Noise-induced suppression of nonlinear distortions in a bistable system with biharmonic excitation in vibrational resonance

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This paper is a report of the experimental evidence of suppression of vibrational higher-order harmonics in a bistable vertical-cavity surface-emitting laser driven by two harmonic signals with very different frequencies in the phenomenon of vibrational resonance when an optimal amount of white, Gaussian noise is applied. A quantitative characterization of the suppression is given on the basis of the coefficient of nonlinear distortions. The behavior of the coefficient of nonlinear distortions is studied in wide ranges of the added noise intensity, the dc current, and the amplitude of the harmonic signals. The experimental results are compared with a numerical simulation of a Langevin model showing good agreement.

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Noise is known to play an important role in the behavior of nonlinear dynamical systems in different fields of physics, chemistry, biology, etc. This topic is a subject of much current interest. One example of an important role of noise is the phenomenon of stochastic resonance (SR), where the response of a bistable system to the effect of a low-frequency (LF) periodic signal can be amplified by the presence of a certain amount of noise [1–3]. When a harmonic signal of large enough amplitude is used, one can observe in the phenomenon of SR the generation of ultraharmonics of the LF signal. This was a subject of a number of theoretical and experimental studies [4–12]. The generation of higher-order harmonics results in nonlinear distortions in the transmission of the LF signal through bistable systems in the regime of SR. Such distortions appear due to the fact that the temporal shape of the output signal is changed significantly in the regime of switching between two states. However, in the context of SR, it was found that suppression of higher harmonics of the LF signal can be observed for a certain value of the noise intensity [4–6]. In addition, it was shown that a sufficiently strong noise may induce linearization in overdamped bistable systems, which manifests itself in the fact that the ensemble-averaged output signal may reproduce the input signal [13,14]. The effect of strong additive noise on the response of driven bistable systems was also studied theoretically in the context of checking the linear-response theory, where the suppression of higher-order harmonics for the input signal of a large amplitude was shown as well [15].

The generation of higher harmonics is a characteristic feature of driven dynamical systems with a nonlinear transfer function, and it can be observed in different fields. When the transfer function has sharp nonlinearities, as, for instance, in a bistable system in the switching regime induced by the periodic modulation, the shape of the output signal will be different from that of the input signal. This becomes especially apparent in the case when the input signal is a sinusoid. This leads to a generation of the ultraharmonics of the input signal (see, for instance, [15]). Recently, it was shown that vibrational resonance (VR) in a bistable vertical-cavity surface-emitting laser (VCSEL) with biharmonic excitation is accompanied by the appearance of higher-order harmonics of the LF signal [16]. In the noiseless case, it shows up in the parameter space (the level of asymmetry, the amplitude of a high-frequency signal) as clear-cut structures with multiple local maxima on the higher harmonics of the LF signal. This leads to strong harmonic distortions in the transmission of the harmonic LF signal through a bistable system in the regime of VR [16]. The phenomenon of VR is a deterministic analog of the phenomenon of SR where noise is replaced by the high-frequency (HF) signal [17]. This results in the resonancelike behavior of the LF response of bistable systems depending on the amplitude or the frequency of the HF signal. In fact, the phenomenon of VR can be considered as a parametric amplification near the bifurcation point corresponding to the transition from bistability to monostability controlled by the HF signal [18,19]. The phenomenon of VR has been evidenced experimentally in bistable analog circuits [20] and in a bistable VCSEL [21]. Further investigations have revealed the phenomenon of VR in various nonlinear systems from different fields. Specifically, the phenomenon of VR was found and studied theoretically in neural systems of varying complexity [22–25], coupled oscillators [26–29], a multistable system with a periodic potential [30], delayed multistable systems [31], fractional-order systems [32], and nonlinear maps [33–35]. The generation of the second harmonic in VR in a symmetrical bistable system was studied theoretically [36].

The effect of noise on VR was also studied in different systems both theoretically [20,37–45] and experimentally [39,40,46,47]. All of these studies concentrated on the effect of noise on the first harmonic of the LF response of nonlinear systems. The main finding in these investigations is that the addition of noise results in a decrease in the LF response amplitude, a broadening of the LF response curve, and a shift of the optimal value of the HF amplitude as the noise strength increases. In particular, scaling laws for a noise-induced decrease of the gain factor, the signal-to-noise ratio, and the shift of the optimal HF amplitude were found theoretically for an overdamped bistable system and verified experimentally in a VCSEL [39,40]. It is significant that the regime of vibrational resonance leads to higher signal-to-noise ratios than the regime of stochastic resonance [40,41].

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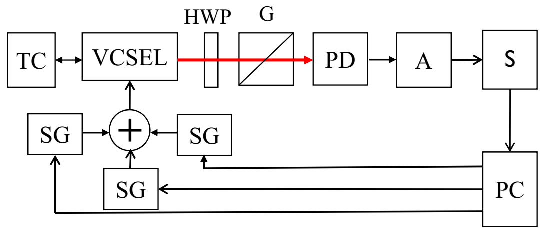

FIG. 1. (Color online) Experimental setup. VCSEL, vertical cavity surface emitting laser; HWP, half-wave plate; G, Glan's prism; PD, photodiode; A, amplifier; S, USB oscilloscope; PC, computer; SG, signal generator; TC, thermocontroller.

Here, in contrast with previous studies, the effect of noise on the higher-order harmonics in VR is investigated. It is shown experimentally that an addition of some optimal amount of noise in a bistable system suppresses nonlinear distortions, or, in other words, harmonic distortions in VR when the LF signal is the sinusoidal one. An experimental study was performed in a bistable VCSEL driven by two harmonic signals with frequencies differing by two orders of magnitude. Throughout the paper, the nonlinear distortion factor (NDF) is used for quantitative characterization of the effect of noise. Two cases were studied. In the first one, the amplitude of the HF signal was set nearly to the optimal value, corresponding to the maximal response in VR, and harmonic distortions were studied depending on the noise intensity and the level of asymmetry of a bistable potential. In the second set of experiments, the distortions were evaluated for the symmetrical configuration of a bistable potential depending on the amplitude of the HF signal and the noise intensity. The experimental results obtained in a VCSEL are in agreement with the results of the numerical simulation performed in the model of the overdamped bistable oscillator with a biharmonic excitation.

Investigations were performed on an experimental setup similar to that recently used to study the resonant behavior on higher-order harmonics in VR (Fig. 1) [16]. A 850-nm VCSEL (Honeywell HFE4080-321) with a threshold current $J_{\rm th} \approx 5.6$ mA was used. The temperature of the laser diode was controlled to an accuracy of 0.01 °C. The measurements were performed with the temperature of the laser diode set to 17.5 °C. A half-wave plate and a Glan's prism were used to split the collimated laser emission into two polarization components. The temporal laser responses on one selected polarization were recorded by a fast photodiode and a USB oscilloscope. Two harmonic signals from function generators with frequencies $f_L = 0.5$ kHz and $f_H = 50$ kHz and different amplitudes A_L and A_H were added to the dc current $j_{\rm dc}$. Both driving frequencies are less than the cutoff frequency of the amplitude-frequency characteristic of the polarization-resolved laser response. The amplitude of the LF signal A_L was set to 7.5 mV. The Gaussian white noise from an arbitrary waveform generator with a bandwidth of 2 MHz with different amplitude σ_N (rms) was also added to the dc current. In what follows, the noise strength is defined as $D_{\rm exp} = \sigma_N^2$. The parameters controlled from a computer during the experiments were the dc current, the noise intensity, and the amplitude of the HF periodic signal.

A theoretical description of the dynamics of the polarization switchings induced by a deterministic modulation and noise in the VCSEL can be performed in the framework of a Langevin equation with a double-well potential [48,49]. This model is widely used in studying noise-induced phenomena in different fields. Therefore, the effect of noise on higher-order harmonics in the phenomenon of VR was studied using the model of an overdamped bistable oscillator. In this case, the equation reads

$$\frac{\partial x}{\partial t} = -V'(x) + A_L \sin 2\pi f_L t + A_H \sin 2\pi f_H t + \zeta(t), \quad (1)$$

where $V'(x)$ is the derivative with respect to x of a bistable potential function $V(x)$, f_L and f_H are low and high frequencies, respectively, and $\zeta(t)$ is a white, Gaussian noise with $\langle \zeta(t)\zeta(t')\rangle = 2D\delta(t-t')$ and mean $\langle \zeta(t)\rangle = 0$. We have used a bistable potential function $V(x)$ in the following form: $V(x) = x^4 - 2x^2 + \Delta x$, where Δ is a level of asymmetry. In the simulation, we have used $f_L = 0.001$ and $f_H = 0.1$, which are less than the intrawell relaxation frequency $f_r = 4/\pi$ in the symmetrical configuration ($\Delta = 0$) of a bistable potential. This consideration corresponds to the adiabatic regime when both periodic forces were very slow with respect to the intrawell relaxation times.

The effect of noise on the higher harmonics of the LF signal was investigated experimentally from the spectra of the Fourier transformed time series of the laser intensity on the selected polarization at the frequencies kf_L ($k = 1,2,\ldots,9$) depending on the noise intensity σ_N^2 in a wide range of the dc current $j_{\rm dc}$ and the amplitude of the HF signal. Some typical amplitude spectra for different values of σ_N are shown in Fig. 2 for the case of the symmetrical configuration of the bistable potential. In this case, only odd harmonics of the LF signal are observed. One can see that some higher harmonics disappear as the noise intensity increases [Fig. 2(b)]. For a certain value of the noise intensity σ_N^2, all ultraharmonics of the LF signal vanish, leading to the suppression of nonlinear distortions [Fig. 2(c)].

The response amplitude $R_L^{(k)}$ at the frequencies kf_L was studied here versus the injection current $j_{\rm dc}$ and the noise

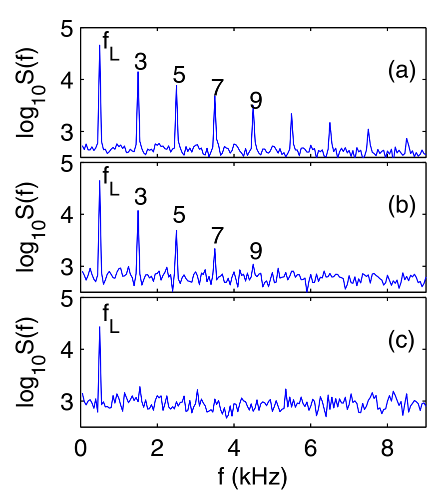

FIG. 2. (Color online) Experiment. Amplitude spectrum $S(f)$ of the laser response on the selected polarization shown for three different values of the noise intensity σ_N^2 [$j_{\rm dc} = 17.78$ mA; $\sigma_N = 0.005$ (a), 0.075 (b), and 0.23 (c) V].

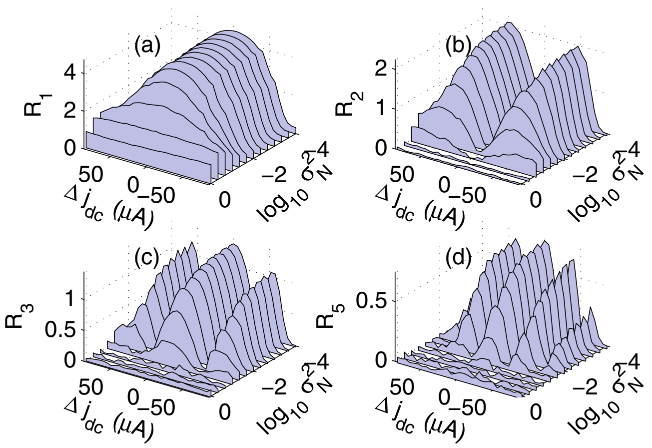

FIG. 3. (Color online) Experiment. The laser response amplitude R_k on the frequency $f_k = kf_L$ ($k = 1,2,3,5$) vs the level of asymmetry $\Delta j_{\rm dc}$ and the noise intensity σ_N^2.

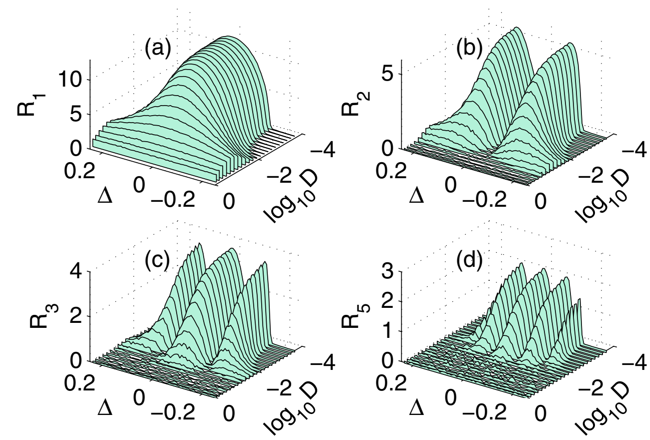

FIG. 4. (Color online) Simulation. The response amplitude R_k on the frequency $f_k = kf_L$ ($k = 1,2,3,5$) vs the level of asymmetry Δ and the noise intensity D.

intensity σ_N^2. The response amplitude $R_L^{(k)}$ in the experimental and numerical studies was estimated as the height of the peaks in the spectra of the laser responses. Since the measurements were performed with different noise intensities added to the LF signal, the noise intensity at the frequency of ultraharmonics as the interpolated level of the background was also measured. Throughout the paper, the response amplitude was defined as $R_k = R_L^{(k)} - N_L^{(k)}$, where k is the harmonic number of the LF signal, $R_L^{(k)}$ is the overall response, and $N_L^{(k)}$ is the noise background. Such an extraction of the noise component allows one to evaluate the true values of the amplitudes of the LF harmonics in the spectra of the output signal.

Harmonic distortions in VR were quantitatively characterized by the nonlinear distortions factor (NDF), which was recently introduced in the noiseless case [16]. The NDF is defined here as follows:

$$\chi = \left[\sum_{k=2}^{m} R_k^2\right]^{1/2} \Big/ R_1, \qquad (2)$$

where R_k is a response amplitude of the kth harmonics. Both in the experiment and in the simulation, the amplitudes of the first nine harmonics ($m = 9$) were measured.

First, the effect of noise was studied for the case when the amplitude of the HF signal was fixed and set close to the optimal value ($A_H = 80$ mV) respective to the maximal response amplitude of the laser for the given amplitude of the LF signal ($A_L = 7.5$ mV). Figures 3 and 4 demonstrate experimental and numerical response amplitudes R_k as a function of the added noise intensity and the level of asymmetry ($\Delta j_{\rm dc}$ and Δ, respectively). The level of asymmetry $\Delta j_{\rm dc}$ in the experimental studies is defined as $\Delta j_{\rm dc} = j_{\rm dc} - j_{\rm sym}$, where $j_{\rm dc}$ is a current value and $j_{\rm sym}$ is the value of $j_{\rm dc}$ that corresponds to the symmetrical configuration of the bistable potential. The value of $j_{\rm sym}$ is determined as a minimal value of the switching amplitude of the HF periodic signals depending on $j_{\rm dc}$. There is a close similarity between the experimental and numerical results presented in Figs. 3 and 4, respectively. Therefore, in what follows, both experimental and numerical results are discussed together. Both Figs. 3 and 4 show the strong diminution and broadening of the response amplitudes R_k upon increasing the noise intensity. However, the structure of the response R_k on ultraharmonics is not changed due to the fact that in the experiment and the simulation, we have used noise with a zero mean, which does not change the symmetry of the potential. The suppression of the response amplitudes R_k occurs for the different optimal values of the noise intensity. Figure 5 illustrates this fact for the case of the symmetrical configuration of the bistable potential, where the experimental and numerical results are directly compared for the fixed optimal values of the HF amplitude, showing a rather good agreement between them. In this figure, the normalized noise intensity is used for both experimental and numerical data. The normalized amplitude of the LF signal is about 0.1 in units of the switching threshold for this case. One can see from Fig. 5 that the optimal value of the noise intensity for each ultraharmonics is different, but more importantly the amplitude of the fundamental harmonic ($k = 1$) still remains large enough, whereas all

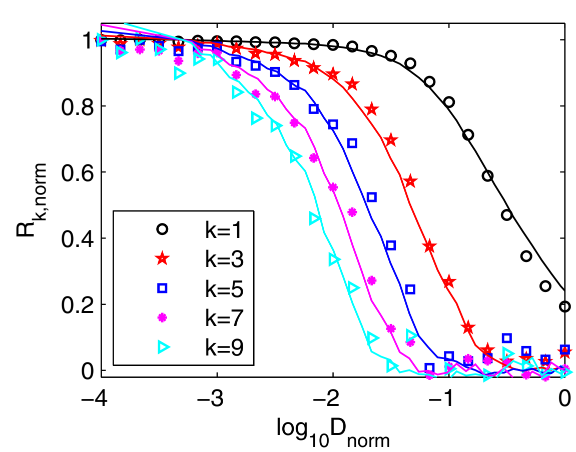

FIG. 5. (Color online) The normalized response amplitude $R_{k,\rm norm}$ on the frequency $f_k = kf_L$ ($k = 1,3,\ldots,9$) vs the normalized noise intensity $D_{\rm norm}$ for $\Delta = 0$. Symbols, experiment; solid lines, numerical simulation.

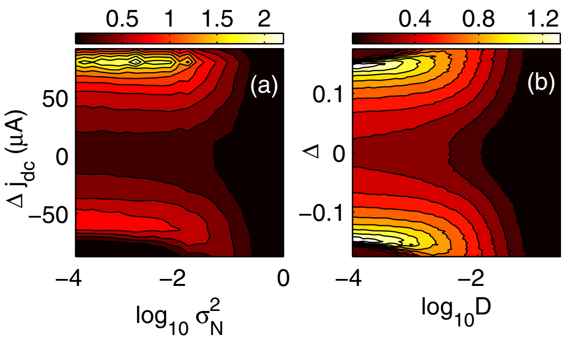

FIG. 6. (Color online) Contour plot for the nonlinear distortion factor χ versus the noise intensity σ_N^2 (D) and the level of asymmetry Δj_{dc} (Δ): (a) experiment and (b) simulation. Brightest colors represent the highest NDFs.

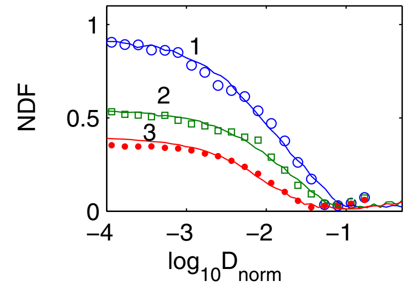

FIG. 8. (Color online) The nonlinear distortion factor NDF (χ) vs the normalized noise intensity D_{norm} shown for three values of the level of asymmetry Δj_{dc} (Δ): Symbols correspond to the experiment [$\Delta j_{\text{dc}} = -0.058$ (1), -0.034 (2), and 0 (3) mA]; the solid lines are the numerical results [$\Delta = -0.12$ (1), -0.07 (2), and 0 (3)].

other harmonics are suppressed. This means that the harmonic LF signal can be transmitted in the regime of VR without distortions with a rather high amplification of about 0.6 from the maximal value of R_1 in the noiseless case.

A quantitative characterization of the suppression is shown in Figs. 6–8, where the coefficient of nonlinear distortion is presented. The contour plot in Fig. 6 gives a general idea of how the NDF depends on the noise intensity and the level of the asymmetry of the bistable potential. In the absence of noise, minimal distortions in VR are observed for the symmetrical configuration of the bistable potential. The same can be said when noise is added to the system. Figure 7 shows the NDF depending on the level of asymmetry $\Delta_{\text{exp}}(\Delta)$ for different values of the noise intensity $\sigma_N^2(D)$. One can note good qualitative agreement between experimental and numerical results. Some asymmetry in the experimental results appears due to the fact that the shapes of the wells of the bistable potential in the experiment do not agree with the result showing that the response amplitudes in the wells are different.

Figure 8 presents a quantitative characterization of the noise-induced suppression of harmonic distortions where the experimental and the numerical dependencies of χ (NDF) as a function of the normalized noise intensity D_{norm} are shown together for different levels of asymmetry. One can note good agreement between them. For a weak level of noise, the NDF weakly decreases as the noise intensity increases. For a moderate level of noise, the NDF linearly diminishes in a semilog scale depending on the noise intensity until it reaches the minimal value. In this region of noise intensity, the fitting of experimental and numerical data yields $\chi \sim D^{-\gamma}$, where $\gamma \approx 0.49$ (1), 0.36 (2), and 0.25 (3) for both experimental and numerical data.

The numerical simulation revealed also that the optimal value of the noise intensity D_{opt} corresponding to suppression of harmonic distortions depends on the amplitude of the LF signal ϵ defined as $\epsilon = A_L/A_{L,\text{th}}$, where $A_{L,\text{th}}$ is the switching threshold. Figure 9 demonstrates the dependencies of NDF for different LF amplitudes as a function of the added level of noise. The fitting of the numerical results shows that D_{opt} is scaled as $D_{\text{opt}} \sim A_L^{\beta}$, where β is a function of the LF amplitude. In the range of $A_L \in [0.05, 0.4]$, the value of β is ≈ 0.8, and it decreases to $\beta \approx 0.57$ in the range of $A_L \in [0.04, 0.8]$. The value of β was evaluated on the level of $\chi \approx 0.05$.

The results presented above correspond to the case when the HF amplitude was fixed close to the optimal value from the standpoint of reaching the maximal response in the noiseless case. In this case, minimal distortions are obtained for the symmetrical configuration of the bistable potential. Previously [16] it was shown experimentally in the noiseless case that for certain values of the HF amplitude, the suppression of the higher harmonics occurs similar to the phenomenon of

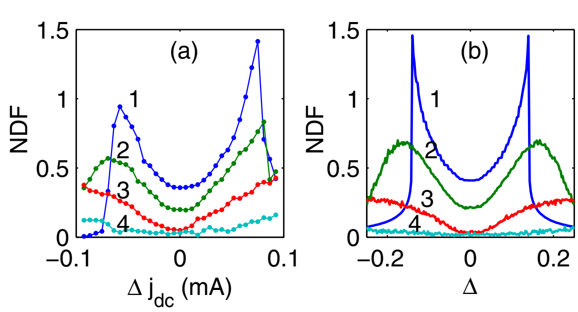

FIG. 7. (Color online) The nonlinear distortion factor NDF (χ) vs the level of asymmetry Δj_{dc} (Δ) shown for different values of the level of noise σ_N (D): (a) experiment [$\sigma_N = 0$ (1), 0.046 (2), 0.15 (3), and 0.32 (4) V] and (b) simulation [$D = 0$ (1), 0.006 (2), 0.037 (3), and 0.122 (4)].

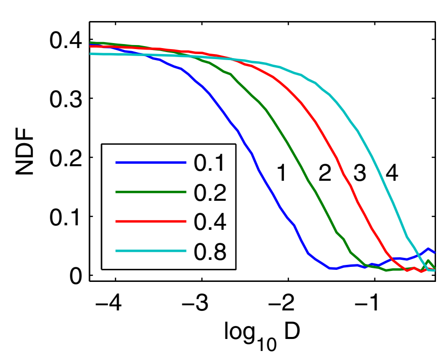

FIG. 9. (Color online) Simulation. The NDF vs the noise intensity D shown for different values of the normalized LF amplitude ϵ [$\epsilon = 0.1$ (1), 0.2 (2), 0.4 (3), and 0.8 (4)].

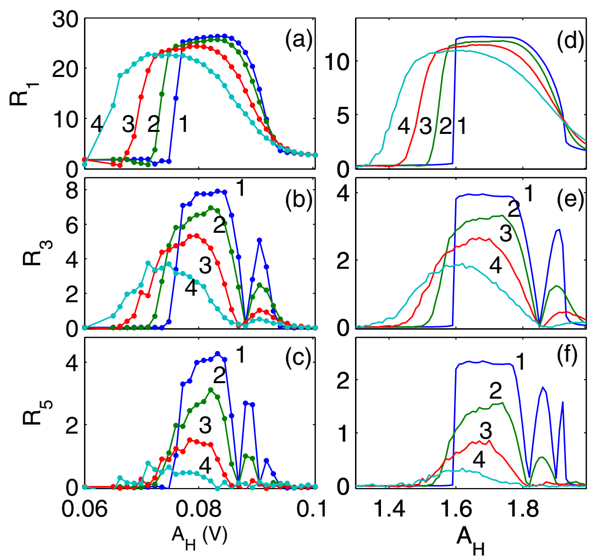

FIG. 10. (Color online) The response amplitude R_k on the frequency $f_k = kf_L$ ($k = 1,3,5$) vs the amplitude A_H of the HF signal shown for different levels of noise σ_N (D) in the symmetrical configuration of a bistable potential. Left column: experiment [$\sigma_N = 0$ (1), 0.0316 (2), 0.1 (3), and 0.215 (4) V]; right column: numerical simulation [$D = 0.001$ (1), 0.05 (2), 0.09 (3), and 0.136 (4)].

noise-induced resonance [5]. Further, the influence of the HF amplitude and the noise intensity on distortions is investigated for this case. Figure 10 shows the response amplitude R_k ($k = 1,3,5$) when the different amounts of noise are added. In the noiseless case, the suppression of the third (R_3) and fifth (R_5) harmonics is clearly seen in Figs. 10(b)–10(f) (curves 1) for both experimental and numerical data having two-humped and three-humped resonance curves, respectively, depending on the HF amplitude. The number of humps increases upon increasing the ultraharmonic numbers k [16]. The addition of noise results in the broadening of the response curve on the ultraharmonics, and it diminishes their amplitudes, but more importantly, the amplitudes of the higher harmonics decrease much faster than the response on the fundamental frequency ($k = 1$) upon increasing the noise intensity. For large enough noise intensity, the response on the ultraharmonics becomes practically negligible. This fact results in a strong diminution of harmonic distortions. A contour plot in Fig. 11 gives a general view of the dependence of the NDF as a function of the HF amplitude and the noise intensity. For all values of A_H in the VR region, one can see that noise suppresses harmonic distortions. However, the minimal noise intensity needed for the suppression is observed for the amplitude of the HF signal beyond the maximal response of R_1 in the vicinity of the exceptional point for the third harmonic, where its strong suppression occurs. In Fig. 12, several examples of the dependence of NDF for certain fixed values of the noise intensity are presented as a function of the amplitude of the HF signal. Curve 3 in Figs. 12(a) and 12(b) corresponds to the optimal value of the noise intensity for which the suppression of harmonic distortions is observed close to the exceptional point for the third harmonic of the LF signal. It should be noted for this optimal value of the noise intensity that the

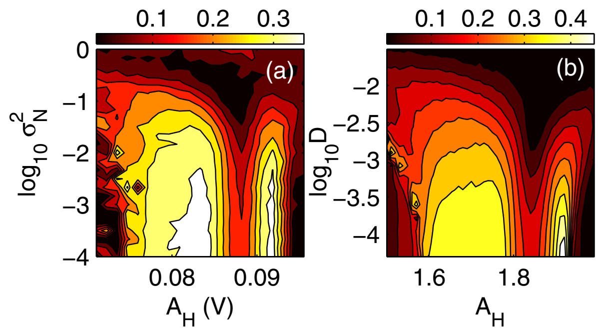

FIG. 11. (Color online) The nonlinear distortion factor χ vs the HF amplitude A_H and the noise intensity σ_N^2 (D): (a) experiment and (b) simulation.

amplitude of the fundamental harmonic ($k = 1$) still remains significant (approximately 0.6 from the maximal response of R_1 in the noiseless case). Comparing with the first case when the amplitude of the HF signal was set close to the optimal value, one can note that in both cases the amplification of the LF signal is practically the same. The only difference is in the optimal value of the noise intensity, which is slightly less for the second case.

The results presented above clearly demonstrate a disappearance of the higher-order harmonics induced by noise when the amplitude of the HF signal is fixed to some value. Presumably, the following inherent processes may clarify the origin of such a suppression. First, the increase in the intensity of additive noise results in a broadening of the response curve at a fundamental frequency ($k = 1$) and a shift of the critical HF amplitude, which corresponds to the maximal response. This critical value can be associated with a shifted bifurcation point for which the phenomenon of VR occurs in the presence of noise. For a large enough level of additive noise for which a complete suppression of higher harmonics is observed, the system can be moved into the range of parameters where bistability disappears and the response of the system can correspond to the monostable one-well system. It should be noted that a noise-induced shift of the bifurcations to different values of the control parameter with respect to their deterministic values is well known for nonlinear dynamical systems, and it was demonstrated for different types of bifurcations [50–53]. On the other hand, a strong

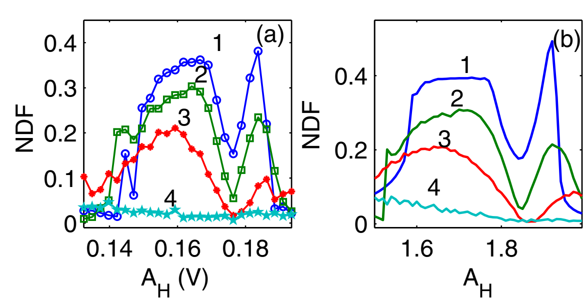

FIG. 12. (Color online) The nonlinear distortion factor χ vs the HF amplitude A_H ($\Delta j_{dc} = 0$ and $\Delta = 0$): (a) experiment [$\sigma_n = 0.005$ (1), 0.089 (2), 0.192 (3), and 0.34 (4) V] and (b) simulation [$D = 0.01$ (1), 0.055 (2), 0.11 (3), and 0.20 (4)].

level of noise may result in the linearization of a nonlinear transfer characteristic of the system. Linearization by noise is a general effect, and it was studied in excitable systems [54,55] and threshold systems with a static nonlinearity [56,57]. In this case, additive noise reshapes the transfer function in threshold systems, leading to linearization of the output. In our case, this follows from an analysis of the experimental and numerical dependencies of the response of the bistable system to the effect of a single periodic modulation depending on its amplitude in the absence and presence of additive noise with different amplitudes. In this case, a sharp dependence in the vicinity of the bifurcation point is replaced by a smoother dependence of the response on the amplitude of the periodic modulation. This leads to a decrease in the amplitudes of higher-order harmonics. Therefore, the interplay between the noise-induced shift of the bifurcation point and the effect of the linearization may result in the suppression of the higher-order harmonics.

To conclude, we have presented experimental and numerical results that demonstrate that the addition of the optimal amount of noise results in the suppression of vibrational higher-order harmonics in a bistable system with biharmonic excitation. The same approach can be extended to the arbitrary LF signal when the highest frequency in the spectrum of the LF signal will be less than the cutoff frequency of the bistable system. The results presented here can also be useful from the point of view of the optimization of the transmission of the harmonic signal through bistable systems or the network of bistable systems without distortions in the regime of vibrational resonance with a rather high amplification.

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