Annealed and quenched disorder in sand-pile models with local violation of conservation

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In this paper we consider the Bak, Tang, and Wiesenfeld (BTW) sand-pile model with local violation of conservation through annealed and quenched disorder. We have observed that the probability distribution functions of avalanches have two distinct exponents, one of which is associated with the usual BTW model and another one which we propose to belong to a new fixed point; that is, a crossover from the original BTW fixed point to a new fixed point is observed. Through field theoretic calculations, we show that such a perturbation is relevant and takes the system to a new fixed point.

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I. INTRODUCTION

Power law behavior is observed in many natural phenomena without tuning external parameters such as temperature. The behavior is widely seen in nature such as earthquakes, stock markets, and forest fires [1–3]. In these phenomena, although the interactions are short ranged, the dynamics naturally takes the system to a critical state where a small perturbation may lead to a large event and therefore a long-ranged correlation is found within the system. Such systems are called self-organized critical (SOC) systems. Bak, Tang, and Wiesenfeld (BTW) [4] proposed a very simple model that had the main features of SOC systems. The model, which is called the BTW sand-pile model, together with its variants, has given a very helpful framework to study self-organized criticality. After the Abelian structure of the model was found by Dhar [5], the model has been usually called the Abelian sand-pile model (ASM).

Much analytical and computational work has been done on the model. The exact results obtained in the model are mostly on static properties of the model. To name a few one can mention a number of steady state configurations, height probabilities, and height correlation functions [5–8]. These achievements are obtained due to the Abelian structure of the model. Also the one-site probabilities of the height variable and the two-point correlation functions have been derived analytically [6,8–10] and the relation of the model to $c = -2$ conformal field theory (CFT) is established [11–13]. However, few exact results are in hand on dynamical properties of the model and they are mostly obtained through simulations. The avalanche distribution is studied in [14–18] and the effect of insertion of dissipation on these probability distributions is studied in [12,19–22], although some efforts have been made to calculate the exponents using some assumptions or with the renormalization scheme [23,24].

The BTW model is a nondissipative model; that is, in each step of dynamics the number of sand grains is conserved. Only at boundaries of the system are sand grains allowed to leave the system. It has been shown that adding bulk dissipation to the system takes it off the criticality and no finite size scaling is observed after a certain length scale [12,20,25]. In [20] the authors have considered a model in which the conservation of sand grains is violated in two different ways: locally or globally. In global violation, all the sites dissipate sand grains. However, in local violation of sand grain conservation, at each time step sand grains may be dissipated or be produced in a way that on average the number of sand grains is conserved. Therefore the total number of sand grains fluctuates. These fluctuations are considered to be due to some internal degrees of freedom and have an annealed randomness; that is, at each time step it is decided how many sand grains are dissipated or produced. They have shown that such a system is still critical but the exponents are different, therefore it belongs to a different universality class.

In many cases, these internal degrees of freedom may have a quenched randomness. Some sites may always dissipate and some others may always produce sand grains. In this paper we consider a BTW model in which some dissipative sites (sinks) and some productive sites (sources) exist. Through investigating probability distributions of avalanche properties, we find that in such a system the system is still critical, but the obtained exponents are different from the BTW model and the model introduced in [20]. In the end, we perform a field theoretic calculation to show that such a perturbation to the original BTW model actually takes the system to a new fixed point.

II. BTW MODEL WITH SINK AND SOURCE SITES

We consider the well-known BTW model on a $L \times L$ square lattice. The height variables h_i are assigned to each site on the lattice. These variables are usually regarded as the number of grains of sands in the sites and can take their values from the set $\{1,2,3,4\}$. The dynamics of the system is as follows: at each time step a grain of sand is added to a random site i. As a result the height of the site may become more than the threshold value $h_c = 4$ and the site becomes unstable and topples: four grains of sand are transferred from the site to the neighboring sites. This may cause some of the neighboring sites to become unstable and a chain of topplings may happen in the system, which is called an avalanche. While all the bulk sites are conservative and no grains of sand are lost, the boundary sites are dissipative; that is, when they topple some grains of sand leave the system.

The bulk dissipation in the BTW model is considered in several papers [12,20,25], however a system with sites that have negative dissipation has been considered much less often. Negative dissipation causes the system to be unstable and therefore cannot be studied. In [20] it has been considered that during each toppling the number of sand grains removed from the toppled site is $4 + \xi$, where ξ is a random number taking integer values between -3 and 3 with equal probabilities. In this way the local conservation of the sand grains is violated, however on average no site dissipates sand grains or adds sand grains to the system. It was observed that with this modification the distribution function of the avalanche sizes still has a power law behavior but with a different exponent from the original BTW model. In this model the local conservation of sand grains is violated using an annealed randomness; that is, during the dynamics of the system, every time a site topples it is decided if the toppling site is going to add a few grains of sand to the system or a few grains of sand are dissipated. It is possible to violate the conservation through a quenched randomness. In our model we introduce such a randomness to the system in the following way: we take an ordinary BTW on a square lattice and let a fraction of the sites of the lattice have positive dissipation, and the same fraction of lattice sites have negative dissipation. We call these sites sinks and sources, respectively. The source and sink sites are distributed randomly throughout the lattice at the beginning and are not changed during the dynamics. To avoid instability, we cannot take the number of sources to be more than the number of sinks. Also, if the number of sinks are taken to be more than sources, their effect may become dominant in large avalanches, and as dissipation is a relevant perturbation to the BTW model [12,20,25] it takes the system off the criticality. In this way, the bulk of the system is conservative on average, however the boundary sites are dissipative as in usual the BTW model.

The usual way to have a dissipative site is to change its threshold height: the site becomes unstable when its height gets more than $4 + m$, and as a result of toppling it loses $4 + m$ grains of sand and only four grains of sand are transferred to the neighboring sites, then during the toppling m grains of sand are lost (or $-m$ grains of sand are produced in the case $m < 0$). In this way sinks or sources can be introduced in the system. But such a construction has a problem. For simplicity take $m = \pm 1$; that is, in sinks and sources just one grain of sand is lost or produced. Now, in order to activate a sink site its height has to become more than 5, while for a sink site the height should become only more than 3. Therefore source sites are activated more easily and more frequently, leading to instability. We have observed such instabilities in simulations. In an improved model, we constructed the sinks and source sites in the following way: when the height of a sink (source) site becomes more than the critical height $h_c = 4$ it loses four grains of sand and three (five) grains of sands are added to the neighboring sites. In the case of sink (source) sites, one of the neighboring sites is chosen randomly and zero (two) grains of sands are added to it. Now both source sites and sink sites are activated, arriving at the height 5.

If the sink and source sites are distributed randomly in the bulk, there is a finite probability that a few source sites gather in a region in which no sink sites exist. In such cases the source sites may activate one another repeatedly and an avalanche may never end. To avoid such situations we have considered that source and sink sites come in pairs. In other words we add a number of pairs of adjacent sites which are sink and source sites. Such a model is stable, at least for a small number of pairs, and we are able to collect numerical data.

III. NUMERICAL DATA

To begin, we have considered a system very similar to the system discussed in [20] and repeated their simulations. We have considered square lattices with sizes up to 1024 and have examined the statistics of at least 10^6 nonzero avalanches. To have more control on the system, we have considered that each toppling can be nonconservative with a probability p, and if it is nonconservative it will dissipate or produce one grain of sand with equal probabilities. We will call this model local nonconservative BTW with annealed randomness.

For very small values of p, the system is more or less like BTW, however the system considered by [20] corresponds to relatively large amounts of p. Figure 1 shows the probability distribution of avalanche sizes for different values of p and for different system sizes. In Fig. 1(a), the probability distribution function is sketched for a system with size $L = 1024$ and for different values of p ranging from 2×10^{-4} to 1. Looking carefully one observes that in the graphs there are two distinct linear parts with slightly different slopes. Such a phenomenon was observed before, in the cases where a relevant perturbation is added to the original BTW model [26]. This phenomenon was not seen in the model considered in [20]; the reason will become clear below.

These two regimes are more clear in Fig. 1(b), where the same probability distribution is sketched for $p = 0.02$ and system sizes from 128 to 1024. Finite size scaling is observed in the system and therefore still the largest length scale of the system is the system size; that is, the system is critical, although there exists an intermediate size scale, s^*, which determines where the second linear portion of the diagram starts. As observed in the diagrams, this point shows no dependence on L, but moves forward as we change p.

Let us call the slope of the first and second linear parts ν_1 and ν_2. We have used the method introduced in [27] to find the exponents. For large values of p the first linear part is too short to have a reliable estimate for ν_1, and for very small p the second linear part becomes very short and no reliable value for ν_2 is at hand. Table I shows the obtained values of these exponents where we had relatively small error bars. The exponent ν_1 gradually increases as p is increased, while the second exponent ν_2 is more or less constant.

The behavior of the exponents associated with the smaller avalanches can be interpreted as follows: there is little chance to have a nonconservative toppling in a small avalanche (this is why in [20] this part of the system was not observed; in their model even in small avalanches the local violation of the sand grains exists). For example, for $p = 10^{-3}$ the avalanche size should be of the order of 10^3 so that a nonconservative toppling is found. Therefore the behavior of the smaller avalanches should be given by the usual BTW model. In the BTW model, the exponent associated with the avalanche size, τ_s, depends on the size of the system [28] through the relation

$$\tau_s(L) = \tau_s(L \to \infty) - \frac{c}{\ln L}, \qquad (1)$$

where L is the size of the system and c is a constant. In fact it has been shown that not the size of the system but the number of topplings needed so that a grain of sand is lost is the important parameter that controls the value of the

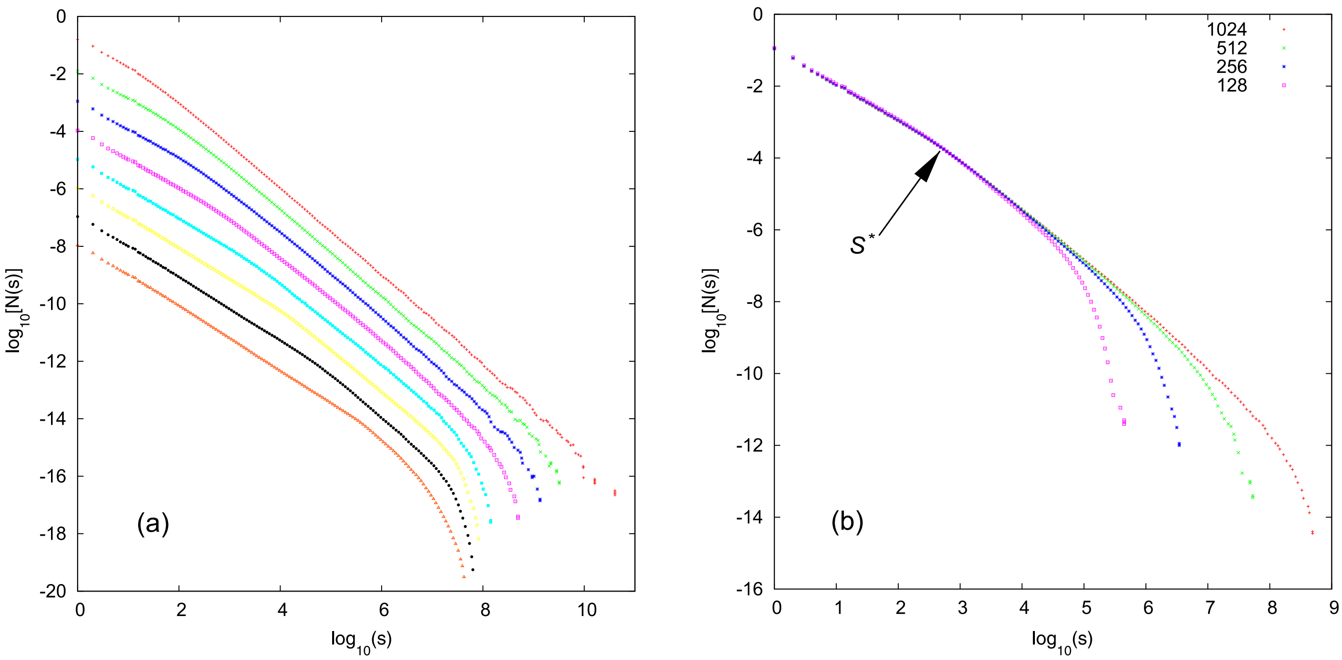

FIG. 1. (Color online) (a) The log-log plot of the distribution function of avalanche sizes for a system of size 1024 with $p = 1, 2.5 \times 10^{-1}$, 6.25×10^{-2}, 2×10^{-2}, 4×10^{-3}, 1×10^{-3}, 2×10^{-4}, and 0. The curves with $p < 10^{-1}$ are shifted downward. (b) The same plot for $p = 2 \times 10^{-2}$ and system sizes $L = 128, 256, 512$, and 1024.

exponent [29]. Therefore, in the model we have considered, the inverse of p plays the role of L in the above equation and the gradual changes of ν_1 are due to the behavior of the BTW model. It is observed that as $p \to 0$ the exponent ν_1 tends to $\tau_s(L)$, defined above. This means that if we consider systems with small sizes or focus on very small values of p the effect of system size on this exponent will become apparent.

Let us now consider the local nonconservative model with quenched randomness, where pairs of sink and source sites are added to the original BTW model. Here the control parameter p is the ratio of the number of sink and source pairs to the total number of sites. We have investigated very low pair densities, because if the number of source and sink sites becomes high the separation of time scales of input and output of mass, which is a key feature of SOC, is ruined. We have examined values of pair density ranging from about 10^{-4} to 10^{-2}. We have mainly focused on the avalanche size distribution function.

Figure 2 shows a typical log-log plot of such a system. In this figure the size of the lattice is $L = 1024$ and $p = 4 \times 10^{-3}$. As in the annealed random case, two distinct linear parts separating at the point s_c can be identified, and two exponents can be assigned to such graphs. We call the exponents associated with smaller avalanches τ_1 and the exponent associated with larger avalanches τ_2. Figure 3(a) shows the avalanche size distribution function for different values of pair density in a system with size $L = 1024$ and Fig. 3(b) shows the same plot for $p = 2 \times 10^{-2}$ and systems with different sizes. Clearly the system obeys finite size scaling and s_c, the characteristic size scale of the avalanches defined above, depends only on p.

To find the numerical estimates for the exponents τ_1 and τ_2 we have used the method introduced in [27]. The values obtained are shown in Table II. As in the annealed random case, the first exponent τ_1 gradually decreases as p is increased while τ_2 is nearly constant. The same argument as above can be applied in the quenched random case to interpret where the exponent τ_1 comes from and why it decreases as p becomes larger.

The second exponent τ_2 which determines the large scale behavior of the system is nearly constant and equal to 1.38 ± 0.03. This value is completely different with τ_1, even with its value at the limit $p \to 0$ and $L \to \infty$, which is 1.25. As the system obeys finite size scaling [see Fig. 3(b)] it is still at a critical point. This means that the addition of pairs of source and sinks should be a relevant perturbation that takes the system to a new fixed point with different exponents.

Perturbations that may take the BTW model to a new fixed point have been observed before [20,26]. For example, if we violate the local conservation law we will have a model with different scaling exponents [20], or if we add randomness to the way the toppled site redistributes its grains of sand we will arrive at the Manna model that may belong to a different universality class. It has been observed that even a small

TABLE I. Values of the exponent ν_1 and ν_2 for the local nonconservative BTW system with annealed randomness. The system size is 1024.

p	5×10^{-4}	2×10^{-3}	8×10^{-3}	2×10^{-2}	6.25×10^{-2}	2.5×10^{-1}	1
ν_1	1.10 ± 0.01	1.07 ± 0.01	1.03 ± 0.01	1.02 ± 0.02			
ν_2	1.45 ± 0.04	1.45 ± 0.03	1.42 ± 0.03	1.44 ± 0.03	1.46 ± 0.03	1.49 ± 0.03	1.53 ± 0.03

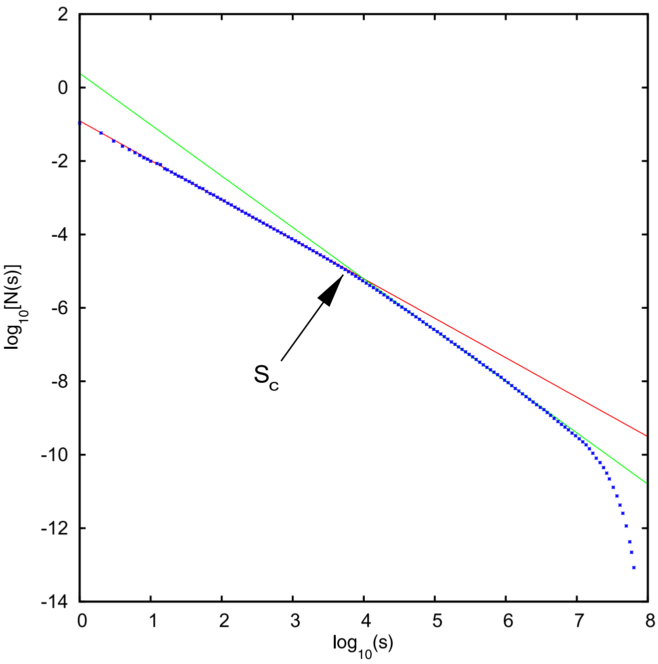

FIG. 2. (Color online) The log-log plot of the distribution function of avalanche sizes for a system of size 1024 with $p = 4 \times 10^{-3}$.

amount of Manna-type perturbation has an effect just like the one observed above: for small avalanches BTW exponents are found and statistics of larger avalanches obeys a power law with the exponents of the Manna model [26].

One might think that the effect we have observed is due to the fact that we have added some randomness to the theory: while redistributing the grains of sands of sources (sinks), a neighbor is selected randomly to have an extra grain (or no grains of sand). To test if the existence of source and sink pairs is the main cause of the new exponent, we have redefined our model in the following way: each grain of sand is supposed to be made of four quarter-grains. When a site has more than 16 quarter-grains, it topples. Sixteen quarter-grains of sand are removed from the site, and for ordinary sink and source sites 4, 3, and 5 quarter-grains of sand are added to all the neighbors of the site. In this way sink sites dissipate one whole grain of sand during a toppling and source sites add one grain of sand to the system during each toppling, and there is no randomness in the law of toppling. The new system has the same behavior as the one considered before with the very same exponents. Therefore stochasticity plays no role in the model.

We have also studied how s_c and s^* depend on pair density and probability of being nonconservative, respectively. Due to the above explanations, one expects in each case as p is increased that the maximum size of an avalanche that does not see the effect of pairs or being nonconservative will grow and hence s_c and s^* will be proportional to $1/p$. Of course, there will be a correction due to the fact that some sites may topple several times. In fact, one has to use the exponents related to conditional expectation values γ_{as} as introduced in [30]. According to the results in [22], this exponent is 0.94 ± 0.05. Figure 4 shows a log-log plot of the measured s_c and s^* versus p for 1024×1024 systems. The best fit to Fig. 4(a) reveals $s_c \sim p^{-u_q}$ with $u_q = 1.0 \pm 0.1$ and the best fit to Fig. 4(b) gives $s^* \sim p^{-u_a}$ with $u_q = 0.8 \pm 0.1$, which are consistent with the above argument within the error bars.

To conclude, we have found that adding source and sink to the BTW model causes the large-scale statistics of the system to change; the system still is at criticality, therefore we suggest that we have arrived at a new critical point. In the next section, using perturbative conformal field theory techniques, we find theoretical support for our suggestion.

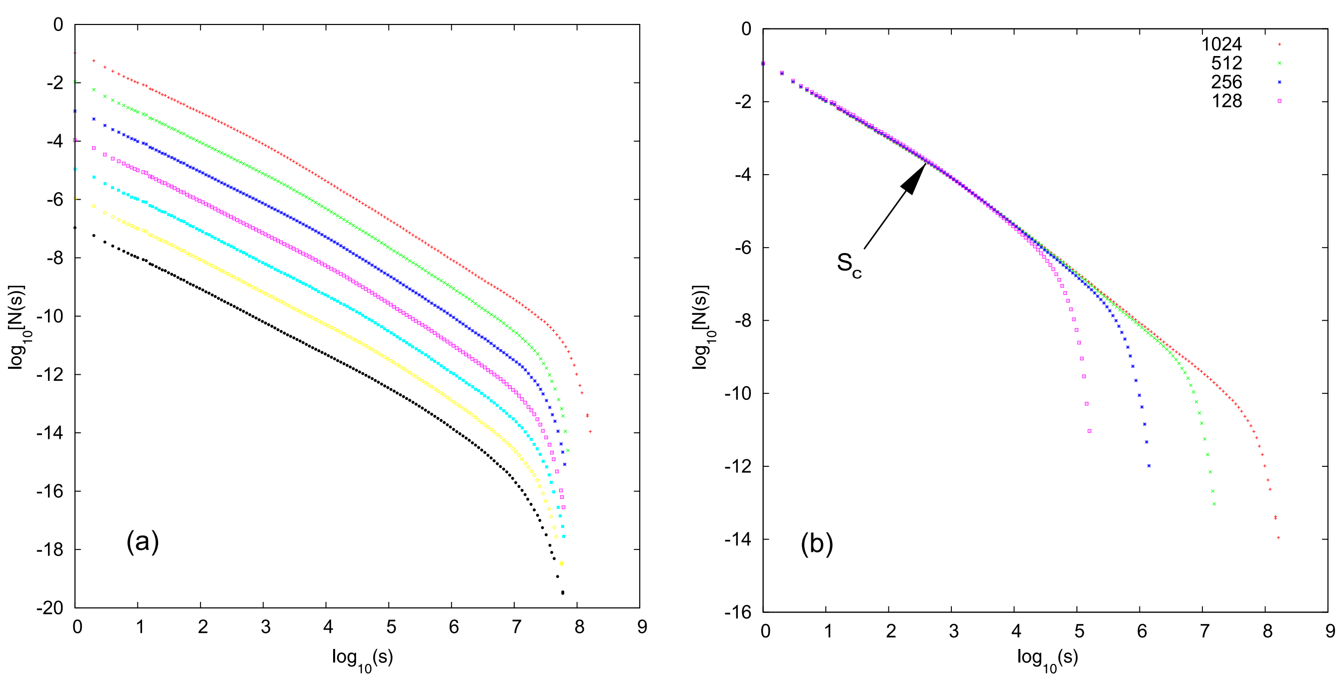

FIG. 3. (Color online) (a) The log-log plot of the distribution function of avalanche sizes for a system of size 1024 with $p = 2 \times 10^{-2}$, 8×10^{-3}, 4×10^{-3}, 2×10^{-3}, 1×10^{-3}, 5×10^{-4}, and 2×10^{-4}. The curves with $p < 10^{-1}$ are shifted downward. (b) The same plot for $p = 2 \times 10^{-2}$ and system sizes $L = 128$, 256, 512, and 1024.

TABLE II. Values of the exponent τ_1 and τ_2 for the system with quenched randomness.

p	5×10^{-4}	1×10^{-3}	2×10^{-3}	4×10^{-3}	8×10^{-3}	2×10^{-2}
τ_1	1.10 ± 0.01	1.08 ± 0.01	1.07 ± 0.01	1.05 ± 0.01	1.03 ± 0.01	1.02 ± 0.01
τ_2	1.38 ± 0.03	1.39 ± 0.03	1.39 ± 0.03	1.40 ± 0.03	1.37 ± 0.03	1.36 ± 0.03

IV. PERTURBATIVE CONFORMAL FIELD THEORY DESCRIPTION OF THE MODEL

Conformal field theory (CFT) has proved to be very powerful in the description of equilibrium critical models in two dimensions. Many statistical models have been studied by means of conformal field theory techniques [31]. The Abelian sand-pile model is one of such models that are found to have a close connection with a specific CFT model. It has been shown that in steady state ASM can take only a subspace of its configurations. These configurations, which are called recurrent configurations, happen with the same probability and their total number is just $\det \Delta$ [5], where the matrix Δ is the toppling matrix whose dimensions are equal to the number of sites in the sand-pile and the element Δ_{ij} is equal to the number of sand grains removed from the site i if the site j is toppled. In the BTW model the toppling matrix is the discrete Laplacian matrix; that is, $\Delta_{ii} = 4$ and $\Delta_{ij} = -1$ if i and j are neighbors and other elements of Δ are zero. Therefore the partition function of the system can be written as $Z = \det \Delta$. This partition function can be reexpressed as Gaussian integrals over Grassman variables:

$$Z = \det \Delta = \int \prod (d\theta_i d\bar{\theta}_i) \exp(\theta_i \Delta_{ij} \bar{\theta}_j), \qquad (2)$$

through which one is able to introduce an action for the model [32]. As the matrix Δ is the discrete Laplacian matrix in the BTW model, in the continuum limit, the above mentioned action becomes $S_0 = (1/\pi) \int dz\, d\bar{z} \partial\theta \bar{\partial}\bar{\theta}$, which is the action of $c = -2$ conformal field theory. It was stated that the connection of the two models (BTW and $c = -2$) was established because both of them had connections with the $q = 0$ Potts model and spanning trees.

Our model is a perturbation of the BTW model: there exist some sites that are dissipative and some other sites that are productive. In other words, effectively some of the diagonal elements of the matrix Δ are different from 4; we write these diagonal terms as $4 + m$, where m can take nonzero values, a positive value for m means that the site is a sink site, and a negative m indicates that the site is a source site (in fact, in the model we have introduced, we have changed Δ_{ij} for neighboring sites i and j, but it is equivalent with changing Δ_{ii} while keeping Δ_{ij} fixed). In this way, in the continuum limit, the action of the model will become

$$S = \frac{1}{\pi} \int dz d\bar{z} (\partial\theta \bar{\partial}\bar{\theta} + m(z)\theta\bar{\theta}) = S_0 + \int \epsilon(z)\phi(z), \qquad (3)$$

where $\phi = \theta\bar{\theta}$ and $\epsilon = (1/\pi)m$. We want to treat the above action as a perturbed action of $c = -2$ and find out if this perturbation is a relevant one or not. It is clear from field theoretic arguments that if m is positive for all sites (in all the space) the theory will lose its criticality. This has been shown through correlation functions of height variables in [12]. On the other hand, if m take only negative values, the system becomes unstable. We are interested in the case in which the variables $\epsilon(z)$ have a quenched randomness where their mean value is vanishing but have a nonzero variance. In the simplest case we take ϵ variables to be independent variables with probability distribution $P(\epsilon) \sim \exp(-\epsilon^2/2g_0)$. Here g_0 determines the width of the probability distribution; our goal is to determine the behavior of this parameter under a renormalization group (RG). If $g_0 = 0$ turns out to be a stable fixed point, then the above perturbation is irrelevant and one expects the model to be just the same as the BTW model. Otherwise there should be another fixed point that determines the critical behavior of the system.

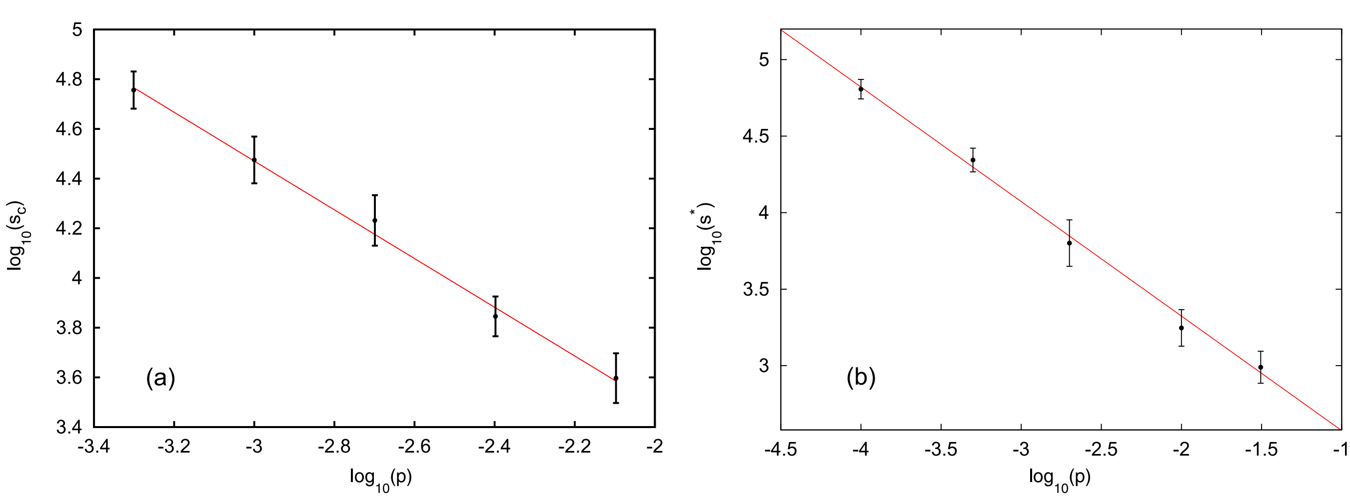

FIG. 4. (Color online) The log-log plot of characteristic avalanche size scale as a function of the probability p (a) in a quenched random nonconservative system and (b) in an annealed random nonconservative system.

One of the most powerful methods in such systems where a quenched randomness exists is the replica method [33]. The averaging over quenched random variables should be done after calculation of the free energy or the logarithm of the partition function. Such averaging is done in the replica method in the following way: we write $-\beta F = \log_{10} Z = \lim_{n\to 0}(Z^n - 1)/n$ and calculate the right-hand side of the equation for integer values of n. In the end we formally take the limit and find the free energy.

To calculate Z^n we write

$$Z^n = \prod_{a=1}^{n} \int D\theta_a D\bar{\theta}_a e^{S_{0,a} + \int \epsilon(z)\phi_a(z) d^2 z}, \tag{4}$$

where the index a counts the replica number. Now we are ready to average over different configurations of the quenched random variables $\epsilon(z)$. Doing the Gaussian integrations we arrive at

$$\langle Z^n \rangle_\epsilon = \int \prod D\theta_a D\bar{\theta}_a e^{\sum_a S_{0,a} + g_0 \int d^2 z \sum_{a,b} \phi_a(z)\phi_b(z)}. \tag{5}$$

Regarding $\phi_a\phi_b$ as a perturbation and following the usual RG steps [33] we are able to find the flow of g_0 under the RG. In order to do that, we have to expand $\exp\,(g_0 \int d^2 z \sum_{a,b} \phi_a(z)\phi_b(z))$ and do the averaging with the weight defined by S_0. If we collect the terms up to second order in g_0 we will have the following terms:

$$\begin{aligned} g \int d^2 z \sum_{a,b=1}^{n} \phi_a(z)\phi_2(z) \\ = g_0 \int d^2 z \sum_{a,b=1}^{n} \phi_a(z)\phi_2(z) \\ + \frac{1}{2}\left(g_0 \int d^2 z \sum_{a,b=1}^{n} \phi_a(z)\phi_2(z)\right)^2 + \dots, \end{aligned} \tag{6}$$

where g is the renormalized parameter. The above equation could be written in the form $g = g_0 + A_2 g_0^2 + \dots$, where A_2 is found to be $2(n-2)\int \langle \phi(z)\phi(w)\rangle$. Knowing the correlation functions of ϕ fields, A_2 could be calculated. The correlation function diverges as $z \to w$ and the above term becomes infinite. Therefore one has to put a cutoff r and do the integrals for $|z - w| > r$. In this way the renormalized parameter is obtained to be

$$g = (r^2 \ln r^2 - r^2)\big(g_0 + A(r) g_0^2\big), \tag{7}$$

where $A(r)$ is $2\pi(n-2)(r^2 \ln r^2 - r^2)$. Note that we multiply the result by $(r^2 \ln r^2 - r^2)$ in order to obtain a dimensionless coupling constant. Hence the beta function of the theory turns out to be

$$\beta(g) = 2g + 4\pi(n-2)g^2. \tag{8}$$

In the limit $n \to 0$ we have $\beta(g) = 2g - 8\pi g^2$, which shows that $g = 0$ is an unstable fixed point and even a small perturbation takes the system to the new fixed point with $g^* = 1/4\pi$.

We have to express that from such field theoretic calculation we are able to find only the static properties of the system and cannot obtain the exponents of the avalanche distribution function. The above calculations only express that we will have a crossover to a new fixed point and therefore the exponents would be changed.

V. CONCLUSION

We have shown both numerically and analytically that introducing sink and source sites to a BTW system causes the system to fall into a new fixed point. In simulations we have observed that adding even a few pairs of sink sites and source sites changes the statistics of the large avalanches and a new exponent is observed, yet the system shows finite size scaling which means the system is still critical. We have done a field theoretic calculation in support of this result. Adding the perturbing operator originated from sink and source sites to the original theory, we have used the replica method to find the renormalization group plow and have observed that the BTW fixed point is unstable under RG transformation and a new stable fixed point exists in the model.

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