Quantum heat baths satisfying the eigenstate thermalization hypothesis

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A class of autonomous quantum heat baths satisfying the eigenstate thermalization hypothesis (ETH) criteria is proposed. We show that such systems are expected to cause thermal relaxation of much smaller quantum systems coupled to one of the baths local observables. The process of thermalization is examined through residual fluctuations of local observables of the bath around their thermal values predicted by ETH. It is shown that such fluctuations perturb the small quantum system causing its decoherence to the thermal state. As an example, we investigate theoretically and numerically thermalization of a qubit coupled to a realistic ETH quantum heat bath.

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I. INTRODUCTION

Quantum thermodynamics has attracted much attention in recent years. On the one hand, there has been much progress in understanding the origin of thermalization in closed quantum systems [1]. Thermalization is meant to be based purely on the phenomenology of quantum mechanics without any need for any external sources of heat to cause thermal relaxation of local observables in a quantum system [2]. It was shown that it is the properties of eigenstates of an isolated quantum system whose local observables show thermal behavior [3,4]. Later, a general canonical principle was introduced [5], stating that local observables thermalize for almost all pure states of a sufficiently big closed system under a global constrain. In this situation, the whole system with respect to local observables can be regarded to be in a microcanonical ensemble. On the other hand, in the studies of quantum thermal machines the emphasis is put on the properties of small systems which are coupled to some macroscopic sources of heat [6]. The heat baths are treated phenomenologically using the methods of open quantum systems. They are usually a collection of an infinite number of harmonic oscillators, which are presupposed to be in thermal equilibrium [7].

The smallest thermal machines are supposed to work at atomic scales and be of benefit in nanotechnology [8]. To be able to do useful work they need to extract heat from heat baths. So far the role of heat baths has been played by macroscopic objects such as electromagnetic radiation or beams of atoms [9]. An interesting question arises whether it is possible to realize isolated quantum heat baths as sources of heat for such small machines. Combining together such small machines and microscopic heat baths it would be possible to create genuine quantum heat devices. A quantum heat bath must satisfy certain conditions to be able to thermalize a small quantum system coupled to it. Among them is the requirement that energies of the bath must be much denser than energies of the system and its density of states must be an increasing function of energy. Thermalization of the small system is then achieved by choosing a bath operator that couples to the system to be a random matrix [10]. The combined bath-system Hamiltonian is then itself random with typically thermal behavior [2]. Indeed, it has been shown in Ref. [11] that a large class of isolated quantum systems in a pure state can serve as a heat bath for its subsystem. The crucial limitation of these works is that the bath and the system could not be separated; they were studied together as a closed quantum system. According to the second law of thermodynamics it is not possible to extract work from a single bath in a complete cycle; we need at least two baths to be able to build a thermal machine. The need for autonomous heat baths arises. In this paper, we present a class of finite realistic quantum heat baths; those thermal properties are dictated solely by the properties of their eigenstates. We show that such systems, whose eigenstates satisfy the ETH criteria, cause thermal relaxation of smaller systems weakly coupled to it. The ETH bath is thermal with some well-defined temperature regardless whether a small system is coupled to it or not. Therefore, the ETH bath is autonomous, the case which has not been addressed previously. This opens the possibility to use such baths to build thermal machines. ETH concerns the thermal properties of local observables of the bath, while the qubit is a separate system not covered by the ETH requirements. In this paper we show when and how the ETH bath thermalizes the qubit coupled to it.

ETH states that under certain conditions almost all eigenstates of a closed quantum system with some Hamiltonian $\hat{H}_B$ have the properties of a thermal state. Let $|E_k\rangle$ be an eigenstate of such system with eigenvalue E_k and $\hat{O}$ its local operator. According to ETH if the diagonal elements $\mathcal{O}(E_k) \equiv \langle E_k|\hat{O}|E_k\rangle$ are smooth functions of E_k, while the off-diagonal elements $\langle E_k|\hat{O}|E_l\rangle$ are negligibly small, then for an initial state $|\phi_0\rangle$ with energy E_B and small energy variance the expectation value $\langle\phi|\hat{O}|\phi\rangle$ in the pure state $|\phi\rangle = \exp(-it\hat{H}_B/\hbar)|\phi_0\rangle$ relaxes to a steady thermal value and stays there for almost all times [3]. ETH predicts that the relaxed value tends to an appropriate microcanonical ensemble average $\langle\phi|\hat{O}|\phi\rangle \approx \frac{1}{\mathcal{N}}\sum_k^{\prime}\langle E_k|\hat{O}|E_k\rangle$. Here we sum over $\mathcal{N}$ energy eigenstates near energy E_B of the system, $E_k \in [E_B-\delta, E_B+\delta]$, with fixed $\delta \ll E_B$. This yields that the whole isolated system with respect to a local observable can be regarded to be in the microcanonical ensemble with the density matrix

$$\hat{\rho}_m = \frac{1}{\mathcal{N}}\sum_k^{\prime}|E_k\rangle\langle E_k|. \tag{1}$$

For a finite quantum system, the values of the matrix elements $O_{kl} \equiv \langle E_k|\hat{O}|E_l\rangle$ are not ideally smooth functions of eigenvalues E_k but exhibit some residual fluctuations:

$$O_{kl} = \mathcal{O}(E_k)\delta_{kl} + \mathcal{R}_{kl}, \tag{2}$$

where $\mathcal{R}_{kl}$ are random with zero mean and small according to ETH. The semiclassical analysis shows that the values $|\mathcal{R}_{kl}|^2$ are characterized by a smooth function [12]

$$|\mathcal{R}_{kl}|^2 \simeq \mathcal{S}(E_k - E_l)/2\pi\rho(\overline{E}), \tag{3}$$

where $\mathcal{S}(x)$ is the spectral function (see Ref. [12] for more details), $\rho(x)$ is the density of states, and $\overline{E} = (E_k + E_l)/2$. The density of states grows exponentially with the number of particles; therefore, the fluctuations are small if the number of particles is large. Recent developments have demonstrated that this behavior pertains to chaotic quantum systems lacking classical analogs [13]. The fluctuating behavior in Eq. (2) manifests itself on the expectation value of the observable

$$\langle\phi|\hat{O}|\phi\rangle = \sum_k |\alpha_k|^2 \mathcal{O}(E_k) + \sum_{k,l} \alpha_k^* \alpha_l e^{it(E_k - E_l)/\hbar} \mathcal{R}_{kl}. \tag{4}$$

The first term here is the expected relaxed thermal value, while the second term averages to zero at long times. However, the fluctuations $\langle\phi|\hat{O}^2|\phi\rangle - \langle\phi|\hat{O}|\phi\rangle^2 \simeq \sum_{kl} |\alpha_k|^2|\alpha_l|^2|\mathcal{R}_{kl}|^2$ are not negligible. They can be interpreted as residual thermal fluctuations [12].

II. ETH BATH

Let an arbitrary small quantum system with Hamiltonian $\hat{H}_S$ be coupled to a larger quantum system satisfying ETH. For brevity, we call the latter ETH bath. The combined system Hamiltonian is given by

$$\hat{H} = \hat{H}_{\rm S} + \hat{H}_I + \hat{H}_{\rm B}, \tag{5}$$

where $\hat{H}_I = g\hat{X}_{\rm S} \otimes \hat{O}$ is the interaction between the system and the bath. Here g is an interaction strength; $\hat{X}_S$ is a system operator which couples to the bath operator $\hat{O}$. Let $|\epsilon_l\rangle$ be the eigenstates of the system and $|E_k\rangle$ be the eigenstates of the bath. The basis of the combined system is chosen to be $|\mathcal{E}_{lk}\rangle = |\epsilon_l\rangle \otimes |E_k\rangle$. Using Eq. (2), the combined Hamiltonian assumes the form of the sum of a "smooth" part with the elements $\langle lk|\hat{H}|l'k'\rangle = \delta_{kk'}[\delta_{ll'}(\epsilon_l + E_k) + gX_{ll'}\mathcal{O}(E_k)]$ and the "irregular" part with the elements $gX_{ll'}\mathcal{R}_{kk'}$. Here, we denoted $X_{ll'} = \langle\epsilon_l|\hat{X}_S|\epsilon_{l'}\rangle$.

The smooth part is a block matrix which contains for a given index k a smaller matrix $h_{ll'}(E_k) = \delta_{ll'}(\epsilon_l + E_k) + gX_{ll'}\mathcal{O}(E_k)$. The terms $gX_{ll'}\mathcal{O}(E_k)$ simply shift the energies of the unperturbed system by an amount $\delta\epsilon_l(E_k)$, which are some smooth functions of E_k. For an ETH bath the energy variance is small around mean energy E_B and we can assume that its eigenenergies $E_k \approx E_B$. Since the coupling g is small we may apply perturbation theory to estimate the energy shift

$$\delta\epsilon_l(E_B) \approx gX_{ll}O(E_B) + g^2O^2(E_B)\sum_{l'} \frac{|X_{ll'}|^2}{\epsilon_l - \epsilon_{l'}}. \tag{6}$$

The irregular part, on the other hand, intertwines blocks with different indices k in a chaotic manner due to randomness of $\mathcal{R}_{ll'}$. This has a more profound effect on the dynamical evolution of the small quantum system. One can imagine that these random terms disturb the phase of the quantum state of the small system leading to its dephasing. Additionally, the off-diagonal nature of the irregular part allows energy exchange between the bath and the small system leading to dissipation. Both effects, dephasing and dissipation, lead to decoherence of the small system [14]. We will show that the ETH bath causes the system to decohere to a thermal state.

III. QUBIT COUPLED TO ETH BATH

Consider a specific example of great interest: a qubit coupled to the above ETH quantum bath. The Hamiltonian of the qubit with two available states $|1\rangle$ and $|2\rangle$ and corresponding energies ϵ_1 and ϵ_2 reads

$$\hat{H}_{\rm S} = \epsilon_1|1\rangle\langle 1| + \epsilon_2|2\rangle\langle 2|, \tag{7}$$

where $\Delta \equiv \epsilon_2 - \epsilon_1 > 0$. We allow energy exchange between the qubit and the bath considering for simplicity $\hat{X}_S = \sigma_+ + \sigma_-$, where $\sigma_+ = |2\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 2|$ are the raising and lowering operators of the qubit. Following the formalism of open quantum systems [7], we move into the interaction picture $\hat{H}_I(t) = e^{it(\hat{H}_S+\hat{H}_B)/\hbar}\hat{H}_I e^{-it(\hat{H}_S+\hat{H}_B)/\hbar}$, which yields $\hat{X}_S(t) = e^{i\Delta t/\hbar}\sigma^+ + e^{-i\Delta t/\hbar}\sigma^-$ and $\hat{O}(t) = \sum_{k,l} e^{it(E_k-E_l)/\hbar}O_{kl}|k\rangle\langle l|$. As the next step, we need to calculate the correlation function of the bath operators $C(t,s) = {\rm tr}[\hat{O}(t)\hat{O}(s)\hat{\rho}_B]$. As we discussed above, the bath satisfying ETH can be assumed to be in the microcanonical ensemble $\hat{\rho}_B = \hat{\rho}_m$, where the density matrix $\hat{\rho}_m$ is given in Eq. (1). This allows us to evaluate the correlation function

$$C(t,s) = C(t-s) = \frac{1}{\mathcal{N}}\sum_k{}' \sum_l e^{i(E_k-E_l)(t-s)/\hbar}|O_{kl}|^2. \tag{8}$$

The coupling is assumed to be small and the bath is large enough to satisfy ETH requirements. Under these conditions we use the Born-Markov approximation to study the dynamics [7]. In fact, it was shown in Ref. [15] that randomness of the coupling renders the second order approach valid. Under these assumptions we obtain the following Schrödinger picture master equation:

$$\begin{aligned}\hbar\frac{\partial}{\partial t}\hat{\rho}_S(t) &= -i[\hat{H}_S,\hat{\rho}_S(t)]\\ &\quad - g^2\int_0^\infty d\tau[\hat{X}_S,\hat{X}_S(-\tau)\hat{\rho}_S(t)]C(\tau)\\ &\quad - g^2\int_0^\infty d\tau[\hat{\rho}_S(t)\hat{X}_S(-\tau),\hat{X}_S]C(-\tau).\end{aligned} \tag{9}$$

To solve this equation we need to evaluate the integrals $\int_0^\infty d\tau\, \hat{X}_S(-\tau)C(\pm\tau)$, which arise in the second and third lines of Eq. (9). This boils down to taking the integrals $I(\Delta) = \int_0^\infty d\tau\, e^{\pm i\Delta\tau/\hbar}C(\tau)$. By using the identity $\int_0^\infty dt\, e^{\pm i\epsilon t} = \pi\delta(\epsilon) \pm i\frac{\mathcal{P}}{\epsilon}$, where $\mathcal{P}$ denotes the principal value, we obtain

$$\begin{aligned}I(\Delta) &= \mathcal{S}(\Delta)\frac{1}{2\mathcal{N}}\sum_k{}' \rho(E_k \pm \Delta)/\rho(E_k \pm \Delta/2)\\ &\quad + \frac{1}{\mathcal{N}}\mathcal{P}\sum_k{}'\sum_l \frac{|O_{kl}|^2}{E_k - E_l \pm \Delta}.\end{aligned} \tag{10}$$

Here, $\rho(\epsilon) = \sum_l \delta(\epsilon - E_l)$ is the density of states of the bath and we used Eqs. (2) and (3) in deriving Eq. (10). The first term in $I(\Delta)$ is equal to $\mathcal{S}(\Delta)/2e^{\pm\beta\Delta/2}$, where

$\beta = \frac{\partial \log \rho(E_B)}{\partial E_B}$ is the inverse temperature of the bath. We have assumed that $\Delta \ll E_B$, which allowed us to expand $\rho(E_k \pm \Delta) \approx \rho(E_k)(1 \pm \beta\Delta) \approx \rho(E_k)e^{\pm\beta\Delta}$ and similarly for $\rho(E_k \pm \Delta/2) \approx \rho(E_k)e^{\pm\beta\Delta/2}$. The second term in $I(\Delta)$ is significant if $E_k = E_l$ when it is equal to $\pm i\frac{1}{N}\sum_k' O^2(E_k)/\Delta \approx \pm i O^2(E_B)/\Delta$.

From Eq. (9) we can now obtain two coupled equations for the diagonal elements of the density matrix describing the process of thermalization:

$$\hbar\frac{\partial\rho_{11}}{\partial t} = -g^2\mathcal{S}(\Delta)e^{-\beta\Delta/2}\rho_{11} + g^2\mathcal{S}(\Delta)e^{\beta\Delta/2}\rho_{22},$$
$$\hbar\frac{\partial\rho_{22}}{\partial t} = -g^2\mathcal{S}(\Delta)e^{\beta\Delta/2}\rho_{22} + g^2\mathcal{S}(\Delta)e^{-\beta\Delta/2}\rho_{11}. \tag{11}$$

The qubit relaxes to the expected thermal steady state $\lim_{t\to\infty}\rho_{11}(t)/\rho_{22}(t) = e^{\beta\Delta}$. The specific form of fluctuations $|\mathcal{R}_{kl}|^2$ used in the derivation of this result proves to be crucial to yield the correct thermal steady state. The remaining equations are for the off-diagonal elements describing the process of decoherence:

$$\hbar\frac{\partial\rho_{12}}{\partial t} = i(\Delta + 2\delta\Delta)\rho_{12} + i2\delta\Delta\rho_{21} - \gamma(\rho_{12} - \rho_{21}),$$
$$\hbar\frac{\partial\rho_{21}}{\partial t} = -i(\Delta + 2\delta\Delta)\rho_{21} - i2\delta\Delta\rho_{12} - \gamma(\rho_{21} - \rho_{12}), \tag{12}$$

where we denoted $\delta\Delta = g^2O^2(E_B)/\Delta$ and $\gamma = g^2\mathcal{S}(\Delta)\cosh(\beta\Delta/2)$. It can be shown that the population relaxation occurs at the time scale $1/(2\gamma)$, while the loss of coherence occurs at the time scale $1/\gamma$. Thermalization of the qubit is thus twice faster than decoherence, which is a standard result in the context of atomic decay and quantum optics [7]. An example of this dynamics is shown in Fig. 1. The off-diagonal terms can be shown to behave as $\sim \exp(it\sqrt{\Delta(\Delta + 4\delta\Delta) - \gamma^2}/\hbar)\exp(-\gamma t/\hbar)$. At small g we have $\gamma \ll \delta\Delta \ll \Delta$; therefore, the energy levels of the qubit are shifted by the amount $\sqrt{\Delta(\Delta + 4\delta\Delta) - \gamma^2} - \Delta \approx 2\delta\Delta \equiv 2O^2(E_B)/\Delta$. This can be easily seen also from Eq. (6), since in this case $|X_{12}| = 1$ and $\epsilon_2 - \epsilon_1 \equiv \Delta$.

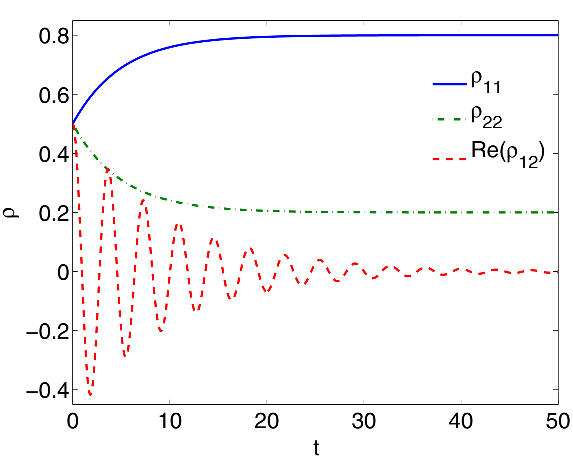

FIG. 1. (Color online) Elements of the density matrix of the qubit, which were obtained by solving Eq. (11) and (12) with the initial state $\rho_{11} = \rho_{22} = \rho_{12} = \rho_{21} = 0.5$ and $\exp(\beta\Delta) = 4$, $\gamma = 0.1$, $\delta\Delta = 0.5$, and $\Delta = 1$. The qubit relaxes to the thermal state $\rho_{11}/\rho_{22} = 4$ and $\rho_{12} = \rho_{21} = 0$. The process of thermalization (relaxation of diagonal elements) is twice faster than the process of decoherence (relaxation of off-diagonal elements).

We study now the case $\hat{X}_S = \sigma_z$, which does not allow energy exchange between the bath and the qubit. Following the same procedure as above we arrive at the following master equations:

$$\hbar\frac{\partial\rho_{12}}{\partial t} = -2g^2\mathcal{S}(0)\rho_{12} + i\Delta\rho_{12},$$
$$\hbar\frac{\partial\rho_{21}}{\partial t} = -2g^2\mathcal{S}(0)\rho_{21} - i\Delta\rho_{21},$$

while $\partial\rho_{11}/\partial t = \partial\rho_{22}/\partial t = 0$. This leads to dephasing of the off-diagonal elements $|\rho_{12}(t)| = |\rho_{21}(t)| \sim \exp[-2g^2\mathcal{S}(0)t/\hbar]$ but not to thermalization, since ρ_{11} and ρ_{22} do not change.

These results are in line with the analysis in the paragraph below Eq. (4): the first term in Eq. (2) causes energy level shift of the qubit, while the residual thermal fluctuations lead to the loss of coherence but not necessarily to thermalization (cf. also Ref. [16]). Therefore, energy exchange between the system and the bath is crucial to obtain thermalization of the system.

The Markovian assumption might not be strictly satisfied for a concrete system. In this case, for example, the upper limit of the time integral in Eq. (9) cannot be extended to infinity but only up to time t [7]. This renders more fine-grained dynamics especially at short times. As we are concerned with thermalization of a qubit, we are interested in the asymptotic state at $t \to \infty$. It is thus expected that the above formalism is adequate for the present studies.

IV. NUMERICAL EXPERIMENT

To realize the proposed heat bath, we consider a two-band double-well potential filled with cold bosons. A complex interplay between the tunneling of bosons and their mutual interactions makes it possible to satisfy the ETH criteria and show thermalization [17]. We use this system as a heat bath for a qubit coupled to it. The Hamiltonian of the bath reads [17,18]

$$\hat{H}_B = -\sum_{r\neq r',l} J^l\hat{b}_r^{l\dagger}\hat{b}_{r'}^l + \sum_{r,l} U^l\hat{n}_r^l(\hat{n}_r^l - 1) + \sum_{r,l} E_r^l\hat{n}_r^l + U^{01}\sum_{r,l\neq l'}(2\hat{n}_r^l\hat{n}_r^{l'} + \hat{b}_r^{l\dagger}\hat{b}_r^{l\dagger}\hat{b}_r^{l'}\hat{b}_r^{l'}), \tag{13}$$

where $\hat{b}_r^l$ ($\hat{b}_r^{l\dagger}$) are the annihilation (creation) operators of a boson with mass m_B in the well $r = L,R$ on the energy level $l = 0,1$. For a given double well potential $V(x)$ the corresponding single-particle wave functions $\phi_r^l(x)$ can be found. The coefficients in the above equation can be then evaluated [18] as follows. The single-particle tunneling rate is $J^l = \int dx\,\phi_L^{l*}(x)(-\frac{\hbar^2}{2m_B}\nabla^2 + V(x))\phi_R^l(x)$, on-site interaction strength is $U^l = g_B\int dx|\phi_r^l(x)|^4$, single-particle energies are $E_r^l = \int dx\,\phi_r^{l*}(x)(-\frac{\hbar^2}{2m}\nabla^2 + V(x))\phi_r^l(x)$, and induced interaction between levels is $U^{01} = g_B\int dx|\phi_r^0(x)|^2|\phi_r^1(x)|^2$. Here, g_B is the two-body interaction coupling between bosons.

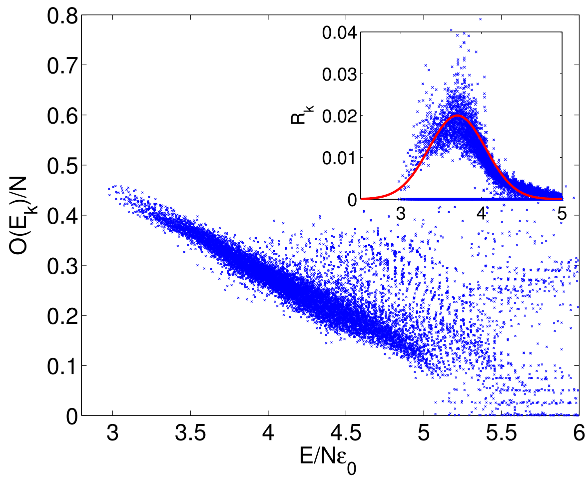

FIG. 2. (Color online) Example of expectation values of the bath operators in the eigenenergy basis, $\langle E_k|\hat{b}_L^{0\dagger}\hat{b}_L^0|E_l\rangle = \mathcal{O}(E_k)\delta_{kl} + \mathcal{R}_{kl}$. At low energies the distribution of the diagonal elements $\mathcal{O}(E_k)$ resembles smooth function in accordance with ETH, while at larger energies their behavior is more chaotic. Inset: off-diagonal elements $R_k = |\mathcal{R}_{kk_0}|^2$ with fixed k_0 corresponding to $E_{k_0} \approx 3.65N\epsilon_0$. They are much smaller than the diagonal elements in accordance with ETH and their distribution resembles a smooth symmetric function around E_{k_0} (shown in red) in accordance with the semiclassical estimate.

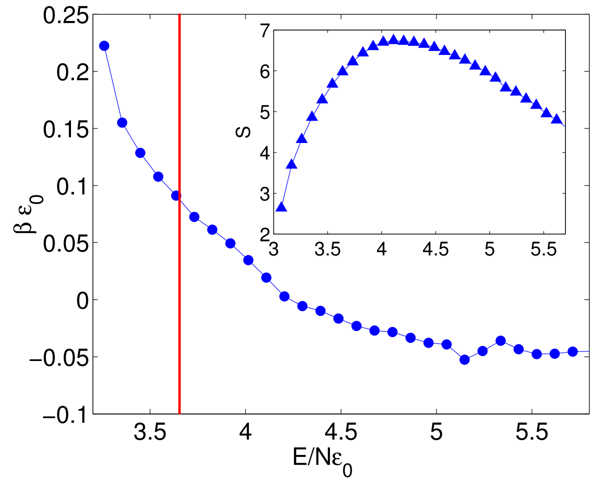

FIG. 3. (Color online) Inverse microcanonical temperature of the bath as a function of energy. It can be inferred from the entropy showing in the inset using the relation $\beta(E) = \partial S(E)/\partial E$. Dashed line shows energy of the bath used in the calculations.

The double well is created by splitting the harmonic potential $m_B\omega_0^2x^2/2$ by the focused laser $10\hbar\omega_0\exp(-x^2/2\sigma^2)$ with the width $\sigma = 0.1\sqrt{\hbar/m_B\omega_0}$. We choose energy units $\epsilon_0 = \hbar\omega_0$. For $g_B = 0.3\epsilon_0$ we obtain $J^0 = 0.26\epsilon_0$, $J^1 = 0.34\epsilon_0$, $U^0 = 0.14\epsilon_0$, $U^1 = 0.1\epsilon_0$, $U^{01} = 0.06\epsilon_0$, $E_0 = 1.25\epsilon_0$, and $E_1 = 3.17\epsilon_0$.

The Hamiltonian of the bath can be easily diagonalized numerically for $N = 30$ bosons and the eigenenergies $|E_k\rangle$ with the corresponding eigenvalues E_k can be extracted. The bath satisfies the ETH criteria at low energies as it is shown in Fig. 2. We define the entropy of the bath at energy E as $S(E) = -\text{Tr}(\hat{\rho}_m \ln\hat{\rho}_m)$ and the corresponding inverse temperature as its derivative $\beta(E) = \partial S(E)/\partial E$. The results are presented in Fig. 3.

A two level qubit is represented via a single particle with mass m_S trapped in a harmonic potential $m_S\omega^2x^2/2$ with two energy states $\psi_n(x)$ $(n = 0,1)$ available to it. The corresponding single-particle Hamiltonian is

$$\hat{H}_S = \hbar\omega\sum_{n=0}^{1}(n+1/2)\hat{a}_n^\dagger\hat{a}_n. \tag{14}$$

We choose $\Delta \equiv \hbar\omega = 1\epsilon_0$. The particle interacts with the bosons in the double well potential via contact interaction g. The Hamiltonian of the combined system is given in Eq. (5), where the last term describes the interaction between the qubit and the bath:

$$\hat{H}_I = g\sum_{nn'll'=0}^{1}\sum_{r,r'=L}^{R} C_{rr'}^{nn'll'}\hat{a}_n^\dagger\hat{a}_{n'}\hat{b}_r^{l\dagger}\hat{b}_{r'}^{l'}, \tag{15}$$

where $C_{rr'}^{nn'll'} = \int dx\,\psi_n^*(x)\psi_{n'}(x)\phi_r^{l*}(x)\phi_{r'}^{l'}(x)$. The interlevel transitions of the qubit between $n \neq n'$ are allowed, such that the qubit and the bath may exchange energy. We expect the qubit to relax to a thermal state with the microcanonical temperature of the bath.

We simulate the quantum dynamics by creating the Hamiltonian in the Fock basis of localized wave functions and propagate an initial state $|\psi(0)\rangle$ by solving the Schrödinger equation, $|\psi(t)\rangle = \exp(-iHt/\hbar)|\psi(0)\rangle$. The initial state of the entire system is a Fock state of the bath $|n_L^0,n_R^0,n_L^1,n_R^1\rangle$ times an initial state of the qubit. We choose two initial states for the qubit: $(1,0)^T$ corresponding to the particle occupying initially the lowest state of the trapping potential and $(1,1)^T/\sqrt{2}$ corresponding to the superposition of the lowest and the first excited states. The initial Fock state of the bath is chosen such that the energy of the bath $\langle\psi(0)|\hat{H}_B|\psi(0)\rangle \approx 3.65N\epsilon_0$ satisfies the ETH. The energy of

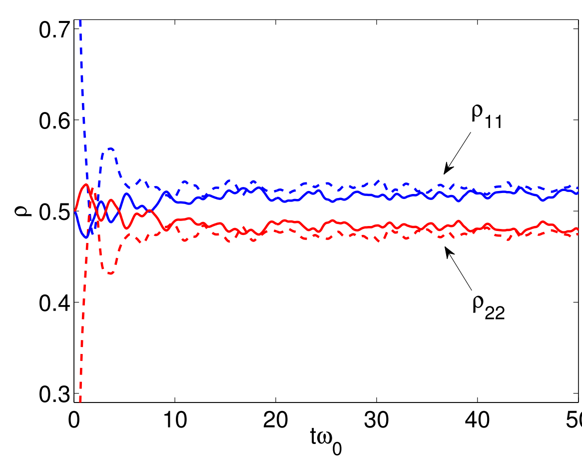

FIG. 4. (Color online) Diagonal elements of the reduced density matrix of the qubit. It relaxes to the thermal state $\rho_{11}/\rho_{22} = \exp(\beta\Delta)$. Solid lines correspond to the initial state of the qubit $(1,1)^T/\sqrt{2}$ and dashed lines to $(1,0)^T$, respectively. For our simulations we chose $\Delta = 1\epsilon_0$ and we found $\beta \approx 0.8\epsilon_0^{-1}$. This is in very good agreement with the inverse microcanonical temperature of the bath; cf. Fig. 3. Inset: corresponding energies of the bath.

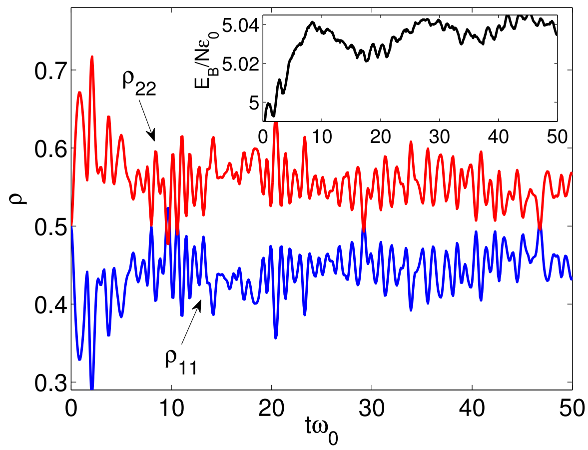

FIG. 5. (Color online) Diagonal elements of the reduced density matrix of the qubit. The initial state of the qubit is $(1,1)^T/\sqrt{2}$ and the initial value of the bath energy $E_B \approx 5N\epsilon_0$. ETH is not satisfied at this energy; cf. Fig. 2. The qubit does not relax to the thermal state. Inset: corresponding energy of the bath.

the bath $\langle\psi(t)|\hat{H}_B|\psi(t)\rangle$ changes in time slightly from this value, since the qubit is coupled weakly to the bath and its Hilbert space is much smaller than that of the bath. At this energy the inverse temperature of the bath can be found from Fig. 3, $\beta \approx 0.8\epsilon_0^{-1}$. As expected, the reduced density matrix of the qubit $\hat{\rho}(t) = \mathrm{Tr}_B|\psi(t)\rangle\langle\psi(t)|$ relaxes to the thermal state $\lim_{t\to\infty} \rho_{11}(t)/\rho_{22}(t) = \exp(\beta\Delta)$ as shown in Fig. 4. On the contrary, thermalization is not observed for higher energies of the bath where ETH is not satisfied as it is evident from Fig. 5. Therefore, ETH is crucial to obtain thermalization of a small quantum system.

V. CONCLUSIONS

We have demonstrated that isolated quantum systems satisfying the criteria of ETH can serve as finite autonomous heat baths for smaller quantum systems. As an example, we studied theoretically and numerically the thermal relaxation of a qubit weakly coupled to a realistic ETH heat bath. We believe ETH heat baths will be of benefit in building thermal machines working genuinely on the microscopic level, where not only the engine but also heat bath is treated quantum mechanically.

The qubit is believed to be the smallest quantum engine. It is an interesting open question of how large the Hilbert space of the thermalized system can get.

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