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Particle-density fluctuations and universality in the conserved stochastic sandpile

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We examine fluctuations in particle density in the restricted-height, conserved stochastic sandpile (CSS). In this and related models, the global particle density is a temperaturelike control parameter. Thus local fluctuations in this density correspond to disorder; if this disorder is a relevant perturbation of directed percolation (DP), then the CSS should exhibit non-DP critical behavior. We analyze the scaling of the variance V_ℓ of the number of particles in regions of ℓ^d sites in extensive simulations of the quasistationary state in one and two dimensions. Our results, combined with a Harris-like argument for the relevance of particle-density fluctuations, strongly suggest that conserved stochastic sandpiles belong to a universality class distinct from that of DP.

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Sandpile models have attracted great interest in both their self-organized [1–5] and conserved versions [6–13]. The conserved sandpile has the same local dynamics as the corresponding driven sandpile, but a fixed number of particles. It is characterized by an order parameter (the activity density) that is coupled to a conserved field (the particle density) whose evolution is arrested in regions without activity [14]. The critical behavior of conserved stochastic sandpiles (CSS) has been studied extensively using both particle models [15–21] and continuum representations [22,23], leading to the conclusion that these models belong to a universality class called *conserved directed percolation* (CDP), distinct from directed percolation (DP). The existence of the CDP class was nevertheless questioned by Basu *et al.* [24], who claim that the one-dimensional CSS belongs to the DP class.

The claim of DP-like critical behavior in the conserved stochastic sandpile rests on two assertions: (1) at long times, and on large length scales, the particle density becomes uniform, so that the propagation of activity occurs as in DP; and (2) the use of random initial conditions leads to incorrect estimates of the critical point and of critical exponents [24]. These results were reexamined by Lee [25,26], who presented numerical evidence favoring non-DP scaling in the CSS and allied models in one and two dimensions.

In the present work we take a different approach, by asking: If the propagation of activity in the CSS *were* DP-like, would fluctuations in particle density represent a relevant perturbation? This motivates our study of particle-density fluctuations in the critical CSS. Scaling properties of such fluctuations were recently reported by Hexner and Levine, who find that they are characterized by universal critical exponents [27]. Here, using a Harris-Luck-like argument along with results for DP subject to diffusive disorder, we obtain a criterion for relevance. Our numerical results for the growth of particle-number fluctuations with length scale, which are in general agreement with those of [27], imply that this disorder should in fact be relevant, leading to the conclusion that, if CSS critical behavior were DP-like, it would be *unstable to the fluctuations generated by its own dynamics*, leading to a contradiction. The observed particle-number fluctuations represent the stationary dynamics of the CSS and are not colored by the initial configuration: random, natural, and uniform initial configurations all lead to indistinguishable results for particle-density fluctuations and other quasistationary (QS) properties. In what follows, we define the model, discuss the theoretical background related to particle-number fluctuations, and then report simulation results for the particle-number variance and correlation function, closing with a summary of our results.

We study a conserved stochastic sandpile, related to Manna's model [28], called the *restricted-height sandpile* [18,20,21,29]. The model is defined on a d-dimensional lattice of L^d sites, with periodic boundaries; the configuration is specified by the number of particles, $z_i = 0$, 1, or 2, at each site i. Sites with $z_i = 2$ are *active*, while those with $z_i \leqslant 1$, are *inactive*. No site may harbor more than two particles in the restricted CSS.

The temporal evolution consists of a series of *toppling* events, in which two particles attempt to hop from an active site to one or more of its nearest neighbors. The target sites for the two particles are chosen independently, with equal probabilities, from the set of nearest neighbors. If a particle attempts to jump to a site already bearing two particles, it returns to the toppling site. The evolution follows a continuous-time Markovian dynamics in which each active site has a transition rate of unity to topple. At each step of the evolution, one of the N_a currently active sites is chosen at random to topple; the time increment associated with each step is $\Delta t = 1/N_a$. In this way, each active site waits, on average, one time unit before toppling.

In conserved sandpiles, the global particle density, $p = N/L^d$, serves as a temperaturelike control parameter [6]. Below the critical value, p_c, the system eventually reaches an absorbing configuration ($N_a = 0$). For $p > p_c$, by contrast, the activity continues indefinitely ($N_a > 0$), in the infinite-size limit. The order parameter associated with the phase transition

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is the activity density, given by the fraction of active sites, $\rho = N_a/L^d$. Although activity must continue indefinitely if $p > 1$, p_c is in fact well below unity. For the models studied here, the best estimates for the critical density are $p_c = 0.929\,780(7)$ ($d = 1$ [18]), and $0.711\,268\,7(2)$ (square lattice [21]). (Figures in parentheses denote the uncertainty in the final digit or digits).

In a coarse-grained or continuous description, the evolution of the activity density $\rho(\mathbf{x},t)$ in a CSS is similar to that of directed percolation, except that ρ is coupled to a conserved field (the excess particle density) $\zeta(\mathbf{x},t)$ which evolves diffusively in the presence of activity and is frozen in its absence [7,8,13]. If ζ were somehow made uniform and time independent, the scaling behavior of the CSS would be that of DP. So, indeed, argued Basu *et al.* [24], in justifying their assertion of DP scaling in the conserved sandpile: at long times, the particle density is sufficiently uniform that the coupling of the order parameter to this field is unimportant. Here we examine this assertion by asking, "Is DP stable to the particle-density fluctuations generated by the CSS?" If so, DP-like behavior remains a possibility for the CSS; if not, DP-like behavior in the CSS is impossible. Let N_ℓ be the number of particles in a hypercube of ℓ^d sites. The departure from uniformity is characterized by $V_\ell \equiv \mathrm{var}[N_\ell]$. If $v_\ell = V_\ell/\ell^d$ tends to zero rapidly enough as $\ell \to \infty$, then variations in the particle density will not affect the critical behavior. In finite-sized systems, of course, $V_L = 0$ due to particle conservation. We are therefore interested in how V_ℓ scales for $1 \ll \ell \ll L$. We shall see that in this regime, V_ℓ follows an approximate power law, $V_\ell \sim \ell^\phi$, in the critical CSS. Since $\phi < d$, the particle fluctuations are said to be "hyperuniform," in the sense that they grow more slowly than does the variance of a sum of ℓ^d independent random variables [27,30].

To frame the issue clearly, we note that in the CSS, the global particle density plays the role of a temperaturelike parameter at an absorbing-state phase transition [6,7], analogous to the creation rate λ in the contact process (CP), which belongs to the DP universality class [31]. In the clean CP, λ is the same at each site; a contact process with fixed, random values λ_i at each site i corresponds to the CP with quenched disorder in the temperaturelike control parameter. The fluctuation of the local particle density, V_ℓ/ℓ^d, likewise corresponds to disorder in the temperaturelike parameter of the CSS.

In order to assess how rapidly fluctuations in the temperaturelike parameter must decay with the region size ℓ to be irrelevant, we recall several established results. First, quenched uncorrelated disorder ($V_\ell \sim \ell^d$) is relevant for DP [32,33]. Second, uncorrelated disorder that relaxes diffusively is also relevant to DP, as observed in the contact process with mobile vacancies (CPMV) [34]. In this model, fluctuations in the vacancy density correspond to *diffusing disorder* in the temperaturelike parameter of the CP, similar to particle-density fluctuations in the CSS. In the CPMV, $V_\ell \sim \ell^d$ on short scales, since each diffusing entity moves independently, but $V_L = 0$ on a lattice of L^d sites, due to conservation of the number of diffusing entities. Finally, we note that disorder that is uncorrelated in space *and* time is an *irrelevant* perturbation of DP [35].

In the CSS, fluctuations in the particle density relax diffusively. Diffusive relaxation in conserved sandpiles is associated with particle-number conservation: since there is no creation or annihilation of particles, changes in the particle density occur via particle *transfer*, i.e., the toppling events discussed above. Since these events transfer particles from a given site to one of its nearest neighbors, relaxation of the particle configuration is diffusive and local, in active regions. We note that fluctuations relaxing via rapid, long-range diffusion of the background field are likely to be irrelevant to DP, since they will then be essentially uncorrelated in both space and time, despite there being a global conservation law.

If the diffusion rate in the CSS were uniform in space and time, the fluctuations in particle density would be similar to those of the temperaturelike parameter in the CPMV [34]. In the CSS, however, the diffusion rate is proportional to the activity density [20,21,36,37], which is highly nonuniform at criticality. Regions with higher particle density, being more active, tend to transfer particles to neighboring regions having a lower particle density, which tend to be less active, or inactive. As a result, V_ℓ grows more slowly than for independent fluctuations. This is the origin of hyperuniformity in sandpiles [27].

We now develop a Harris-like criterion [38,39] for the relevance of particle-number fluctuations following $V_\ell \sim \ell^\phi$. Consider a system exhibiting a continuous phase transition at $p = p_c$, where p is a temperaturelike control parameter, and p_c is the critical value for the disorder-free system. Letting $\Delta = p - p_c$, we have as usual $\xi \sim |\Delta|^{-\nu_\perp}$ for the correlation length in the neighborhood of the transition, where $\nu_\perp$ is a critical exponent. Now suppose that the variance of the particle number over regions of size ℓ scales as ℓ^ϕ; this means that the standard deviation of Δ over regions of length ℓ scales as $u_\ell \sim \ell^{\phi/2-d}$. Since the pure system is correlated over regions of length ξ, the width of the probability distribution of Δ is u_ξ. If u_ξ/Δ tends to zero with Δ, disorder is irrelevant, and vice versa. Thus, since $u_\xi/\Delta \sim \xi^{\phi/2-d+1/\nu_\perp}$, disorder is relevant if

$$\phi > 2\left(d - \frac{1}{\nu_\perp}\right). \tag{1}$$

For uncorrelated disorder ($\phi = d$) the above relation reduces to Harris' criterion: disorder is relevant if $d\nu_\perp < 2$. Our scaling analysis is equivalent to that of Luck [40], who defines a wandering exponent ω via the relation $v_\ell \equiv V_\ell/\ell^{2d} \sim \ell^{-d(1-\omega)}$. Thus we have $\phi = 2d\omega$, so that Eq. (1) is equivalent to Luck's criterion, $\omega > 1 - 1/(d\nu_\perp)$, for the relevance of correlated disorder. Using the known values $\nu_\perp = 1.096\,854(4)$ and $0.734(4)$ for DP in one [41] and two [42] spatial dimensions, respectively, the criterion yields $\phi > 0.1766$ and $\phi > 1.271$ for particle-number fluctuations to be relevant in one and two dimensions.

The above Harris-like argument holds for quenched disorder. In Ref. [34], however, it is argued that if quenched disorder is relevant in a given system, then diffusive disorder is also relevant provided that, in the pure system, the dynamic critical exponent satisfies $z = \nu_\parallel/\nu_\perp < 2$, where the exponent $\nu_\parallel$ governs the divergence of the correlation time τ via $\tau \sim |\Delta|^{-\nu_\parallel}$. The reason is that the disorder configuration affecting a correlated region of the activity field has a relaxation time $\tau_d \sim \xi^2$, whereas the relaxation time of the activity follows $\tau \sim \xi^z$. Thus if $z < 2$, as the system approaches the critical

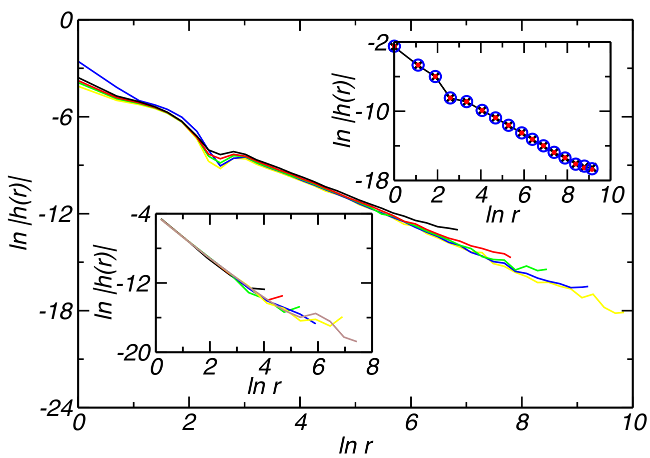

FIG. 1. (Color online) Stationary correlation function $|h(\ell)|$ in the one-dimensional CSS at criticality for system sizes $L = 2000$, 5000, 10 000, 20 000, 40 000, and 80 000. Upper inset: $|h(\ell)|$ in the one-dimensional critical CSS, system size $L = 20\,000$, for random ($\times$), natural (points), and uniform (circles) initial configurations. Lower inset: $|h(\ell)|$ in the two-dimensional CSS at criticality, system sizes $L = 128,256,\ldots,4096$.

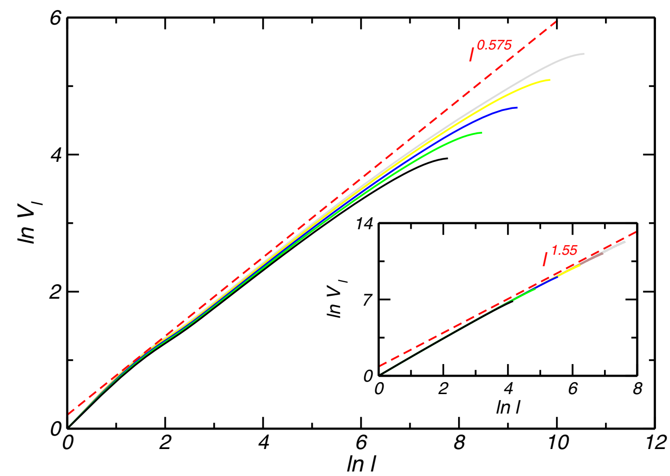

FIG. 2. (Color online) Particle-number variance V_ℓ over strips of size ℓ in the one-dimensional CSS at criticality. The curves have been shifted by subtracting V_1 from each result. System sizes as in Fig. 1. Inset: particle-number variance V_ℓ over squares of side ℓ in the two-dimensional CSS at the critical point, system sizes $L = 128,256,\ldots,4096$. Curves have been shifted by subtracting V_1 from each result. The dashed lines correspond to the power laws, $V_\ell \sim \ell^\phi$, reported in [27], with $\phi = 0.575(25)$ in one dimension and $\phi = 1.55(3)$ in 2D.

point, the ratio $\tau_d/\tau \to \infty$, so that the disorder is effectively quenched on large scales [34]. The condition $z < 2$ holds for DP in dimensions $d < 4$. Note as well that in the CSS, in which diffusion is conditioned on activity, relaxation of the disorder is *slower* than for disorder that diffuses independently of the activity field, so that the tendency toward effectively quenched disorder is even stronger than in the CPMV. The above observations lead us to expect particle-density fluctuations to be relevant if the inequality of Eq. (1) is satisfied.

We found it useful to study the particle-number correlation function, $h(j) \equiv \mathrm{cov}(z_i,z_{i+j}) = \langle z_i z_{i+j}\rangle - p^2$. As shown in Appendix A,

$$V_\ell = \ell h(0) + 2\sum_{k=1}^{\ell-1}(\ell-k)h(k), \qquad (2)$$

where $h(0) \equiv \mathrm{var}[z_i] \simeq p_c(1-p_c) + O(L^{-\beta/\nu_\perp})$. We further show that if $h(r) \sim r^{-\psi}$, then the exponent governing the growth of the particle-number variance obeys the scaling relation $\phi + \psi = 2d$ in d dimensions.

We study the restricted CSS using quasistationary simulations, which permit arbitrarily long evolution times [43]. (Details on simulation times and preparation of initial conditions are provided in Appendix B). The particle density $p = N/L^d$ is set as close to the critical value p_c as possible.

Figure 1 shows simulation results for $|h(\ell)|$ in one and two dimensions, averaged over intervals that grow $\propto e^\ell$, so that they are uniformly spaced in $\ln\ell$ (logarithmic binning). In one dimension, the correlation function decays roughly as a power law for ln $\geqslant 3$ ($\ell \geqslant 20$), until $\ell \simeq L/4$. [The periodic boundaries imply that $h(\ell)$ must be symmetric about the point $\ell = L/2$, where it takes its minimum value]. In two dimensions, apparent power-law decay holds even for small ℓ. The decay exponent ψ is approximately 1.40(2) in one dimension (1D), and 2.35(4) in two dimensions (2D). These values reflect the behavior at *small* ℓ: corrections to scaling and finite-size effects make it difficult to obtain a precise value, which, in any case, is not required for the present discussion. Although the correlations decay algebraically, their amplitude is quite small; for $\ell = 20$, $|h(\ell)|$ is already of order 10^{-4}. The correlation function is negative for $\ell > 1$, as might be expected from particle number conservation. In one dimension, there is a minimum in $|h(\ell)|$ for $\ell \simeq 10$. This minimum reflects details in the local transfer dynamics in one dimension and does not affect the large-ℓ scaling behavior of V_ℓ; it is not observed in 2D.

In one dimension, we use the simulation data for $h(\ell)$ to calculate the particle-number variance V_ℓ via Eq. (A1). In two dimensions, V_ℓ is calculated directly from the number of particles in squares of side $\ell = 2,4,8,\ldots,L/2$; statistics are taken for all L^2 possible positions of the squares.

The results for V_ℓ for different lattice sizes are quite similar, for $\ell < L/4$, as shown in Fig. 2, in both one dimension (main graph) and two dimensions (inset). [In this plot the initial value $V_{\ell=1} = h(0)$ is subtracted from each curve]. The variance grows roughly as a power law with the size of the region, but with an apparent exponent, $\phi(\ell)$, which decreases with increasing ℓ. Fits to the 1D data on the region $3 < \ln\ell < 4$, the beginning of the apparent power-law regime, yield values for ϕ that grow systematically with system size, leading to the estimate: $\phi(0) \simeq 0.60(2)$. In two dimensions, for small ℓ, we find $V_\ell \sim \ell^\phi$, with an exponent $\phi(0) = 1.71(1)$ (here the fitting interval is $2 \leqslant \ell \leqslant 10$). (Thus the scaling relation $\phi + \psi = 2d$ is satisfied to within uncertainty). In the interest of comparing our results with those of [27], we plot in Fig. 2 the power laws reported by these authors; they are generally consistent with our results, although deviations from simple power laws are evident.

In 1D, the maximum variance, $V_{L/2}$, grows systematically with system size, following $V_{L/2} \sim L^{\overline{\phi}}$ with $\overline{\phi} = 0.483(2)$.

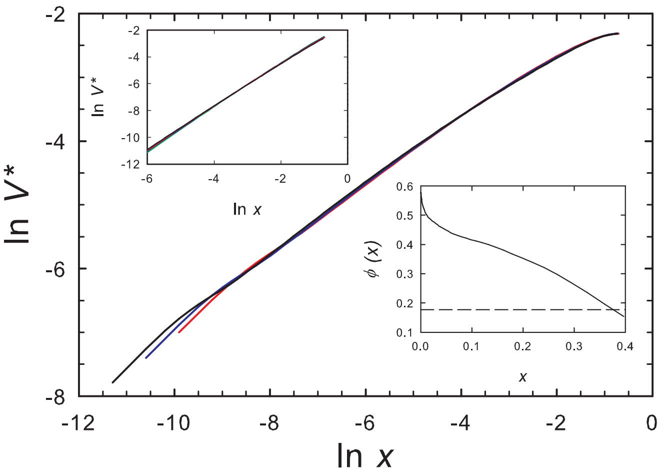

FIG. 3. (Color online) Scaling plot of particle-number variance in the one-dimensional CSS at criticality using scaling variables $V^* \equiv V_\ell/L^{0.483}$ and $x \equiv \ell/L$. System sizes (lower to upper at left) $L = 20\,000$, 40 000, and 80 000. Upper inset: scaling plot for the two-dimensional CSS, using $V^* \equiv V_\ell/L^{1.60}$, with $L = 512$, 1024, 2048, and 4096. Lower inset: effective growth exponent $\phi(x)$ versus x in one dimension. The dashed line denotes the value of ϕ above which particle-density fluctuations are relevant.

This motivates us to seek a scaling form for V_ℓ; in one dimension, we find that the data for the three largest sizes follow,

$$V_\ell = L^{\overline{\phi}} V^*(\ell/L). \tag{3}$$

The scaling function V^* is plotted in Fig. 3. While there is a good data collapse, we note that the exponent used to obtain the data collapse, 0.483, is considerably smaller than the value $\phi(0) = 0.60(2)$ describing the growth of V_ℓ on short scales. Thus the scaling function is not a simple power law: it exhibits significant variations in slope (on logarithmnic scales) well before the maximum at $x \equiv \ell/L = 1/2$. A similar scaling is found in two dimensions, using $\overline{\phi} = 1.60(2)$ (see upper inset of Fig. 3). Our analysis confirms the scaling of particle-density fluctuations with system size, which in turn permits us to assess the relevance of such fluctuations to critical behavior. In the infinite-size limit, we have $V_\ell \sim \ell^{\phi(0)}$, and since $\phi(0) > 0.1766$, the particle-number variance grows rapidly enough with ℓ to be relevant by the criterion of Eq. (1). In fact, the lower inset of Fig. 3 shows that $\phi(x)$ exceeds the value required for relevance over most of the range of variation of x. The same conclusion holds in two dimensions: the effective exponent $\phi(x)$ remains well above the threshold for relevance ($\phi = 1.271$) for $0 < x < 1/2$.

According to Basu *et al.* [24], the nature of the initial configuration has an important influence on the scaling properties of the CSS. We therefore repeat the study of $h(\ell)$ for natural and uniform initial configurations, generated as described in Appendix B. The results for $h(\ell)$ are identical (see Fig. 1, inset). Thus the observed particle-density correlations (and fluctuations) are generated by the dynamics of the system, and do not reflect the initial distribution, just as was verified by Lee [24–26]. We further verify that the QS values of the activity density $\overline{\rho}$ and of the moment ratio $m = \langle \rho^2 \rangle / \overline{\rho}^2$ obtained using different kinds of ICs agree to within uncertainty. For example, for $L = 80\,000$ and $N = 74\,382$, we find QS activity densities of $\rho = 0.015\,89(8)$ and $0.015\,93(8)$ for random and natural ICs, respectively; the corresponding values of the moment ratio are $m = 1.150(3)$ and 1.153(3). In two dimensions, we again verify that the results for V_ℓ, and for the QS activity density $\overline{\rho}$ and the moment ratio m are the same, whether we use random, natural or uniform ICs.

In summary, we study particle distribution statistics in a restricted-height conserved stochastic sandpile in one and two dimensions, focusing on the particle number variance on regions of increasing size. Our results show that the growth in the variance is sufficiently rapid for particle-density fluctuations to be a relevant perturbation of directed percolation. This means that if the critical behavior were DP-like, the fluctuations in particle density, *generated by the dynamics of the sandpile itself*, would *alter* the critical behavior. We thus have quite general reason for believing that the CSS does not belong to the DP universality class. Our argument depends on Harris' criterion for the relevance of disorder. Although the validity of this criterion is well established for DP, we note that several apparent violations of the criterion have been observed in phase transitions to an absorbing state [44], including the CSS [25,26] and kinetic Ising cellular automata in the parity-conserving class [45]. Understanding these observations is an important subject for future study.

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APPENDIX A: PARTICLE-NUMBER VARIANCE AND CORRELATIONS

To begin, we recall the relation between particle-number fluctuations and the associated correlation function. Consider the one-dimensional CSS. Denoting stationary averages by $\langle \cdots \rangle$, we have $\langle z_i \rangle = p$. Define the correlation function $h(j) \equiv \mathrm{cov}(z_i, z_{i+j}) = \langle z_i z_{i+j} \rangle - p^2$. Then, since $N_\ell = \sum_{i=1}^{\ell} z_i$, we have

$$\begin{aligned} V_\ell &= \left\langle \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1} z_i z_j \right\rangle - (p\ell)^2 \\ &= \sum_{k=-(\ell-1)}^{\ell-1} (\ell - |k|) \langle z_0 z_k \rangle - (p\ell)^2 \\ &= \ell \langle z_0^2 \rangle + 2 \sum_{k=1}^{\ell-1} (\ell - k) \langle z_0 z_k \rangle - (p\ell)^2 \\ &= \ell h(0) + 2 \sum_{k=1}^{\ell-1} (\ell - k) h(k). \end{aligned} \tag{A1}$$

Thus the particle number variance can be obtained from the occupancy correlation function $h(j)$.

Our simulation results show that at the critical point, the correlation function follows an approximate power law, $h(r) \approx r^{-\psi}$. The exponent ϕ governing the growth of V_ℓ is related to ψ as follows. Since the total particle number is fixed, we have

$$V_L = Lh(0) + 2\sum_{k=1}^{L-1}(L-k)h(k) = 0. \tag{A2}$$

Due to the periodic boundary condition, we have $h(L-j) = h(j)$, which allows us to write

$$\begin{aligned}\sum_{k=1}^{L-1}(L-k)h(k) &= \sum_{k=1}^{L/2}(L-k)h(k) + \sum_{k=L/2+1}^{L-1}(L-k)h(k)\\ &= \sum_{k=1}^{L/2}(L-k)h(k) + \sum_{k=0}^{L/2-1}kh(k)\\ &= L\sum_{k=1}^{L/2-1}h(k) + \frac{L}{2}h(L/2).\end{aligned} \tag{A3}$$

Equation (A2) then implies the sum rule,

$$2\sum_{k=1}^{L/2-1}h(k) + h(L/2) = -h(0). \tag{A4}$$

Using Eqs. (A1) and (A4) we may write

$$\begin{aligned}\frac{V_\ell}{\ell} &= h(0) + 2\sum_{k=1}^{\ell-1}h(k) - \frac{2}{\ell}\sum_{k=1}^{\ell-1}kh(k)\\ &= -h(L/2) - 2\sum_{k=\ell}^{L/2-1}h(k) - \frac{2}{\ell}\sum_{k=1}^{\ell-1}kh(k).\end{aligned} \tag{A5}$$

We note that $h(0) \equiv \mathrm{var}[z_i] = \mathrm{Prob}[z_i = 1] + 4\,\mathrm{Prob}[z_i = 2] - p^2$. But finite-size scaling implies that $\mathrm{Prob}[z_i = 2] = \rho \simeq AL^{-\beta/\nu_\perp}$ at the critical point (A is a critical amplitude), and using $\mathrm{Prob}[z_i = 1] + 2\,\mathrm{Prob}[z_i = 2] = p$, we find that at the critical point $h(0)$ tends to $p_c(1-p_c)$ with a correction term $\propto L^{-\beta/\nu_\perp}$.

Suppose that for $1 \ll k \ll L$, the correlation function follows $h(k) \simeq -\overline{h}k^{-\psi}$, where $\overline{h} > 0$. Approximating the sums in Eq. (A5) by integrals, we find, for k sufficiently large, but $k \ll L$, that

$$V_\ell \sim \ell^{2-\psi}, \tag{A6}$$

so that $\phi = 2 - \psi$. Extending the argument to a d-dimensional hypercubic lattice yields $\phi = 2d - \psi$.

APPENDIX B: SIMULATION DETAILS

In one dimension we study rings of 2000, 5000, 10 000, 20 000, 40 000, and 80 000 sites, calculating averages over a set of five to ten realizations. In two dimensions we study systems of L^2 sites, with $L = 128$, 256, 512, 1024, 2048, and 4096; the number of realizations ranges from 90, for the smallest system, to 4, for the largest.

In one dimension, even for the largest systems studied, the activity density reaches a stationary value after about 10^7 time units. We nevertheless discard the first 10^9 time units and perform averages over the subsequent interval of 2×10^9 units. In two dimensions we use 6×10^8 time units for relaxation, and calculate QS averages over 10^9 units.

We employ QS simulations, which permit arbitrarily long evolution times [43]. This method probes the quasistationary probability distribution by restarting the evolution in a randomly chosen active configuration whenever the absorbing state is reached. A list of N_c such configurations, sampled from the evolution, is maintained. The list is renewed by exchanging one of the saved configurations with the current one at rate p_r. In one dimension we use $N_c = 500$, and $p_r = 8/L$. In two dimensions, due to memory limitations, we use $N_c = 10$, and $p_r = 0.1$. In both cases, during the relaxation phase, we use a value of p_r that is ten times greater, to eliminate the vestiges of the initial configuration from the list.

Random initial configurations (ICs) are generated by inserting N particles randomly on a ring of L sites or a square lattice of $L \times L$ sites, subject to the restriction $z_i \leqslant 2$. "Natural" ICs are generated by running the process until it reaches an absorbing configuration, and then performing L^d diffusion events, in which a particle is chosen at random and made to hop to a nearest-neighbor site i, provided this does not result in $z_i > 2$. The diffusion events provide activity without significantly altering particle-number fluctuations on large scales. In one dimension, uniform ICs are prepared by placing a single particle at each site, and then vacating every kth site, with $k = [L/(L-N)]$. (Square brackets denote the integer part). If necessary, a few additional particles are removed from uniformly spaced sites to achieve the desired particle number. We follow a similar procedure to generate uniform initial conditions in two dimensions. The resulting configurations are highly uniform but inactive; to generate an initial population of active sites, we perform L^d diffusion events as in the preparation of natural ICs.

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