PHYSICAL REVIEW E **92**, 022135 (2015)

Source-enhanced coalescence of trees in a random forest

A. A. Lushnikov
Geophysical Center of Russian Academy of Science, 3, Molodezhnaya Street, 119296 Moscow, Russia
and National Research Nuclear University MEPhI, 31, Kashirskoye Road, 115409 Moscow, Russia

(Received 28 May 2015; published 24 August 2015)

The time evolution of a random graph with varying number of edges and vertices is considered. The edges and vertices are assumed to be added at random by one at a time with different rates. A fresh edge connects either two linked components and forms a new component of larger order g (coalescence of graphs) or increases (by one) the number of edges in a given linked component (cycling). Assuming the vertices to have a finite valence (the number of edges connected with a given vertex is limited) the kinetic equation for the distribution of linked components of the graph over their orders and valences is formulated and solved exactly by applying the generating function method for the case of coalescence of trees. The evolution process is shown to reveal a phase transition: the emergence of a giant linked component whose order is comparable to the total order of the graph. The time dependencies of the moments of the distribution of linked components over their orders and valences are found explicitly for the pregelation period and the critical behavior of the spectrum is analyzed. It is found that the linked components are γ distributed over g with the algebraic prefactor $g^{-5/2}$. The coalescence process is shown to terminate by the formation of the steady-state γ spectrum with the same algebraic prefactor.

DOI: 10.1103/PhysRevE.92.022135 PACS number(s): 02.50.Ga, 02.50.Ey, 05.40.−a

I. INTRODUCTION

A random forest is a collection of linked random trees (linked graphs without cycles). The linked tree is a tree in which each vertex is connected by one or several edges with any other vertex belonging to it. The interest to the structures of random forests is connected to their great resemblance to a wide range of systems in nature and society [1–6]. Most frequently cited examples are polymers and the polymerization processes [7–12], social nets, telephone nets and their developments, the Internet, neural nets in living creature (including the human brain) [13–16]. While the polymers are the most natural objects for describing them by random graphs, other nets are organized by different principles. Still comparing their structure with that of random graph gives an information on the deviation from randomness and thus helps us to recover the causes for these deviations [16].

The random graph with M vertices and P edges is characterized by its spectrum of linked components (the numbers $N_{g,\nu}$ of linked components with g vertices and ν edges). The structure of the graph changes as P increases. There is another way for generating the random graph. Assuming that the extra edges appear randomly one at a time, we watch for the time changes of the graph structure. Here “time” is not real time (in seconds), but a parameter connected with the total average number of edges in the graph. In this case it is possible to formulate an evolution equation describing the time evolution of a random graph. This very approach will be used in this paper.

In previous works [7–23] the authors considered the free evolution process, where newly appeared edges connected initially bare vertices. Free coalescence of arbitrary initial graphs with the vertices of limited valence was studied in my work [23].

The close resemblance of the dynamics of evolving random graphs to many dynamical processes such as polymerization [7,8,12–14], evolution of social networks [16], percolation [17–19], phase transitions [24], and spreading of infections [25–28] stimulated the studies of the time evolution of various types of random graphs. most of which relied upon the combinatorial analysis [1,3–5]. Here I apply the kinetic approach, where the edges and the vertices are randomly added to the graph with different rates. In this case the graph evolution is analogous to the source-enhanced coagulation process [6,29]. In such a graph the edges connecting initially empty vertices lead to the emergence of a collection of linked components of different sizes (number of vertices in the component). The issue is to find the distribution of the linked components over their sizes.

The efficiency of the Smoluchowski equation for studying the evolution of random graphs had been clearly demonstrated in Refs. [11,13,21,22], where the analogy between the particle coalescence and the graph transformation by randomly adding the edges to the evolving graph. Indeed, if we consider a graph comprising $N_1, N_2 \ldots N_g \ldots$ linked components, then any extra edge either converts a pair of components to one or produces an additional cycle if it connects two vertices in one component. So far the above consideration assumed the infinite valence of the vertices, i.e., the number of outgoing edges was not limited. It is clear that if the valence of each vertex in the evolving graph is finite then the resulting structures will occur more loose.

This paper aims at the consideration of the evolution of a random graph with the vertices of finite valence in the presence of a source of vertices. In the next Sec. II the kinetic equation describing the evolution of the spectrum of linked components is formulated. The *ansatz* of Ref. [29] applies for solving this equation in Sec. III. In Sec IV the time dependencies of the lower moments of the spectrum are found. It is shown that the second moment diverges at a critical time, which is evidence of the sol-gel transition in evolving random forests. The postgel behavior of the random forest is analyzed in Sec. V. Sections VI and VII summarize the main results and discuss the specific features of the model considered in this paper. The critical behavior of the forest is shown to differ from that in other models of the graph evolution: a residual steady-state distribution of small graphs forms at the final stage of the coalescence process.

II. BASIC EQUATIONS

The time evolution of a random graph is considered as a chain of pair coalescences of linked components of the graph by randomly appearing edges connecting their vertexes. Each vertex is assumed to have a finite valence, i.e., not more than s edges are permitted to join to it. Any extra edge diminishes the valence of each component by one. This process is represented by the scheme,

$$(g_1,\sigma_1+1)+(g_2,\sigma_2+1)\to(g_1+g_2,\sigma_1+\sigma_2). \tag{1}$$

Here g is the order of the coalescing component and σ is its valence. In what follows the rate of this process is denoted as $\varkappa$.

Another process that contributes to the forest evolution adds the bare vertices of valence s with the constant rate I.

The probability of coalescence is proportional to the product of total valences of two linked components. The maximal valence of a vertex is denoted as s. In the process of coalescence the valence of each linked component changes. Let $N_{g,\sigma}(t)$ be the number of linked components of order (size) g and valence σ. Then we can write down the balance equation that governs the time evolution of the graph spectrum,

$$\frac{dN_{g,\sigma}(t)}{dt}=Iq_{g,\sigma}+\frac{\varkappa}{2}\left[\sum_{l,\eta}(\sigma-\eta+1)(\eta+1)N_{g-l,\sigma-\eta+1}(t)N_{l,\eta+1}(t)-\sigma N_{g,\sigma}(t)\sum_{l,\eta}\eta N_{l,\eta}(t)\right]. \tag{2}$$

Here I is the productivity of the source of vertices, the function $q_{g,\sigma}$ describes the dependence of the source on the order of the respective linked component and its valence, and $\varkappa$ is the rate of coalescence of a pair of linked trees due to adding an extra edge. Rescaling time by the factor $\sqrt{I\varkappa}$ and introducing the concentrations $c_{g,\sigma}=N_{g,\sigma}/\sqrt{I/\varkappa}$ allows us to write down the kinetic equation for $c_{g,\sigma}$,

$$\frac{dc_{g,\sigma}(t)}{dt}=q_{g,\sigma}+\frac{1}{2}\sum_{l,\eta}(\sigma-\eta+1)(\eta+1)c_{g-l,\sigma-\eta+1}(t)c_{l,\eta+1}(t)-\sigma c_{g,\sigma}(t)\sum_{l,\eta}\eta c_{l,\eta}(t). \tag{3}$$

Similar equation was formulated in Ref. [23] for the case of free coalescence $I=0$.

It is seen that the evolution process Eq. (1) produces only trees and never leads to cycling, the process that kills two valences in one linked component $(g,\sigma)\to(g,\sigma-2)$.

In what follows we consider the zero initial conditions to Eq. (3),

$$c^{(0)}_{g,\sigma}=0, \tag{4}$$

and the monodisperse monovalent source providing the system with the vertices of valence s,

$$q_{g,\sigma}=\delta_{g,1}\delta_{\sigma,s}. \tag{5}$$

Here $\delta_{i,k}$ stands for Kroneker's δ.

Equation (3) describes the emergence and the growth of the forest, because the process Eq. (1) does not permit for cycling. This fact means that the total valence σ of each linked component (tree) is expressed through its total mass g as follows:

$$\sigma=g(s-2)+2. \tag{6}$$

This equality claims that the total valence of the tree is equal to the maximal possible valence of all its vertices sg minus twice the number of edges in the tree $2(g-1)$. Hence, we can operate with

$$c_\sigma(t)=\sum_g c_{g,\sigma}(t), \tag{7}$$

because

$$c_{g,\sigma}(t)=c_\sigma(t)\delta_{\sigma,g(s-2)+2}. \tag{8}$$

Here δ is the Kroneker δ.

III. GENERATING FUNCTION

Let us introduce the bivariate generating function for $c_{g,\sigma}(t)$,

$$\mathcal{F}(z,\xi;t)=\sum_{g,\sigma}z^g\xi^\sigma c_{g,\sigma}(t). \tag{9}$$

Equation (8) determines the structure of the function $\mathcal{F}$,

$$\mathcal{F}(z,\xi;t)=\xi^2\Phi(z\xi^{s-2},t). \tag{10}$$

The order spectrum $c_g(t)$ is given by the Cauchy formula,

$$c_g(t)=\sum_\sigma c_{g,\sigma}(t)=\frac{1}{2\pi i}\oint\frac{\mathcal{F}(z,\xi;t)dz}{z^{g+1}}\bigg|_{\xi=1}. \tag{11}$$

The integration goes counterclockwise over a small circle surrounding the origin of coordinates in the complex plane z. Applying Eqs. (10) and (11) and replacing the variable $y=z\xi^{s-2}$ yields

$$c_g(t)=\frac{1}{2\pi i}\oint\frac{\Phi(y,t)dy}{y^{g+1}}. \tag{12}$$

On multiplying both sides of Eq. (3) by $z^g\xi^\sigma$ and summing over all g and σ gives the equation for $\mathcal{F}$,

$$\frac{\partial\mathcal{F}}{\partial t}=z\xi^s+\frac{1}{2}\left(\frac{\partial\mathcal{F}}{\partial\xi}\right)^2-\xi\frac{\partial\mathcal{F}}{\partial\xi}S(t). \tag{13}$$

Here

$$S(t)=\sum_\sigma\sigma c_\sigma(t)=\xi\frac{\partial\mathcal{F}}{\partial\xi}\bigg|_{\xi=1} \tag{14}$$

is the total average valence of the forest. Equations (13) and (14) gives

$$\dot{S} = s - S^2. \tag{15}$$

In what follows we omit the argument z in $\mathcal{F}$. The solution to Eq. (13) can be found in the form [29]

$$\mathcal{F}(\xi,t) = Q(\xi,t) - tA(u) - \int_u^\xi \sqrt{2[A(\zeta) - A(u)]}d\zeta, \tag{16}$$

where $u = u(\xi,t)$ is defined from the equality,

$$t = \int_u^\xi \frac{d\zeta}{\sqrt{2[A(\zeta) - A(u)]}}, \tag{17}$$

and the functions $A(\xi)$ and $Q(\xi,t)$ are not yet specified. The function $u = u(\xi,t)$ appeared first in Eq. (16) is introduced in such a way that the partial derivative over u of the right-hand side of Eq. (16) would be zero. Keeping this fact in mind we find $A(\xi)$ and $Q(\xi,t)$ by differentiating both sides of Eq. (16) with respect to ξ and t and substituting the results into Eq. (17). We have

$$\partial_t \mathcal{F}(\xi,t) = -A(u) + \partial_t Q \tag{18}$$

and

$$\partial_\xi \mathcal{F}(\xi,t) = \partial_\xi Q - \sqrt{2[A(\xi) - A(u)]}. \tag{19}$$

Substituting Eq. (16) into Eq. (13) yields

$$\partial_t Q = \xi^s + \tfrac{1}{2}(\partial_\xi Q)^2 - \partial_\xi Q\sqrt{2[A(\xi) - A(u)]} + A(\xi) - \xi S(t)\{-\sqrt{2[A(\xi) - A(u)]} + \partial_\xi Q\}. \tag{20}$$

In order to get rid of $\sqrt{2[A(\xi) - A(u)]}$ we should put

$$\partial_\xi Q - S\xi = 0 \quad \text{or} \quad Q = \tfrac{1}{2}\xi^2 S. \tag{21}$$

The remainder of Eq. (20) gives

$$\dot{Q} = \xi^s + A(\xi) + \tfrac{1}{2}Q_\xi^2 - S\xi Q_\xi. \tag{22}$$

We put

$$A(\xi) = \frac{s\xi^2}{2} - \xi^s \tag{23}$$

and find from the above two equations

$$\frac{\xi^2 \dot{S}}{2} = \frac{s\xi^2}{2} - \frac{1}{2}S^2\xi^2. \tag{24}$$

Equations (14) and (15) provide this equation to hold.

IV. MOMENTS AND CRITICAL TIME

The moments of the bivariate spectrum $c_{g,\sigma}(t)$ are introduced as follows:

$$M_{r,\rho}(t) = \sum_{g,\sigma} c_{g,\sigma}(g)_r(\sigma)_\rho, \tag{25}$$

where $(\nu)_k = \nu(\nu-1)\dots(\nu-k+1)$ is the Pochhammer symbol. At the pregelation stage $t < t_c$ [see Eq. (32)] they are expressed in terms of $\mathcal{F}$,

$$M_{r,\rho}(t) = \frac{\partial^r}{(\partial z)^r}\frac{\partial^\rho}{(\partial \xi)^\rho}\mathcal{F}\bigg|_{z=\xi=1}. \tag{26}$$

The special notation is introduced for the first three moments: $M_{0,0}(t) = N(t)$ is the average number of trees in the forest, $M_{1,0}(t) = M(t)$ is the average total order of the forest, and $M_{0,1}(t) = S(t)$ is its average total valence.

From Eq. (15), we find

$$S(t) = \sqrt{s}\tanh t\sqrt{s}. \tag{27}$$

It is possible to write down the equations for other moments. For the number concentration $N(t) = \sum_{g\sigma} c_{g,\sigma}(t) = \mathcal{F}(1,1;t)$ one has, $\dot{N} = 1 - \frac{1}{2}S^2$ or

$$N(t) = \tfrac{1}{2}[\sqrt{s}\tanh t\sqrt{s} - (s-2)t]. \tag{28}$$

The total mass concentration $M = M_{1,0} = \sum_{g,\sigma} gc(g,\sigma;t) = \partial_z\mathcal{F}(1,1;t)$ linearly grows with time, $M = t$. Other moments of interest are

$$\dot{M}_{0,2} = s(s-1) + (M_{0,2})^2 - 2SM_{0,2}. \tag{29}$$

The solution to this equation is

$$M_{0,2} = S + \gamma\tan\gamma t, \tag{30}$$

where

$$\gamma = \sqrt{s^2 - 2s}. \tag{31}$$

It is seen that $M_{0,2}(t)$ diverges at the critical time

$$t_c = \frac{\pi}{2\sqrt{s^2 - 2s}}, \tag{32}$$

which evidences on the sol-gel transition, i.e., the emergence of a giant tree. Next,

$$\dot{M}_{1,1} = s + M_{1,1}(M_{0,2} - S) = s + M_{1,1}\gamma\tan\gamma t \tag{33}$$

or

$$M_{1,1} = \frac{s}{\gamma}\tan\gamma t. \tag{34}$$

Next,

$$\dot{M}_{2,0} = (M_{1,1})^2. \tag{35}$$

Hence,

$$M_{2,0} = \frac{s^2}{\gamma^3}\int_0^t \tan^2\gamma t' dt' = \frac{s^2}{\gamma^3}[\tan\gamma t - \gamma t]. \tag{36}$$

The above results are valid only at the precritical stage. It is important, however, to mention that $S(t)$ [Eq. (15)] describing the dependence of the total valence of the forest holds during the whole coalescence process $0 \leqslant t < \infty$. We return to the discussion of this issue in Sec. VI.

V. GEL COMES UP

A. Order parameter

According to the definition of the average total valence Eqs. (14) and (19), we have

$$\tilde{S}(t) = S(t) - \sqrt{2[A(1) - A(\tilde{u})]}, \tag{37}$$

where $\tilde{u}(t) = u(1,t)$. We immediately see that the identity $\tilde{S} = S$ is broken unless $\tilde{u} \neq 1$. Below we show that at $t > t_c$ Eq. (17) has a solution different from $\tilde{u} = 1$ at $t > t_c$. In analogy with the theory of phase transition the function $W(t) = 1 - \tilde{u}(t)$ is referred to as the order parameter, because the deviations of all moments of the order distribution from those at the precritical period appear when $W(t) \neq 0$. The function $W(t) = 0$ at the pregelation period [see Eq. (46)].

For the analysis of the critical behavior we find u at ξ close to 1. To this end we introduce

$$B(\zeta) = \frac{s-2}{2} - A(\zeta) \tag{38}$$

and expand $B(\zeta)$ over $1-\zeta$,

$$B(\zeta) = \frac{s(s-2)}{2}(1-\zeta)^2 - \frac{s(s-1)(s-2)}{6}(1-\zeta)^3 + \ldots. \tag{39}$$

In terms of variables $1-\zeta = \eta$, $1-\xi = \theta$, and $1-u = w$ the integral in Eq. (17) is rewritten as

$$t = \int_\theta^w \frac{d\eta}{\sqrt{2[B(w) - B(\eta)]}}. \tag{40}$$

Now we replace the integration variable $y^2 = 2B(\eta)$ and obtain

$$t = \int_{\sqrt{2B(\theta)}}^{\sqrt{2B(w)}} \frac{y\,dy}{B'[\eta(y)]}\frac{1}{\sqrt{2B(w) - y^2}}, \tag{41}$$

with $\eta(y)$ being the root of equation $y^2 = 2B[\eta(y)]$. It is easy to see that

$$\lim_{y\to 0} \frac{y}{B'[\eta(y)]} = \frac{1}{\gamma}. \tag{42}$$

Hence, Eq. (41) can be rewritten as

$$t = \frac{1}{\gamma}\arccos[\sqrt{B(\theta)/B(w)}] + \int_{\sqrt{2B(\theta)}}^{\sqrt{2B(w)}} \frac{R(y)dy}{\sqrt{2B(w) - y^2}}, \tag{43}$$

or

$$\sqrt{B(w)}\cos[\gamma(t - Z)] = \sqrt{B(\theta)}, \tag{44}$$

where

$$Z(w,\theta) = \int_{\sqrt{2B(\theta)}}^{\sqrt{2B(w)}} \frac{R(y)dy}{\sqrt{2B(w) - y^2}}, \tag{45}$$

$R(y) = y/B'[\eta(y)] - 1/\gamma$, and $\theta = 1 - \xi$. At $\theta = 0$ there are two solutions to Eq. (44),

$$w = 0 \qquad \text{and} \qquad t - Z(W,0) = t_c \quad (t_c = \pi/2\gamma), \tag{46}$$

the first of which corresponds to the pregelation regime, while the second one describes the time evolution of the order parameter at the postgelation stage.

Equations (44) and (46) are still exact.

B. Postcritical moments

The postcritical behavior of the moments can be readily found from Eqs. (18) and (19). Equation (19) yields

$$\tilde{S}(t) = S(t) - \tilde{\sigma}(t), \tag{47}$$

where

$$\tilde{\sigma} = \sqrt{(s-2) - 2A(\tilde{u})} = \sqrt{2B(\tilde{u})} \tag{48}$$

is the total valence of the gel. On putting $\xi = 1$ in Eq. (18) we come to the equation for $\tilde{N}(t)$,

$$\frac{d\tilde{N}}{dt} = \frac{1}{2}\dot{S} - A(\tilde{u}), \tag{49}$$

The integration of this equation gives

$$\begin{aligned}\tilde{N}(t) &= \frac{1}{2}S(t) - \int_0^t A[\tilde{u}(t')]dt' \\ &= \frac{1}{2}\sqrt{s}\tanh(\sqrt{s}t) - \frac{s-2}{2}t + \int_0^t B[\tilde{u}(t')]dt'.\end{aligned} \tag{50}$$

The average order of the forest is then found from Eq. (6) by multiplying both its sides by $c_{g,\sigma}$ and summing over all g and σ,

$$(s-2)\tilde{M}(t) = \tilde{S}(t) - 2\tilde{N}(t). \tag{51}$$

Equations (47), (50), and (51) yield

$$\tilde{M} = t - \frac{1}{s-2}\left\{\sigma(t) + 2\int_0^t B[\tilde{u}(t')]dt'\right\}. \tag{52}$$

Equation (52) allows us to find the average order of the giant component,

$$\mu(t) = t - \tilde{M} = \frac{1}{s-2}\left\{\sigma(t) + 2\int_0^t B[\tilde{u}(t')]dt'\right\}. \tag{53}$$

C. Near the critical point

The ratio $y/B'[\eta(y)]$ can be expanded in powers of y,

$$\frac{y}{B'(\xi(y))} = \frac{1}{\sqrt{s(s-2)}} + \frac{(s-1)y}{3s(s-2)} + \ldots = \frac{1}{\gamma} + R(y). \tag{54}$$

At small y we find

$$R(y) \approx \frac{(s-1)y}{3s(s-2)} \tag{55}$$

and [see Eq. (45)]

$$Z \approx \frac{(s-1)}{3\gamma}\sqrt{2[B(w) - B(\theta)]}. \tag{56}$$

In the limit $1 \gg B(w) \gg B(\theta)$ and $|t - t_c| \ll 1$, we can replace $\sqrt{2B(x)} \approx \gamma x$ and $Z \approx (s-1)w/3$. Then,

$$w[\cos(\gamma t - \gamma Z)] = \theta. \tag{57}$$

Now it is possible to formulate a simple algebraic equation for w

$$\frac{\gamma(s-1)}{3}w^2 + w\cos\gamma t - \theta = 0. \tag{58}$$

In deriving this equation we neglected $B(\theta)$ in Eq. (56). Equation (58) is readily solved to give

$$w = \frac{-\cos\gamma t \pm \sqrt{\cos^2\gamma t + 4a\theta}}{2a}, \tag{59}$$

where $a = (s-1)\gamma/3$. The positive solution should be retained. At $\theta = 0$,

$$w = W = \frac{-\cos\gamma t + |\cos\gamma t|}{2a}. \tag{60}$$

At $t < t_c$ (the precritical stage) $B(w) = 0$ and at $t > t_c$ the first term becomes positive and thus $w > 0$. Near $t = t_c$ we approximate $\cos\gamma t \approx \pi\tau/2$, where the parameter $\tau = (t_c - t)/t_c$ characterizes the deviations from the critical point,

$$\int_0^t B(t')dt' \approx \frac{3t_c\pi^2\tau^3}{8(s-1)^2}\theta(\tau) \tag{61}$$

and

$$\sigma \approx \frac{3\pi\tau}{s-1}\theta(\tau). \tag{62}$$

Here $\theta(\tau)$ is the Heaviside step function. As follows from Eqs. (47), (50), and (52) the moments $\tilde{S}(t)$ and $\tilde{M}$ have a cusp at $t = t_c$. The moment $\tilde{N}$ has a disruption at $t = t_c$ only in the third derivative over time.

The function $\xi\partial_\xi\mathcal{F}$ generates $\sigma c_\sigma(t)$. The valence spectrum is thus expressed through the contour integral,

$$c_\sigma(t) = \frac{1}{2\pi i\sigma}\oint\frac{\mathcal{F}'(\zeta)d\zeta}{\zeta^\sigma}. \tag{63}$$

Equation (12) applies for transforming the integral on the right-hand side of this equation,

$$c_\sigma(t) = -\frac{1}{2\pi i\sigma}\oint\frac{\sqrt{2[A(\zeta) - A(u)]}d\zeta}{\zeta^\sigma}. \tag{64}$$

The asymptotic form of the spectrum $c_\sigma(t)$ at large σ is closely related to the analytic properties of the generating function $\mathcal{F}(\zeta,t)$ at $\zeta = \zeta_0(t)$, where $\zeta_0(t)$ is the position of the singularity of the integrand in Eq. (64) nearest to the origin of coordinates in the complex plane ζ [30]. At $\theta \ll w \ll 1$ we find [see Eq. (38)]

$$\sqrt{2[A(\zeta) - A(u)]} = \sqrt{2[B(w) - B(\theta)]} \approx \sqrt{2B(w)} \approx \gamma w(\theta). \tag{65}$$

In the above according to Eq. (57) we ignored the term $B(\theta)$. Hence,

$$\begin{aligned} c_\sigma(t) &= -\frac{\gamma}{2\pi i(4a\sigma)}\oint\frac{\sqrt{\cos^2\gamma t + 4a(1-\zeta)}d\zeta}{\zeta^\sigma} \\ &\approx -\frac{\gamma}{2\sqrt{a}\sigma}\frac{1}{2\pi i}\oint\frac{d\zeta}{\zeta^\sigma}\sqrt{\zeta_0 - \zeta}, \end{aligned} \tag{66}$$

where

$$\zeta_0(\tau) = 1 + \frac{\pi^2\tau^2}{16a} \tag{67}$$

is the position of the singularity of the integrand. It is seen that $\zeta_0 \to 1$ as $\tau \to 0$ (or $t \to t_c$), which means that at $t = t_c$ the spectrum becomes algebraic. Finally, we obtain

$$c_\sigma = \frac{\pi}{4\sqrt{a}}\sigma^{-5/2}\exp(-\sigma\pi^2\tau^2/16a). \tag{68}$$

In order to find the order spectrum we should put $\sigma = g(s-2)$ (at $g \gg 1$).

D. Final stage

At $t \to \infty$, $\tilde{u} \to 0$, and we have

$$F(z,\xi,t) = \xi\sqrt{s} - \sqrt{\xi^2 s - 2z\xi^s}. \tag{69}$$

From here we find the spectrum

$$\begin{aligned} c_g(\infty) &= \frac{(2g-2)!}{[g(s-2)+2]s^{g-1/2}2^{g-1}(g-1)!g!} \\ &\approx \frac{1}{(s-2)\sqrt{2\pi g^5}}\left(\frac{2}{s}\right)^{g-1/2}, \end{aligned} \tag{70}$$

and

$$\tilde{S}(\infty) = \sqrt{s} - \sqrt{s-2}, \tag{71}$$

$$\tilde{N}(\infty) = \frac{\sqrt{s}}{2} - \int_0^1\sqrt{s\zeta^2 - 2\zeta^s}d\zeta. \tag{72}$$

Equation (6) together with Eqs. (71) and (72) yields

$$\tilde{M}(\infty) = \frac{2}{s-2}\int_0^1\sqrt{s\zeta^2 - 2\zeta^s}d\zeta - \frac{1}{\sqrt{s-2}}. \tag{73}$$

VI. RESULTS AND DISCUSSION

The main results of this paper can be summarized as follows:

(1) The kinetic equation [Eq. (3)] describing the time evolution of the order-valence spectrum of linked components of a random forest in the presence of a source of monovalent vertices is formulated in terms of concentrations of linked trees of given order g and valence σ (the spectrum of trees). The equation for bivariate generating function for the spectrum [Eq. (13)] is derived and solved exactly [see Eq. (16)] for initially zero graph.

(2) It is shown that the moments of the tree spectrum can be expressed in terms of the derivatives of the generating function. Moreover, the evolution equations for them are closed, i.e., the equation for the mth moment contains only lower moments. This result holds only for the finite interval of time $t < t_c$, where t_c is the critical time that appears automatically in the exact analytical expression for the second moment [see Eqs. (30) and (31)].

(3) The precritical time dependencies of the lowest moments of the spectrum are found for zero initial conditions. It is demonstrated that the present model successfully reproduces the emergence of the giant component at the critical time $t_c = \pi/2\sqrt{s(s-2)}$.

(4) The time dependence of the moments at the postcritical period is shown to be expressed in terms of the order parameter $W(t)$. The equation for $W(t)$ is formulated and analyzed [Eq. (40)]. It is shown that there are two solutions to this equation describing the precritical and the postcritical regimes, respectively. At the precritical period $W(t) = 0$. The postcritical dependence is given by the inversion of Eq. (17). The time dependence of the order parameter is displayed in Fig. 1. Other lower moments of the tree spectrum (the number concentration $N(t)$, the average order $M(t)$, and average valence $S(t)$) are expressed through the order parameter. Their time dependencies are shown in Fig. 2. Figure 3 displays the average graph valence at different s. The remarkable feature of these curves is the cusp at the transition point. This cusp does not appear in $N(t)$. The reason for this is clear, the expression for $N(t)$ contains the integral $\int B(t)dt \propto \tau^3$ rather

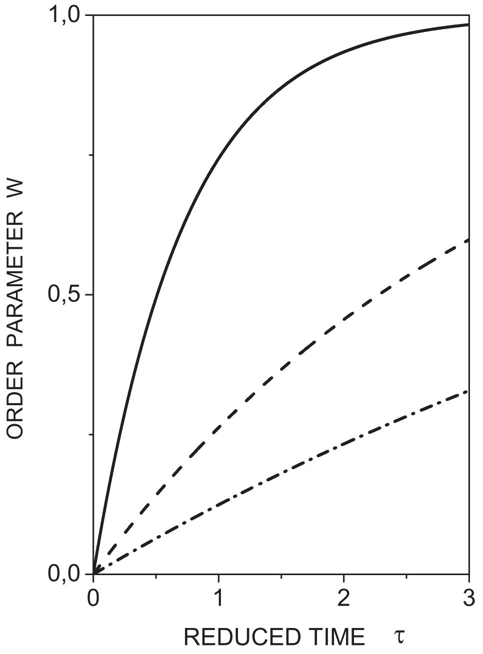

FIG. 1. The phase transition. Shown is the dependence of the order parameter $W(\tau)$, on the reduced time $\tau = (t - t_c)/t_c$, for different values of the vertex valence s. Solid, dash, and dash-dot curves correspond to $s = 3$, $s = 5$, and $s = 7$, respectively.

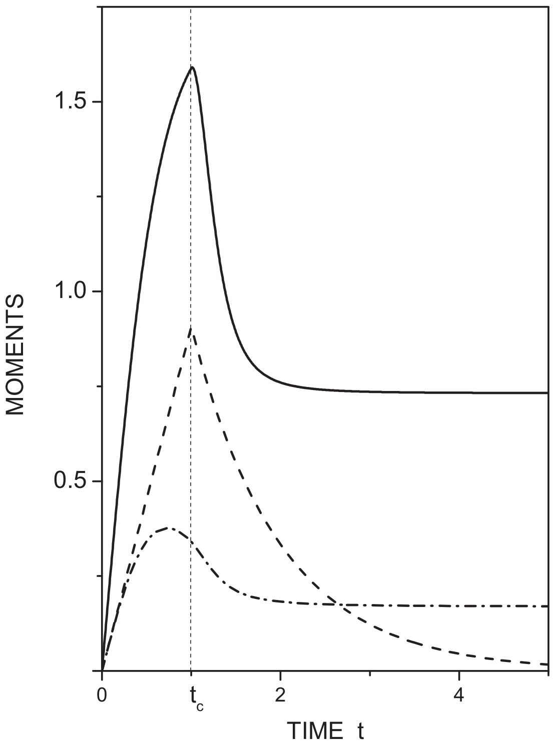

FIG. 2. Average forest valence $\tilde{S}(t)$ (solid curve), average forest order $\tilde{M}(t)$ (dash curve), and the average number of trees $\tilde{N}(t)$ (dash-dot curve) are shown as the functions of time. First two moments $\tilde{S}(t)$ and $\tilde{M}(t)$ have clearly expressed cusps at $t = t_c$, whereas $\tilde{N}(t)$ does not display any singularity at the critical time. The explanation of this fact is given in the text.

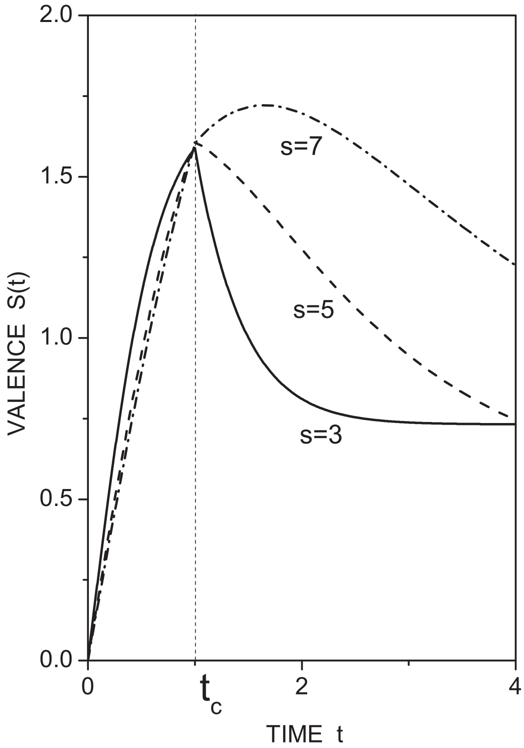

FIG. 3. Time dependence of the average valence $\tilde{S}(t)$ of the forest at different values of the vertex valence s.

than $\sqrt{B(t)} \propto \tau$ [Eqs. (61) and (62)]. Respectively, the cusp in this case is expressed much weaker (see Fig. 2).

(5) The critical behavior of the spectrum does not differ from that of the free coalescing forest. The asymptotic spectrum is shown to comprise two parts: the algebraic multiplier (standard $g^{-5/2}$ dependence) and an exponential factor [see Eq. (68)]. The coefficients in this expression is connected with the second moment of the spectrum. At the critical time the spectrum is algebraic $c_g \propto g^{-5/2}$.

(6) It is important to emphasize that after the critical moment the spectrum again acquires the exponential factor. At the final stage of the coalescence process a steady-state spectrum forms. Nothing like this is observed in the model of source enhanced coalescence [29]. It is remarkable that this final spectrum is not algebraic, $c_g \propto g^{-5/2} \exp(-\alpha g)$.

The structural properties of an evolving random graph change in adding extra edges one at a time. In particular, the distribution of the linked components in such graphs becomes broader with time. The distribution evolves to algebraic one and a giant component emerges after a finite period of time (at $t > t_c$). This pattern is common for many types of random graphs.

The random graph with finite valence of the vertexes is not exclusion. In contrast to widely studied random graphs the time evolution of such graphs is described in terms of bivariate concentrations of linked components over their order g and valence σ. The coalescence rate depends on the total valences of the coalescing components. Each extra edge kills randomly a couple of free valences in the graph. Respectively, the graph jumps from the state with k free valences to the state with $k - 2$ valences. The analogy with the reaction of

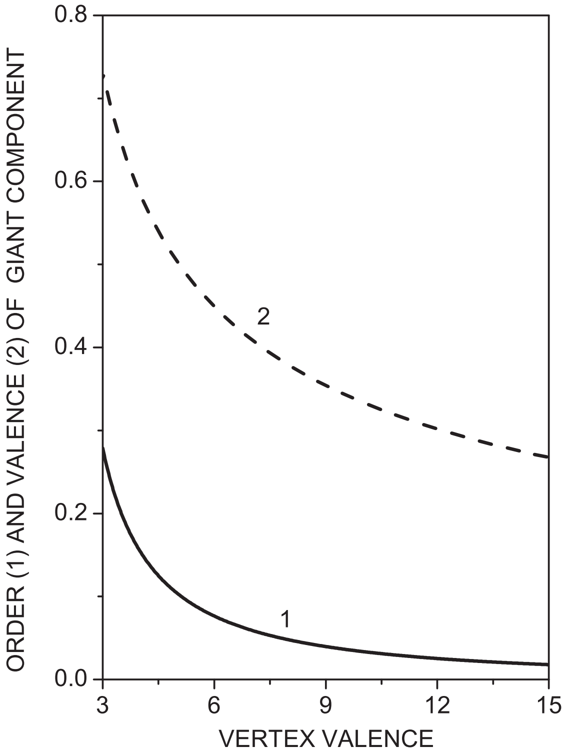

FIG. 4. Shown are the moments $\tilde{S}(\infty)$ and $\tilde{M}(\infty)$ of the residual forest vs. s.

annihilation is clearly seen. The kinetic equation for the total valence thus has the form

$$\dot{\bar{S}} = s - \bar{S}^2. \tag{74}$$

The same result comes from the differentiation of the generating function [Eq. (13)]. At the postcritical period the part of the valent vertexes belongs to the giant component. The valence and order of the gel have been found in Sec. V for the giant and small components. The results are displayed in Figs. 2–4. As expected the critical time is shorter for the graphs with larger vertex valence s.

In the case of initially empty graph the present model reminds the Flory model of polymerization. The point is that at large M when we can ignore cycling the linking of bare vertexes leads to the formation of trees, in which the valence and the mass of each component are connected. Indeed, the number of edges in the tree of mass g is $g - 1$. Each edge reduces the valence of the component by 2. Hence, the valence of the component containing g vertices is $gs - 2(g - 1) = g(s - 2) + 2$. Of course, Eq. (6) reproduces this result.

Another scenario where the linked component of very large order $g > G$ are instantly removed from the graph. Their valence is entirely suppressed and they are unable to influence the run of the coalescence process. This scenario is analogous to the Stockmayer scenario of polymerization [8]. The postcritical solution for the source enhanced systems in this case is yet unknown, although the attempt to do this was reported in Ref. [31]

VII. CONCLUSIONS

In this paper the model describing the dynamics of the coalescing random graph has been proposed. In contrast to commonly accepted approach relying upon the analogy between graph coalescence and the coagulating system with the product kernel [6] the present model considers the coalescence rate proportional to the product of the total valences of coalescing components. The dependence of the coalescence rate on the masses does not enter explicitly into the kinetic equation. This equation has been then formulated in terms of the bivariate generating function that has been found exactly for arbitrary initial conditions. In turn, this result allowed for finding the time dependencies of the valence of giant component and its mass.

The formal chemical kinetic approach Eq. (3) allows for consideration of thermodynamically large systems, where the total numbers of each linked component is of the total order of the graph M. This means that the giant component is indistinguishable in the thermodynamic limit, i.e., its concentration goes to zero as $M \to \infty$. Still we can distinguish this giant component from indirect evidences: its total order and valence contribute to the respective balances after the critical time.

There exists a more straightforward (but much more complex) approach to the time evolution of random graphs. I mean the approach of Ref. [22] that operates with the probability to find the graph in a given definite state. This probability obeys a master equation that had been solved by me for "normal" random graphs in Ref. [22]. It is likely that this approach can also be applied to the problem considered above.

ACKNOWLEDGMENT

This research was supported by the Ministry of Education and Science of Russia (Grant No. 14.607.21.0058).

---

[1] P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[2] F. Harary and E. M. Palmer, *Graphical Enumeration* (Academic Press, New York, London, 1973), pp. 1–271.
[3] J. Pitman, Combinatorial stochastic processes. Technical Report 621, Dept. Statistics, UC Berkeley, 2002. Lecture notes for St. Flour course (2002).
[4] J. Pitman, J. Combinat. Theory A **85**, 165 (1999).
[5] B. Bollobas, *Random Graphs* (Academic Press, London, 1985).
[6] F. Leyvraz, Phys. Rep. **383**, 95 (2003).
[7] P. J. Flory, J. Am. Chem. Soc. **63**, 3083 (1941).
[8] W. H. Stockmayer, J. Chem. Phys. **11**, 45 (1943).
[9] M. Falk and R. E. Thomas, Can. J. Chem. **52**, 3285 (1974).
[10] J. L. Spouge, Can. J. Chem. **62**, 1262 (1984).
[11] J. L. Spouge, J. Stat. Phys. **38**, 573 (1985).
[12] R. M. Ziff and G. Stell, J. Chem. Phys. **73**, 3492 (1980).
[13] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).
[14] E. Ben–Naim and P. L. Krapivsky, J. Phys. A **37**, L189 (2004).

[15] Y. Amit and D. Geman, Neural Comput. **9**, 1545 (1997).
[16] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[17] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[18] G. Grimmett, *What Is Percolation?* (Springer, Berlin, Heidelberg, 1999).
[19] D. Stauffer, Phys. Rep. **54**, 1 (1979).
[20] M. V. Simkin and V. P. Roychowdhury, Phys. Rep. **502**, 1 (2011).
[21] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **71**, 026129 (2005).
[22] A. A. Lushnikov, J. Phys. A **38**, L777 (2005).
[23] A. A. Lushnikov, Phys. Rev. E **91**, 022119 (2015).
[24] C. Kollath, A. M. Lauchli, and E. Altman, Phys. Rev. Lett. **98**, 180601 (2007).
[25] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).
[26] M. E. J. Newman and D. J. Watts, Phys. Lett. A **263**, 341 (1999).
[27] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[28] C. F. Moukarzel, Phys. Rev. E **60**, R6263(R) (1999).
[29] A. A. Lushnikov, Phys. Rev. E **74**, 011103 (2006).
[30] A. A. Lushnikov, J. Colloid Interface Sci. **48**, 400 (1974).
[31] A. A. Lushnikov, J. Phys A. Math. Theor. E **41**, 072001 (2008).