/\sqrt{C_5 e^{-t} + C_6}$ with constants C_5 and C_6 determined by initial condition, and it is apparently not periodic. The nonperiodicity of solution (16) is mainly due to the fact that the switching of sign for g_0 in Eq. (1) makes $\sigma(t)$ an exponential-like function instead of a trigonometric-like function.

In Fig. 2, we give the time evolution of $\sigma(t)$ with the initial condition $\sigma(0) = 1.0$ and $\sigma_t(0) = 0$ through numerical calculation for $N = 2 \times 10^5$ and $T_z = 93.76$ ms for ^{6}Li. We can see that $\sigma(t)$, which determines the soliton evolution, approaches finite values at $t \to \pm\infty$ while it reaches its

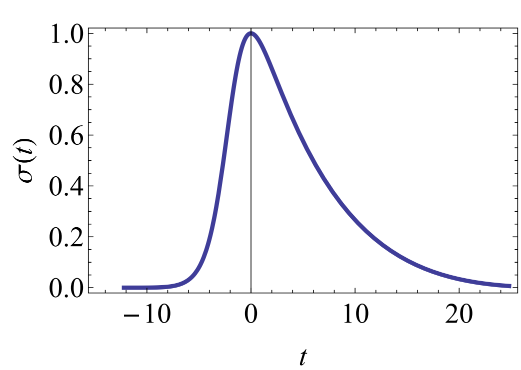

FIG. 2. (Color online) Time evolution of the parametric function $\sigma(t)$ with the initial conditions $\sigma(0) = 1$ and $\sigma_t(0) = 0$ for $N = 2 \times 10^5$ and $T_z = 93.76$ ms with ^{6}Li atoms.

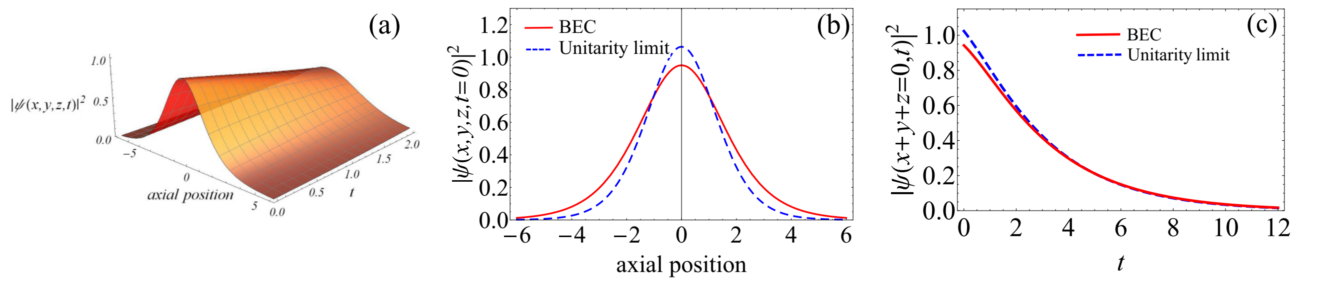

FIG. 3. (Color online) (a) The evolution of $|\psi|^2$ as a function of axial position ($x+y+z$, for example) and t. (b) $|\psi|^2$ at $t=0$ as a function of $x+y+z$ in the BEC limit with $\gamma=1$ (solid line) and unitarity limit with $\gamma=2/3$ (dashed line). (c) $|\psi|^2$ at $x+y+z=0$ as a function of time t in the BEC limit ($\gamma=1$) and the unitarity limit ($\gamma=2/3$).

maximum at $t=0$, exhibiting the nonperiodic feature. We already know that $\sigma(t)$ is a constant when the zero external potential is applied. The varying $\sigma(t)$ contributes to the modulation of soliton width which is induced from the trapping effect of the external harmonic potential.

We show in Fig. 3(a) the 3D evolution of the bright-soliton-like solution in the BEC limit with $\gamma=1$. A comparison of the bright-soliton-like solution between $\gamma=1$ (BEC limit) and $\gamma=2/3$ (unitarity limit) at fixed time $t=0$ and position $x+y+z=0$ is given in Figs. 3(b) and 3(c). From the figures we see that $|\psi(x,y,z,t)|$ decreases with time while its shape is retained. This is due to the existence of dissipation of the system; i.e., $\Gamma_0<0$. We also find that the amplitude (height) of $|\psi(x,y,z,t=0)|$ is smaller (larger) in the $\gamma=1$ ($\gamma=2/3$) case. This is because the interaction between atoms in the BEC limit is weaker than in the unitarity limit. In addition, the maximum of $|\psi(x,y,z,t=0)|$ does not occur in the center. We also see from Fig. 3(c) that when we choose a fixed position $x+y+z=0$, the time evolution of $|\psi|$ for different values $\gamma=1$ and $\gamma=2/3$ tends to approach the same value although their initial values are slightly different.

Notice that based on the availability of the multisoliton solutions of 1D GGPE (4), we can also construct multisoliton solutions of 3D GGPE (1) from ansatz (6). On the other hand, the results we obtained may also be used in the context of optics, where the model is given by the Maxwell equation governing the beam propagation in a bulk medium with the potential induced by the modulation of the refractive index. Also, the oscillatory solution of the 3D GGPE for the bright-soliton case can be explored by using different ansatz with different amplitude and phase transformation, which can be treated as an extension of the result presented in this work.

IV. CONCLUSION

In conclusion, we study the solitonlike solutions and their dynamics in a superfluid Fermi gas trapped in a time-dependent 3D harmonic potential with gain or loss. To obtain 3D analytical solitonlike solutions, we solve the generalized Gross-Pitaevskii equation based on the self-similar approach combined with scale and phase transformations, ruling out any integrable constraints used elsewhere. We study the propagation of both bright- and dark-soliton-like solutions, as well as their dynamics in detail. We find that the amplitude of dark-soliton-like solutions exhibits periodic oscillation, whereas the amplitude of bright-soliton-like solutions shows no periodic behavior. We highlight that the oscillation periods of dark-soliton-like solutions predicted by our theory are matched very well with the recent experiment reported by the MIT group [46], in both BEC and unitarity limits. Our results are useful not only for acquiring a better understanding of the experimental results, but also for guiding relevant experiments in the superfluid regimes, especially from the BEC to the strongly interacting unitarity regime.

ACKNOWLEDGMENTS

The authors thank Prof. Guoxiang Huang and Qingchun Zhou for fruitful discussions. This work was supported by National Science Foundation of China under Grants No. 11105039, No. 11205071, and No. 11475063.

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