Marangoni convection in a thin film on a vertically oscillating plate

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Thermocapillary (Marangoni) convection in a thin film on a plate oscillating with a frequency ranging from ultralow to high is considered. By adjusting the vibration amplitude, the impact of the vibration is kept non-negligible. Using the long-wave approximation framework, the amplitude equations are derived for each frequency interval, and linear and weakly nonlinear stability analyses are performed, supplemented by computations where necessary. In the case of a high vibration frequency, the surface tension effectively increases due to vibration, but the film still ruptures. When the frequency is ultralow, the vibration provides gravity modulation, and the surface deformation emerges subcritically, grows fast, and then decays, all during less than half of the vibration period. In the intermediate regime, the vibration either results in a short-wavelength instability or it does not affect the Marangoni convection.

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I. INTRODUCTION

The stability and dynamics of thin liquid films subject to external forcing continues to attract research efforts [1,2]. Some of the most important factors influencing the dynamics are vibration [3,4], heating [5,6], the surface properties of a substrate or interface [7,8], liquid properties [9,10], or combinations of these effects [11,12].

Use of the long-wave approximation to study thin films is justified by the fact that ordinarily the unperturbed film thickness H_0 is much less than its lateral scale ℓ or the surface wavelength λ, e.g., $\epsilon = H_0/\ell \ll 1$ [1,2]. For vibrated films, there are three basic approaches to the derivation of the model nonlinear dynamics equations [4]:

(i) The dimensionless vibration frequency Ω is small, e.g., $\Omega \sim \epsilon$ [13,14]. In this case, the linear and nonlinear terms of a single evolution equation for the film thickness h are time-dependent.

(ii) When Ω is $O(1)$, e.g., it is finite [15–17], one obtains a single autonomous nonlinear equation that describes the averaged dynamics of h in slow time. The key idea of separating the dynamics into slow and fast scales and the general discussion of the averaging methods can be found in Refs. [11,25,30].

(iii) Using the lubrication approximation [2,4,18,19] at the Reynolds numbers $\mathrm{Re} \ll 1$ and retaining the inertial terms leads to several evolution equations. Most often, two equations emerge, i.e., for the film thickness and for the volumetric flow velocity [4,18,19].

Let us review in more detail the case (iii) above in application to the Faraday instability on the surface of a horizontal film; see, for instance, Refs. [20–25]. Faraday instability (also termed parametric instability) possesses two important characteristics: (i) fluid viscosity is important, and (ii) for a laterally infinite system, the critical wavelength λ_F does not exceed H_0. Because of the latter characteristic, Faraday instability can be classified as short-wave.

In Refs. [18,19], the authors used the expansion in small parameters ϵ and $\mathrm{Re} = \omega H_0^2/\nu$, where ω is the vibration frequency and ν is the kinematic viscosity; since $\mathrm{Re} \ll 1$, $\omega \ll \nu/H_0^2$. The linear stability analysis within this approach gives the Mathieu equation with a dissipation term, and thus it allows one to apply the Floquet theory to study the stability of the time-periodic basic states and also the resonant regimes. The authors considered perturbations of H_0 that are $\sim e^{ikx}$, where x is the horizontal coordinate, thus the perturbation wavelength $\lambda \sim \ell$, which implies $\lambda \gg H_0$ and the instability is long-wave. However, in the numerical solutions of the Mathieu equation and in the experiments [26] (to which the authors of Refs. [18,19] compare their results), the critical wavelength is much smaller than the film thickness, and $\mathrm{Re} \sim O(1)$, e.g., $\omega \sim \nu/H_0^2$. In addition, these authors do not compare their results to the numerical results of more traditional approaches, such as that in Ref. [24].

The authors of Ref. [4] employ the long-wave approximation coupled with the Karman-Pohlhausen approximation (the boundary integral method); the latter is routinely applied in studies of falling films [2,27,28]. This allows one to retain the inertial terms alongside viscous terms in the evolution equation for h. They scale the velocity by some characteristic vertical velocity V_0, the horizontal coordinate by ℓ, and the vertical coordinate by H_0, all along assuming $\epsilon = H_0/\ell \ll 1$ (the lubrication approximation). Also, the perturbation wavelength $\lambda \sim \ell$, the amplitude of the vertical vibration $b \sim H_0$, and $\mathrm{Re} = V_0 H_0/\nu$ is either small or finite [$\sim\epsilon$ or $\sim O(1)$]. The authors recognize that due to the introduction of the parameter V_0, the problem loses consistency, e.g., there are more parameters than necessary, and to regain the latter they rely on the relation $\mathrm{Re}\Omega = \omega H_0^2/\nu$. Thus the dimensionless frequency $\Omega \sim 1/\mathrm{Re}$ is large (when $\omega \sim \nu/H_0^2$). If one uses instead viscous units to make the problem dimensionless, as in Refs. [15,24], then the problem is consistent, and the inertial and viscous terms are of the same order at the vibration amplitude $b \sim \ell^2/H_0$ (the frequency $\omega \sim \nu/H_0^2$ and the vertical velocity component $V_0 \sim \nu/H_0$), and thus the dimensionless vibration amplitude $\sim 1/\epsilon^2$.

Summarizing the above discussion, in Refs. [4,18,19] the simultaneous use of the long-wave and the lubrication approximations gives rise to a contradiction: the formal long-wave instability $\lambda \sim \ell$ is applied to a short-wave Faraday instability $\lambda_F \leqslant H_0$. In doing this, the vibration frequency is considered large, e.g., $\mathrm{Re} = \omega H_0^2/\nu \sim 1$.

For isothermal liquid films, the typical long-wave instability is Rayleigh-Taylor, which emerges in two-layer systems [1,3,25,29]. Theoretical [25] and experimental [29] investigations of the damping of the Rayleigh-Taylor instability by a vertical high-frequency vibration were performed. Also linear analysis of long-wave instability at a moderate vibration frequency and the averaged description at high frequency are conducted in Ref. [3]. In the case of interest for Refs. [4,15,18,19], above the film there is a vacuum or the low-density gas layer, thus the Rayleigh-Taylor instability does not emerge. However, in the nonisothermal situation, long-wave instability can emerge easily—this is the thermocapillary (Marangoni) instability [1,31,32]. The averaged dynamical description of a nonisothermal case is developed in Ref. [31], similarly to Ref. [3] for the isothermal case. In that article, a detailed investigation of the amplitude equation with the Marangoni effect is also presented, however short-wave Faraday instability is absent (the only active modes are long-wave).

The authors of Ref. [23] studied the interaction of Marangoni convection and Faraday waves in an infinitely deep fluid layer with a thermally insulated free surface. They considered the case of small viscosity, high vibration frequency, and heating on the side of the free surface. However, they noted that when heating is from the bulk fluid side, long-wave instability emerges, which can be stabilized by a vertical vibration analogously to the case of Rayleigh-Taylor instability.

In this paper, we plan to consider long-wave Marangoni convection in a liquid film on a vertically vibrated plate. In this system, both long-wave Marangoni instability and short-wave Faraday instability can emerge. Our aim is to extend our theory of averaged fluid motion in vibrated, isothermal, ultrathin films [15,16]. In Ref. [15], a vertical vibration is considered; the conventional small-amplitude, high-frequency approximation of the averaged dynamics is revised due to the number of time and length scales inherent to the ultrathin-film system. In particular, (i) the vibration amplitude b has to be *large* in comparison to the mean layer thickness H_0 in order to provide a finite effect on the film dynamics (see the above discussion of Ref. [4]), whereas (ii) the vibration frequency ω may be of the order of the inverse time of the momentum equilibration across the layer, $\omega \sim \nu/H_0^2$. In contrast, for the longitudinal and tilted vibration [16], the amplitude should be comparable to H_0; therefore, the longitudinal component prevails except for the case of an almost normal vibration.

Operating in a single unified framework, our treatment relies on a consistent asymptotic expansion (see Refs. [15,16]) within the long-wave approximation, which clearly separates the Marangoni and the Faraday instabilities. The interaction of the two instabilities is briefly discussed in Appendix. The main focus in the paper is on the Marangoni instability.

The paper is organized as follows. In Sec. II we formulate the problem. The averaged description is stated and the film dynamics is studied in Sec. III. Also in that section, the reduction to the two well-known limits of Marangoni and Faraday instability is presented. For ultralow frequencies, the corresponding amplitude equation is derived and the dynamics is studied in Sec. IV. The switching between these cases is analyzed in Sec. V via the development of the intermediate asymptotic. In Sec. VI, we present a summary of the results and concluding remarks.

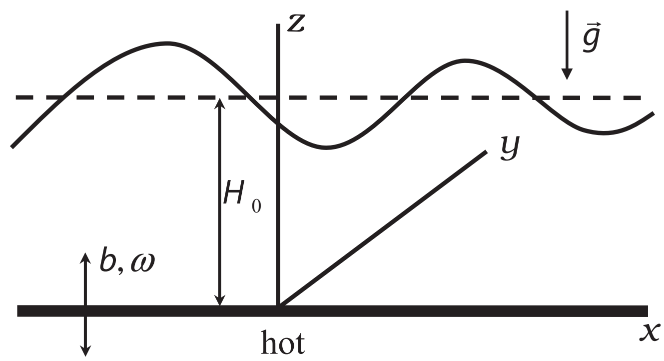

FIG. 1. Problem geometry.

II. PROBLEM STATEMENT

We consider a three-dimensional (3D) thin liquid film of unperturbed height H_0 on top of a planar horizontal plate heated from below (Fig. 1). The plate is at a constant temperature Θ (measured from the temperature of the ambient gas), and it is subject to the vertical harmonic vibration of the amplitude b and the frequency ω. The film height is assumed sufficiently small, so that the conventional averaging procedure fails and the approach developed in Refs. [15,16] should be applied.

We use H_0^2/ν, H_0, ν/H_0, $\rho\nu^2/H_0^2$, and Θ as the units for the time, length, velocity, pressure, and temperature, respectively. (Here ν is the kinematic viscosity and ρ is the density of the liquid.) The Cartesian reference frame is chosen such that the x and y axes are in the plane of the plate and the z axis is normal to this plane. The dimensionless boundary-value problem takes the form

$$\nabla \cdot \mathbf{v} = \mathbf{0}, \tag{1a}$$

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nabla^2\mathbf{v} \tag{1b}$$

$$-(G_0 + B\Omega^2\cos\Omega t)\mathbf{e}_z, \tag{1c}$$

$$P(T_t + \mathbf{v}\cdot\nabla T) = \nabla^2 T, \tag{1d}$$

$$z = 0: \quad \mathbf{v} = \mathbf{0}, \quad T = 1, \tag{1e}$$

$$z = h: \quad h_t + \mathbf{u}\cdot\nabla h = w, \tag{1f}$$

$$\mathbf{n}\cdot\sigma = -\mathrm{Ca}K\mathbf{n} - M\nabla_\tau(T|_{z=h}), \tag{1g}$$

$$\nabla_n T = -\mathrm{Bi}T. \tag{1h}$$

Here, $\mathbf{v} = (\mathbf{u}, w)$ is the fluid velocity (where $\mathbf{u}$ is the velocity in the plane of an oscillating plate and w is the component normal to the plate), T is the temperature, p is the pressure in the liquid, $\sigma = \nabla\mathbf{v} + (\nabla\mathbf{v})^T - p\,\mathbf{I}$ is the stress tensor (the superscript T denotes transposed tensor, and $\mathbf{I}$ is the identity tensor), $h(x,y,t)$ is the dimensionless height of the film, $\mathbf{e}_z$ is the unit vector directed upward, $\mathbf{n} = (\mathbf{e}_z - \nabla h)/\sqrt{1 + (\nabla h)^2}$ is the normal unit vector to the free surface, τ is the tangent vector, and $K = \nabla \cdot \mathbf{n}$ is the mean curvature of the free surface. The problem is characterized by the following dimensionless parameters:

$$\mathrm{Ca} = \frac{\sigma_0 H_0}{\rho \nu^2}, \quad M = -\frac{\sigma_T \Theta H_0}{\rho \nu^2}, \quad G_0 = \frac{g_0 H_0^3}{\nu^2},$$
$$B = \frac{b}{H_0}, \quad \Omega = \frac{\omega H_0^2}{\nu}, \quad \mathrm{Bi} = \frac{q H_0}{\kappa}, \quad P = \frac{\nu}{\chi}, \qquad (2)$$

which are the capillary number, the Marangoni number, the Galileo number, the amplitude and frequency of the vibration, the Biot number, and the Prandtl number, respectively. Here σ_0 is the surface tension, $\sigma_T \equiv d\sigma_0/dT$, q is the heat transfer rate, κ is the thermal conductivity, and χ is the thermal diffusivity of a liquid.

We assume that the capillary number is large, so that $C = \epsilon^2 \mathrm{Ca}$ is $O(1)$. The stretched horizontal coordinates $(X,Y) = \epsilon(x,y)$ and the time scale $t_2 = \epsilon^2 t$, which are conventional for the long-wave Marangoni convection in this case, are also introduced. In what follows, we omit the subscripts "2" for t_2 and the two-dimensional gradient operator with respect to X and Y. We will present an analysis of a wide range of the vibration frequency Ω, adjusting the amplitude B in such a way that the vibration effect is comparable to other important effects (the surface tension damping, thermocapillary, etc.). Any assumptions for other parameters are not required. Three frequency domains featuring qualitatively different dynamics are discussed below:

(i) High vibration frequency: $\Omega = O(1)$, when the averaging technique can be applied; see Sec. III.

(ii) Ultralow vibration frequency: $\Omega = O(\epsilon^2)$, when the vibration modulates the gravity; see Sec. IV.

(iii) Low vibration frequency: $\epsilon^2 \ll \Omega \ll 1$, which matches the asymptotics (i) and (ii); see Sec. V.

III. HIGH VIBRATION FREQUENCY: $\Omega = O(1)$

In this section, we consider the case of finite frequency, $\Omega = O(1)$. Therefore, the vibration period is comparable to the characteristic time of the momentum transfer across the layer, but it is small compared to the typical time of the film evolution. This calls for the application of the averaging technique, even though the conventional restriction inherent in the high-frequency approximation ($\Omega \gg 1$) does not hold. To guarantee that the vibration effect is finite, we assume the vibration amplitude to be large, $B = B_1/\epsilon$, where $B_1 = O(1)$.

A. The amplitude equation

Derivation of the amplitude equation in the case of the above-stated conditions is straightforward [1,15,16,33], thus we omit the details. The equation reads

$$h_t = \nabla \cdot \left[\frac{h^3}{3} \nabla \Pi - \frac{M \mathrm{Bi}\, h^2 \nabla h}{2(1+\mathrm{Bi}\, h)^2} - \frac{B_1^2 \Omega^2}{2} \mathbf{Q} \right], \qquad (3a)$$

$$\Pi = G_0 h - C \nabla^2 h - \frac{B_1^2 \Omega^2}{2} \nabla (f_r h \nabla h), \qquad (3b)$$

$$\mathbf{Q} = Q_1 h^2 (\nabla h)^2 \nabla h + h^3 [Q_{21} \nabla^2 h \nabla h + Q_{22} \nabla h \cdot \nabla \nabla h], \qquad (3c)$$

where

$$Q_1 = \frac{3(2g_1 - \gamma g_2)}{\gamma^2 (\cosh\gamma + \cos\gamma)^2}, \qquad (4a)$$

$$Q_{21} = 6q_1 - 2q_2, \quad Q_{22} = 5q_1 - q_2 - \frac{1}{3}, \qquad (4b)$$

$$f_r = 1 - \frac{\sinh\gamma + \sin\gamma}{\gamma(\cosh\gamma + \cos\gamma)}, \qquad (4c)$$

$$g_1 \equiv \sinh\gamma \sin\gamma, \quad g_2 \equiv \sinh\gamma \cos\gamma + \sin\gamma \cosh\gamma, \qquad (4d)$$

$$q_1 = \frac{\sinh\gamma - \sin\gamma}{\gamma^3(\cosh\gamma + \cos\gamma)}, \quad q_2 = \frac{\cosh\gamma - \cos\gamma}{\gamma^2(\cosh\gamma + \cos\gamma)}, \qquad (4e)$$

$$\gamma = \sqrt{2\Omega} h. \qquad (4f)$$

Here h is the film thickness averaged over the fast time; it depends only on the slow time t. Equation (3a) is the conventional Kopbosynov-Pukhnachev equation [1,33] with the additional vibration-induced terms [15,16].

In the limiting case $\Omega \gg 1$, the oscillatory flow is inviscid; also, the standard averaging technique is applicable. In this limit, Eq. (3a) reduces to Eq. (3.22) from Ref. [31]. In the opposite limiting case, $\Omega \ll 1$, the auxiliary functions f_r, Q_1, Q_{21}, and Q_{22} simplify to the powers of h, and $\mathbf{Q}$, Π in Eq. (3a) take the form

$$\mathbf{Q} = \frac{2\Omega^2 h^6}{315} [h(\nabla^2 h \nabla h + 9 \nabla h \cdot \nabla\nabla h) - 21 (\nabla h)^2 \nabla h], \qquad (5a)$$

$$\Pi = G_0 h - C \nabla^2 h - \frac{B_1^2 \Omega^4}{15} \nabla \cdot (h^5 \nabla h). \qquad (5b)$$

B. Vibration influence on the linear stability and nonlinear dynamics of the film

The linear stability analysis within Eq. (3a) results in the following dispersion relation:

$$\lambda = -k^2 \left[\frac{1}{3} G_0 + \frac{1}{3} k^2 \left(C + \frac{1}{2} B_1^2 \Omega^2 f_r(\gamma_0) \right) - \frac{M \mathrm{Bi}}{2(1+\mathrm{Bi})^2} \right], \qquad (6a)$$

$$\gamma_0 = \sqrt{2\Omega}, \qquad (6b)$$

which couples the growth rate λ and the wave number k of a perturbation. As is seen from Eq. (6a), within the linear stability problem the vibration only effectively increases the surface tension.

The stability boundary ($\lambda = 0$) is determined by the relation

$$M_0 = \frac{2(1+\mathrm{Bi})^2}{3\,\mathrm{Bi}} \left[G_0 + k^2 \left(C + \frac{1}{2} B_1^2 \Omega^2 f_r(\gamma_0) \right) \right]. \qquad (7)$$

Obviously, the critical perturbations are long-wave, with $k = 0$; this mode is insensitive to the vibration. For a confined system with a discrete spectrum of k, the vibration leads to layer stabilization. The growth rate Eq. (6a) and the neutral curve Eq. (7) at $B_1 = 0$ are well-known for the "pure" long-wave Marangoni convection (without the vibration effect [1,33,34]).

At a small supercriticality, $M = M_0 + \delta^2 M_2$, the amplitude of the stationary solution is

$$a^2 = -K_a M_2. \tag{8}$$

We do not present here the cumbersome Landau coefficient K_a. If one introduces the potential V in a similar fashion to Ref. [35] [see Eq. (3.24) therein], then in the limit $V = 0$ the Landau coefficient coincides with the value in that paper. Moreover, again similar to Ref. [35], K_a is positive in the entire parameter domain, i.e., the subcritical bifurcation occurs even in the presence of the vibration.

This growth of a small perturbation does not end in saturation, and the rupture occurs along with the formation of a fractal structure, described in Ref. [36]. The evolution of the near-ruptured film is unaffected by the vibration. Indeed, for small h, γ is also small and the vibration-induced terms are given by Eqs. (5). These terms are very small (at least of the order h^6) in comparison with the contributions from the capillarity and thermocapillarity.

To summarize this section, we note that (i) the vibration stabilizes the film, (ii) the subcritical excitation emerges for any B_1, and (iii) the rupture is not affected by the vibration.

IV. ULTRALOW VIBRATION FREQUENCY: $\Omega = O(\epsilon^2)$

In the limit of ultralow frequency, $\Omega = \epsilon^2 \Omega_2$ with $\Omega_2 = O(1)$, the typical time of the oscillations becomes comparable to the time of the surface evolution. Therefore, the averaging technique used in the previous section is not applicable. In fact, the only effect of the vibration in this case is the gravity modulation within the conventional Kopbosynov-Pukhnachev equation [1,33]:

$$h_t = \nabla \cdot \left[\frac{h^3}{3} \nabla \Pi_2 - \frac{M \mathrm{Bi}\, h^2 \nabla h}{2(1 + \mathrm{Bi}\, h)^2} \right], \tag{9a}$$

$$\Pi_2 = G(t) h - C \nabla^2 h, \tag{9b}$$

where $G(t) = G_0 + B_4 \Omega_2^2 \cos \Omega_2 t$ is the modulated gravity and $B_4 = B\epsilon^4$. Hence, the amplitude of the ultralow frequency vibration has to be large, $O(\epsilon^{-4})$, in order to provide a finite acceleration; notice that this is three orders of magnitude larger than in the previously considered case of a high vibration frequency.

It can be easily shown that the vibration does not change the stability boundary within the Floquet theory (the asymptotic criterion). Indeed, the linearized equation (9a) can be integrated exactly; the perturbation to the flat surface $h_0 = 1$ is proportional to $\exp[ikX + \phi(t)]$, with $\phi(t)$ given by

$$\phi(t) = \lambda_0 t - k^2 \frac{B_4 \Omega_2}{3} \sin \Omega_2 t, \tag{10}$$

$$\lambda_0 = -k^2 \left[\frac{G_0 + Ck^2}{3} - \frac{M \mathrm{Bi}}{2(1 + \mathrm{Bi})^2} \right]. \tag{11}$$

In the vein of the Floquet theory, the growth/decay of the perturbations is determined only by the sign of λ_0; hence, the stability boundary is given by Eq. (7) at $B_1 = 0$, or

$$M_0^{(\mathrm{asy})} = \frac{2(1 + \mathrm{Bi})^2}{3\, \mathrm{Bi}} (G_0 + Ck^2). \tag{12}$$

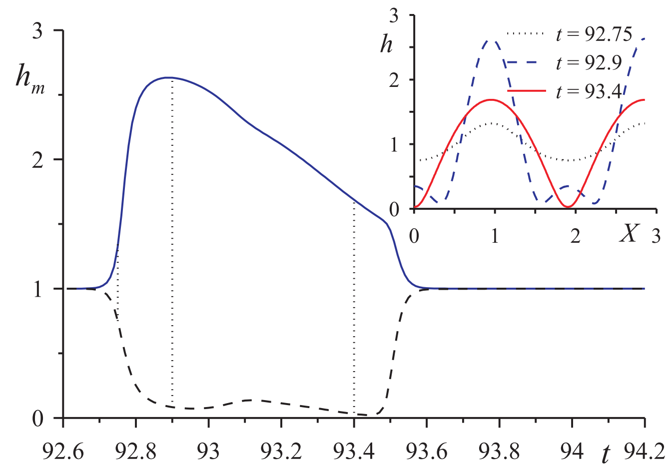

FIG. 2. (Color online) Evolution of the maximum and minimum film thickness $h(X)$ within Eq. (9a); $M_B = M\mathrm{Bi} = 7$, $G_0 = 10$, $C = 1$, $k = 1.1$, $B_4 = 7$, and $\Omega_2 = 2$. The inset shows the film profile at the three time moments marked by the vertical dashed lines.

The vibration results simply in the time-periodic modulation of this decaying/growing perturbation.

However, in the case of low frequency considered here, the Floquet theory, being the asymptotic criterion, often overestimates the stability threshold [37–40]. In contrast, the empirical criterion, which operates with the growth rate obtained directly from Eq. (9a), results in the underestimated stability threshold [assuming the minimum value, $G_0 - B_4 \Omega_2^2$, of $G(t)$ over the vibration period]. The stability boundary within this criterion is given by

$$M_0^{(\mathrm{emp})} = \frac{2(1 + \mathrm{Bi})^2}{3\, \mathrm{Bi}} \left(G_0 - B_4 \Omega_2^2 + Ck^2 \right). \tag{13}$$

Next we describe the computations of the film thickness. We set $C = 1$, which can be achieved by the appropriate rescaling of the horizontal length scale. The computations are performed for a finite domain with the boundary conditions $h_X = h_{XXX} = 0$ at $X = 0$ and $X = \pi/k$. For simplicity, we deal with the limiting case $\mathrm{Bi} \ll 1$, $M \gg 1$, whereas $M\mathrm{Bi} = M_B$ is kept finite. The computations show that a high amplitude surface deformation can emerge from the limit cycle subcritically; the excitation first occurs at the Marangoni number satisfying $M_0^{(\mathrm{emp})} < M < M_0^{(\mathrm{asy})}$. Typical examples of such limit cycles are presented in Figs. 2 and 3. In both cases, the disturbance emerges, grows fast, and then decays during less than a half-period; although the excitation period resembles a homoclinic cycle, the oscillation is strictly periodic in time with the period $2\pi/\Omega$.

In Fig. 2, the Marangoni number is less than $M_0^{(\mathrm{asy})}$, which is suggestive of the subcritical (noise-induced) excitation. In this example, the "wave number" of the pattern remains fixed during the period; the film distortion always contains three half-waves, as seen in the inset. For larger M, the pattern changes during the excitation period: initially the disturbance contains three half-waves [see Fig. 3(b)], but after some rearrangement only one half-wave survives [see Fig. 3(c)]. Also it is important that $M = 8$ is larger than $M_0^{(\mathrm{asy})}$; in the absence of vibration, the thin film is unstable at this value of M

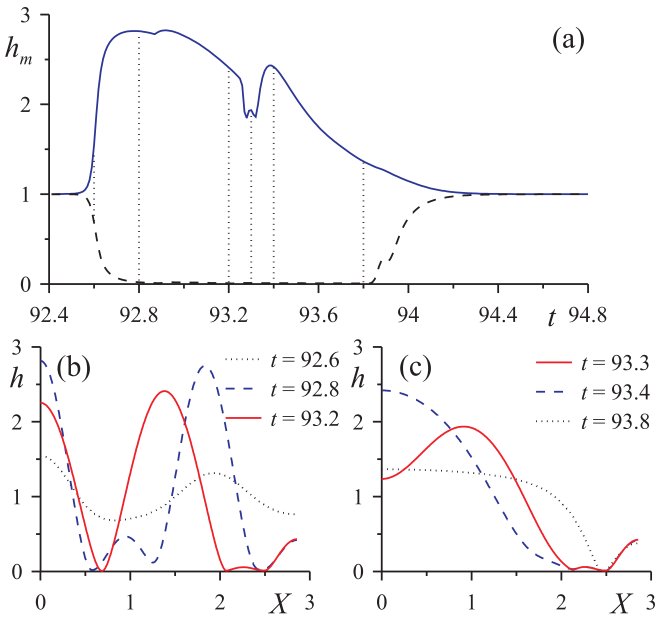

FIG. 3. (Color online) Dynamics of the film thickness within Eq. (9a); $M_B = M\mathrm{Bi} = 8$, $G_0 = 10$, $C = 1$, $k = 1.1$, $B_4 = 7$, and $\Omega_2 = 2$. (a) Evolution of the maximum and minimum of $h(X)$; (b,c) Film profiles for the six time moments marked by the vertical dashed lines in panel (a). Notice that $\min(h) \approx 0.006 > 0$ on the time interval shown, thus the film does not rupture.

and the instability leads to the layer rupture [35,36,41,42]. Therefore, although the vibration makes the flat film unstable at smaller values of M [see Eq. (13)], it also stabilizes the moving film against rupture, which can be important for many microfluidic applications.

Detailed computations not presented here indicate that the characteristics of the stable limit cycle vary substantially as the amplitude and frequency of the vibration change. There are several bifurcations that result in a multistability domain and the dependence of the final state on the initial conditions. However, a detailed discussion of these limit cycles is beyond the scope of this paper.

To summarize this section, we note that (i) the gravity is modulated at ultralow frequency, (ii) the linear stability is affected by vibration, and (iii) the limit cycle emerges subcritically at large $B_4\Omega_2^2$.

V. INTERMEDIATE ASYMPTOTICS AT LOW VIBRATION FREQUENCY: $\epsilon^2 \ll \Omega \ll 1$

In this section, we trace the intermediate asymptotics mediating (i) the averaged model, Eq. (3a), with (ii) the excitation at ultralow frequency, Eq. (9a).

Detailed analysis shows that the intermediate asymptotics with

$$B = B_i/\epsilon^{1+3\beta/2}, \quad \Omega = \Omega_i \epsilon^{\beta}, \tag{14}$$

where $0 < \beta < 2$ (thus $\epsilon^2 \ll \Omega \ll 1$), is special in the sense that small B values produce a small effect on the film dynamics, whereas larger values lead to vibration-induced instability, as we explain below.

Within the scaling given by Eq. (14), the averaged dynamics of the layer thickness is governed by the conventional Kopbosynov-Pukhnachev amplitude equation; see Eq. (3a) at $B_1 = 0$ or Eq. (9a) in the absence of gravity modulation, $B_4 = 0$. Therefore, the vibration does not influence the slow convective motion even at large B_i.

However, one can employ another scaling of the time and space coordinates, viz.,

$$\bar{t} = \epsilon^{\beta} t, \quad \xi = \epsilon^{\alpha} x,$$

where still $0 < \beta < 2$ and $1/2 < \alpha = (2+\beta)/4 < 1$ [compare to the standard long-wave scaling $t_2 = \epsilon^2 t$, $(X,Y) = \epsilon(x,y)$ that was used to arrive at Eqs. (3a) and (9a)]. The new scaling corresponds to short waves, as follows from ξ, and as a result the effect of the vibration is retained. The amplitude equation in this case reads

$$\partial_{\bar{t}} h_s = \tfrac{1}{3}\partial_\xi\left[h_s^3 \partial_\xi\left(B_i \Omega_i^2 \cos\Omega_i \bar{t} h_s - C\partial_\xi^2 h_s\right)\right]. \tag{15}$$

Notice that Eq. (15) is the limiting case $M = G_0 = 0$ of Eq. (9a).

Again, the vibration does not lead to instability within the Floquet theory (the asymptotic criterion) applied to Eq. (15), but instability does occur within the empirical criterion. An example of a noise-induced finite-amplitude regime is shown in Fig. 4; it is qualitatively similar to the oscillations shown in Fig. 3. Subcritical excitation takes place if B_i exceeds a certain value B_*, which depends on the problem parameters, the length of the computational domain, and, generally speaking, the initial conditions.

To summarize, at $\epsilon^2 \ll \Omega \ll 1$ the vibration either does not have an impact on the long-wave Marangoni convection, or it leads to a nonlinear excitation of the regime with a shorter wavelength and faster characteristic time scale. The matching

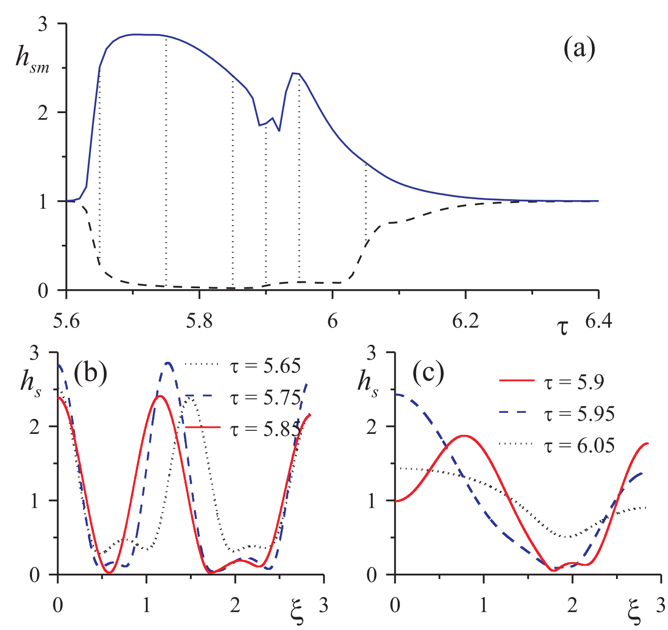

FIG. 4. (Color online) Dynamics of the film thickness within Eq. (15); $C = 1$, $k = 1.1$, $B_i = 5$, and $\Omega_i = 2$. (a) Evolution of the maximum and minimum of $h_s(X)$; (b,c) the film profiles for the six time moments marked by the vertical dashed lines in panel (a).

of the intermediate asymptotics to the averaged description is provided by the limit $B_1 \to 0, \Omega \to 0$ within Eq. (3a) and by the limit $\beta \to 0$ within Eq. (14). In the opposite limit, $\beta \to 2$, the difference between h_s and h disappears (since $\xi = X$ at $\beta = 2$) and the "averaged" (Kopbosynov-Pukhnachev) and "oscillatory" (h_s) effects are simply combined in a single equation. This leads to Eq. (9a).

VI. CONCLUSION

We studied the stability and dynamics of a thin liquid film on top of a heated, vertically vibrating substrate. A wide interval of the vibration frequency ω is considered, from the high (such that the Stokes layer is thin in comparison to the mean film thickness) to the ultralow (such that the vibration period is comparable to the typical time τ of the layer thickness relaxation). Three frequency regimes are analyzed: (i) high frequency [$\omega\tau \gg 1$ or $\Omega = O(1)$], when the averaging technique is applied; (ii) ultralow frequency [$\omega\tau = O(1)$ or $\Omega = O(\epsilon^2)$], when the vibration results only in a gravity modulation; and (iii) the intermediate regime, in which the asymptotics (i) and (ii) are matched. In case (i), the conventional Kopbosynov–Pukhnachev equation [1,33] holds with the additional vibration-generated terms. Linear analysis reveals long-wave instability with the critical wave number $k = 0$, which is insensitive to vibration. Weakly nonlinear analysis shows the film rupture. In case (ii), both the asymptotic and empirical expressions for the stability boundary are derived, and the subcritical excitation of the finite surface deformation is numerically determined. In regime (iii), the vibration either does not have an impact on the long-wave Marangoni convection, or it leads to a nonlinear excitation of the regime with a shorter wavelength and a faster characteristic time scale.

It is well known that the vertical vibration may result in the Faraday instability of a free surface [1,2,24]. Therefore, there are two competitive instability mechanisms: the Marangoni convection and the Faraday instability. For clarity, we display both stability boundaries in Fig. 5. The details on how this sketch was obtained can be found in Appendix.

When the vibration amplitude B is increased, the vibrational Marangoni instability studied in this paper will occur prior to the Faraday instability, since the boundaries of domains (i)–(iii) are below the line demarcating Faraday instability. However, since the boundaries of domains (i) and (iv) are close to the boundary of the Faraday instability, it follows that a high-frequency vibration suppresses the Marangoni instability and gives rise to the Faraday instability.

Finally, we emphasize that a high vibration frequency $\Omega = \omega H_0^2/\nu \geqslant 1$ is considered in Refs. [4,18,19] (or equivalently, $\mathrm{Re}\Omega = \omega H_0^2/\nu \geqslant 1$ in terms of Ref. [4] and $\mathrm{Re} = \omega H_0^2/\nu \geqslant 1$ in terms of Refs. [18,19]). Thus the results obtained by these authors are not in doubt even when the long-wave instability is present [see domains (i) and (iv) in Fig. 5]. However, at a low frequency $\Omega \ll 1$ ($\mathrm{Re} \ll 1$), the long-wave instability emerges first [see domains (ii) and (iii) in Fig. 5] and thus in this case the approach of Refs. [4,18,19] should be used with caution.

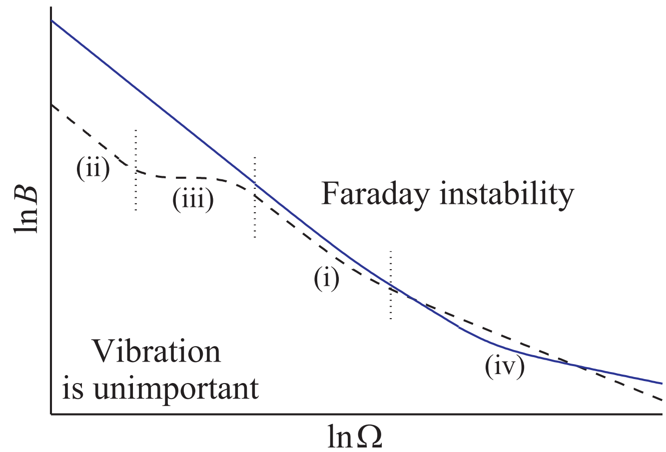

FIG. 5. (Color online) Sketch of the amplitude-frequency variation $\ln B(\ln \Omega)$. The dashed line corresponds to the vibration impact on the Marangoni convection: the averaged description within Eq. (3a) is valid in domain (i); the ultralow frequency equation (9a) applies in domain (ii); the intermediate asymptotics, Eqs. (14) and (15), applies in domain (iii). Domain (iv) corresponds to the high-frequency approximation within the averaged description; see Ref. [31]. The dashed line should be thought of as a stripe, where the corresponding amplitude equations hold. The solid line represents the boundary of the Faraday instability; see Appendix. The film is unstable above the line.

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APPENDIX: INTERACTION OF PARAMETRIC INSTABILITY WITH MARANGONI CONVECTION

A complete analysis of Faraday instability in thin films was performed by Mancebo and Vega (MV) [24], who classified all regimes according to the parameters $C_{\mathrm{MV}} = \nu/\sqrt{gH_0^3 + \sigma H_0/\rho}$, $S_{\mathrm{MV}} = \sigma/(\sigma + \rho g H_0)$, and $\omega_{\mathrm{MV}} = \omega\sqrt{\rho H_0^3/(\rho g H_0^2 + \sigma)}$ (C_g, S, and ω in the notation of Ref. [24], respectively). With our scalings, $C_{\mathrm{MV}} = \epsilon/\sqrt{C}$, $S_{\mathrm{MV}} = 1 - \epsilon^2 G_0/C$, and $\omega_{\mathrm{MV}} = \epsilon\Omega/\sqrt{C}$, thus either case B.1.2 (the long-wave limit, finite Ω) or B.1.3 (the small-frequency limit, $\Omega \ll 1$) is operative.

According to Ref. [24], in case B.1.2 the critical wave number is given by

$$\tilde{k} = \frac{k}{\sqrt{1 - S_{\mathrm{MV}} + C_{\mathrm{MV}}}} = \frac{kC^{1/4}}{\sqrt{\epsilon}}.$$

Thus the wavelength of the critical perturbations, $O(\epsilon^{-1/2})$, is small compared to the typical convective length scale $O(\epsilon^{-1})$. A similar situation occurs for the typical time scales: the characteristic time scale is given by $\tilde{t} = C_{\mathrm{MV}}\sqrt{\mathrm{Ca}}t_* = O(1)$ (thus the rescaled frequency $\tilde{\omega}$ coincides with Ω), which is much smaller than the typical time scale of the Marangoni convection, $t = O(\epsilon^{-2})$.

Intuitively, it seems plausible that the processes that operate on such different time and length scales do not interact. Formally, this can be proved using the idea of frozen coefficients. To prevent the excitation of the Faraday waves, one only has to ensure that the stability conditions found by Mancebo and Vega are valid at any moment and at any point within the layer. Therefore, for our purposes, the parameters introduced in Ref. [24] must be stated using the local thickness $H_0 h$ instead of the mean thickness H_0.

It follows that the stability condition

$$\tilde{a}\tilde{\omega}^2 < A_c(\tilde{\omega}), \tag{A1}$$

where $\tilde{a} = C_{\mathrm{MV}} a = B_1/\sqrt{C}$ is the rescaled vibration amplitude, is restated as follows:

$$\frac{B_1}{\sqrt{C} h^{3/2}} \Omega^2 h^4 = \frac{B_1 \Omega^2}{\sqrt{C}} h^{5/2} < A_c(\Omega h^2). \tag{A2}$$

The function A_c is given in Fig. 4 of Ref. [24]; the case that we consider corresponds to $\gamma = (1 - S_{\mathrm{MV}})/C_{\mathrm{MV}} \sim \epsilon \ll 1$, thus the curve for $\gamma = 0$ is needed. In case B.1.3 [24], the small-frequency limit is $\omega_{\mathrm{MV}} \ll C_{\mathrm{MV}}$, or our rescaled limit is $\Omega \ll 1$. Accounting for the scaling factors, we arrive at the sketch, shown in Fig. 5, for the critical value of the amplitude.

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