Generalized perturbation (*n*, *M*)-fold Darboux transformations and multi-rogue-wave structures for the modified self-steepening nonlinear Schrödinger equation

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In this paper, a simple and constructive method is presented to find the generalized perturbation (*n*, *M*)-fold Darboux transformations (DTs) of the modified nonlinear Schrödinger (MNLS) equation in terms of fractional forms of determinants. In particular, we apply the generalized perturbation (1, *N* − 1)-fold DTs to find its explicit multi-rogue-wave solutions. The wave structures of these rogue-wave solutions of the MNLS equation are discussed in detail for different parameters, which display abundant interesting wave structures, including the triangle and pentagon, etc., and may be useful to study the physical mechanism of multirogue waves in optics. The dynamical behaviors of these multi-rogue-wave solutions are illustrated using numerical simulations. The same Darboux matrix can also be used to investigate the Gerjikov-Ivanov equation such that its multi-rogue-wave solutions and their wave structures are also found. The method can also be extended to find multi-rogue-wave solutions of other nonlinear integrable equations.

I. INTRODUCTION

In general, it is difficult to directly seek exact solutions including solitons of nonlinear wave equations in the soliton theory and integrable system. If one can find a proper Lax pair [1] of a nonlinear wave equation, then, based on its Lax pair, one may find its solutions by means of some transformations such as the inverse scattering transformation [2–4], the Riemann-Hilbert scattering method [5], the Darboux transformation [6–8], and so on. The Darboux transformation [6–8] is regarded as a powerful approach to study solutions of nonlinear integrable systems in terms of their corresponding Lax pairs, which originally arose from Darboux's study [9] of the linear Sturm-Liouville equation, $\psi_{xx} + [\lambda - u(x)]\psi = 0$, which is also called the linear Schrödinger equation with the external potential in quantum mechanism. Until now, the Darboux transformation has been used to investigate many types of solutions of nonlinear integrable equations, including multisoliton solutions, breathers, periodic solutions, and rational solutions. In fact, the Darboux transformation exhibits different forms in nonlinear integrable systems, in which the following three main types of Darboux transformations were usually used: the Darboux transformation without the Darboux matrix (see, e.g., Ref. [10]) [Type (i)], the Darboux transformation with the iteration Darboux matrix (see, e.g., Ref. [6]) [Type (ii)], and the Darboux transformation with the *N*-order Darboux matrix whose elements being the polynomials of the spectral parameters (see, e.g., Refs. [7,8]) [Type (iii)].

Recently, the modified Darboux transformation related to Type (i) was used to find multi-rogue-wave solutions of the self-focusing nonlinear Schödinger (NLS) equation from the initial plane-wave solution [11]. Moreover, the generalized Darboux transformation related to Type (ii) with the initial plane-wave solution was also used to investigate multi-rogue-

wave solutions of the self-focusing NLS equation [12]. In fact, there exist some other methods to study rogue-wave solutions of many nonlinear wave equations with constant or varying coefficients (see, e.g., Refs. [13–17]). It is still an important topic to present more powerful and simple methods to find other types of solutions including multi-rogue-wave solutions of nonlinear integrable equations.

To the best of our knowledge, Type (iii) of the Darboux transformations and their some extensions have not been used to investigate other types of solutions (e.g., multi-rogue-wave solutions) of integrable equations except for their multi-soliton-type solutions. In this paper, we will present an approach to construct the generalized perturbation (*n*, *M*)-fold Darboux transformation of nonlinear integrable equations based on other Type (iii) Darboux transformations and the Darboux matrix in terms of the Taylor-series expansion for the parameter and a limit procedure such that their multi-rogue-wave solutions can be obtained.

The modified nonlinear Schrödinger (MNLS) equation with the self-steepening term is given by the form

$$iq_t + q_{xx} + i(|q|^2 q)_x + 2\rho|q|^2 q = 0, \qquad (1)$$

where $q \equiv q(x,t)$ is the slowly varying complex envelope of the wave, ρ is a real constant, and $i^2 = -1$, where the subscript denotes the partial derivative with respect to the variables x, t and the term $i(|q|^2 q)_x$ is called the self-steepening term, which causes an optical pulse to become asymmetric and steepen upward at the trailing edge [18,19]. When $\rho = 0$, Eq. (1) reduces to the Kaup-Newell type of the derivative NLS equation [20]. In fact, Eq. (1) can also reduce to the derivative NLS equation via a gauge transformation [21]. Equation (1) describes the short pulses propagate in a long optical fiber charaterized by a nonlinear refractive index $n(\omega, \vec{E}) = n(\omega) + n_2|\vec{E}|^2$ [22]. Equation (1) can also be used to describe Alfvén waves propagating along the magnetic field in cold plasmas [23] and the deep-water gravity waves [24]. Equation (1) was also called the perturbation NLS equation

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[25]. Equation (1) was shown to be completely integrable by the inverse scattering transformation [26]. The dynamics of optical solitons with initial phase modulation was studied [27]. The projection matrix method combining the Daroux transformation and the Riemann-Hilbert problem was used to derive the DT and Bäcklund transformation for Eq. (1) such that a new solution was obtained [28]. Its N-soliton solutions were also obtained in terms of the generalized Zakharov-Shabat equation [29] and the bilinear method [30]. Moreover, multiple pole solutions of Eq. (1) were also found [31]. Equation (1) and its higher-dimensional case were also studied analytically or numerically [32–35]. Recently, some rogue-wave solutions of the derivative NLS equation have been investigated [36].

In this paper, we will further investigate multi-rogue-wave solutions of Eq. (1) via the generalized perturbation (n,M)-fold Darboux transformation technique, which is a simple and constructive method. Moreover, this method, with the same formal Darboux matrix, can also be applied to investigate multi-rogue-wave solutions of the Gerjikov-Ivanov equation [37]

$$iq_t + q_{xx} - iq^2 q_x^* + \tfrac{1}{2}|q|^4 q = 0. \qquad (2)$$

The rest of this paper is organized as follows. In Sec. II, we will give a brief introduction of Type (iii) of N-fold DT for Eq. (1) based on the Lax pair in Ref. [28]. In Sec. III, we present an idea to derive the generalized perturbation $(1, N-1)$-fold (using one spectral parameter) and (n,M)-fold (using n spectral parameters) Darboux transformations for the MNLS equation (1) by means of the Taylor expansion and a limit procedure related to the N-fold Darboux matrix. In particular, the generalized perturbation $(1, N-1)$-fold Darboux transformation is used to exhibit its multi-rogue-wave solutions. Moreover, we analyze the wave structures and dynamical behaviors of the multi-rogue-wave solutions for differential parameters using the numerical simulations. In Sec. IV, we illustrate the multi-rogue-wave solutions of the Gerjikov-Ivanov equation [Eq. (2)] in terms of its generalized perturbation $(1, N-1)$-fold Darboux transformation. Finally, we will address the conclusions in Sec. V.

II. THE N-FOLD DARBOUX TRANSFORMATION

In the following we first give the N-fold Darboux transformation of Eq. (1) in order to present our main aim, i.e., its generalized perturbation (n,M)-fold Darboux transformation presented in Sec. III. The modified NLS equation (1) is just a zero-curvature equation $U_t - V_x + [U, V] = 0$ with $[U, V] \equiv UV - VU$ and two 2×2 matrices U and V satisfying the linear isospectral problem (Lax pair) [28],

$$\varphi_x = U\varphi, U = \begin{pmatrix} -\frac{i}{\lambda^2} + i\rho & \frac{q}{\lambda} \\ -\frac{q^*}{\lambda} & \frac{i}{\lambda^2} - i\rho \end{pmatrix}, \qquad (3)$$

$$\varphi_t = V\varphi, \quad V = \begin{bmatrix} -2i\left(\frac{1}{\lambda^2} - \rho\right)^2 + i\frac{|q|^2}{\lambda^2} & 2\left(\frac{1}{\lambda^3} - \frac{\rho}{\lambda}\right)q - \frac{|q|^2 q}{\lambda} + i\frac{q_x}{\lambda} \\ -2\left(\frac{1}{\lambda^3} - \frac{\rho}{\lambda}\right)q^* + \frac{|q|^2 q^*}{\lambda} + i\frac{q_x^*}{\lambda} & 2i\left(\frac{1}{\lambda^2} - \rho\right)^2 - i\frac{|q|^2}{\lambda^2} \end{bmatrix}, \qquad (4)$$

where $\varphi = \varphi(x,t) = (\phi(x,t), \psi(x,t))^{\mathrm{T}}$ is the complex eigenfunction; $\lambda \in \mathbb{C}$ is the spectral parameter; $q = q(x,t)$ denotes the complex potential and is also the solution of Eq. (1); the subscript denotes the partial derivative with respect to the variables x, t; and the star stands for the complex conjugate of the corresponding variables.

In what follows, we introduce the gauge transformation

$$\widetilde{\varphi} = T\varphi, \qquad (5)$$

where $\varphi = (\phi, \psi)^{\mathrm{T}}$ satisfies the Lax pair (3) and (4), T is a 2×2 Darboux matrix to be determined later, and $\widetilde{\varphi} = \widetilde{\varphi}(x,t) = (\widetilde{\phi}(x,t), \widetilde{\psi}(x,t))^{\mathrm{T}}$ is required to satisfy the same formal Lax pair (3) and (4) with U and V replaced by $\widetilde{U}$ and $\widetilde{V}$, that is,

$$\widetilde{\varphi}_x = \widetilde{U}\widetilde{\varphi}, \quad \widetilde{U} = \begin{pmatrix} -\frac{i}{\lambda^2} + i\rho & \frac{\widetilde{q}}{\lambda} \\ -\frac{\widetilde{q}^*}{\lambda} & \frac{i}{\lambda^2} - i\rho \end{pmatrix}, \qquad (6)$$

$$\widetilde{\varphi}_t = \widetilde{V}\widetilde{\varphi}, \quad \widetilde{V} = \begin{bmatrix} -2i\left(\frac{1}{\lambda^2} - \rho\right)^2 + i\frac{|\widetilde{q}|^2}{\lambda^2} & 2\left(\frac{1}{\lambda^3} - \frac{\rho}{\lambda}\right)\widetilde{q} - \frac{|\widetilde{q}|^2 \widetilde{q}}{\lambda} + i\frac{\widetilde{q}_x}{\lambda} \\ -2\left(\frac{1}{\lambda^3} - \frac{\rho}{\lambda}\right)\widetilde{q}^* + \frac{|\widetilde{q}|^2 \widetilde{q}^*}{\lambda} + i\frac{\widetilde{q}_x^*}{\lambda} & 2i\left(\frac{1}{\lambda^2} - \rho\right)^2 - i\frac{|\widetilde{q}|^2}{\lambda^2} \end{bmatrix}, \qquad (7)$$

where $\widetilde{q} = \widetilde{q}(x,t)$ is a new potential function and $\widetilde{q}^*$ is its complex conjugate.

Therefore, according to the compatibility condition $\widetilde{\varphi}_{xt} = \widetilde{\varphi}_{tx}$ due to Eqs. (6) and (7) we have

$$T_x + [T, \{U, \widetilde{U}\}] = 0, \qquad (8a)$$

$$T_t + [T, \{V, \widetilde{V}\}] = 0, \qquad (8b)$$

where we have introduced the generalized Lie bracket $[F, \{G, \widetilde{G}\}] = FG - \widetilde{G}F$. In particular, the generalized Lie bracket reduces to the usual Lie bracket $[F, \{G, \widetilde{G}\}] = [F, G]$ for the case $\widetilde{G} = G$.

Therefore we have

$$\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = T(U_t - V_x + [U, V])T^{-1} = 0, \quad (9)$$

which yields the same equation (1) with $q \to \widetilde{q}$, i.e., $\widetilde{q}$ in the new spectral problem (6) and (7) is a solution of Eq. (1).

Hereby we assume that the Darboux matrix $T(\lambda)$ is of the form

$$T(\lambda) = \begin{bmatrix} T_{11}(\lambda) & T_{12}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda^{2N} + \sum_{j=0}^{N-1} A^{(2j)}\lambda^{2j} & \sum_{j=0}^{N-1} B^{(2j+1)}\lambda^{2j+1} \\ -\sum_{j=0}^{N-1} B^{(2j+1)*}\lambda^{2j+1} & \lambda^{2N} + \sum_{j=0}^{N-1} A^{(2j)*}\lambda^{2j} \end{bmatrix}, \tag{10}$$

with the complex functions $A^{(2j)}$ and $B^{(j+1)}$ $(j = 0, 1, \ldots, N-1)$ solving the linear algebraic system $T(\lambda_k)\varphi_k(\lambda_k) = 0$ $(k = 1, 2, \ldots, N)$, i.e.,

$$\left[\lambda_k^{2N} + \sum_{j=0}^{N-1} A^{(2j)}(\lambda_k)\lambda_k^{2j}\right]\varphi_1(\lambda_k) + \sum_{j=0}^{N-1} B^{(2j+1)}(\lambda_k)\lambda_k^{2j+1}\varphi_2(\lambda_k) = 0, \tag{11}$$

$$-\sum_{j=0}^{N-1} B^{(2j+1)*}(\lambda_k)\lambda_k^{2j+1}\varphi_1(\lambda_k) + \left[\lambda_k^{2N} + \sum_{j=0}^{N-1} A^{(2j)*}(\lambda_k)\lambda_k^{2j}\right]\varphi_2(\lambda_k) = 0, \tag{12}$$

where $\varphi_k(\lambda_k) = (\phi_k(\lambda_k), \psi_k(\lambda_k))^{\mathrm{T}} = (\phi_k, \psi_k)^{\mathrm{T}}$ $(k = 1, 2, \ldots, N)$ is a solution of the spectral problem (3) and (4) for the given spectral parameters λ_k and the initial solution q_0. The spectral parameters λ_k $(\lambda_i \neq \lambda_j, i \neq j, i = 1, 2, \ldots, N)$ are some different parameters suitably chosen such that the determinant of coefficients of Eqs. (11) and (12) for the $2N$ variables $A^{(2j)}$, $B^{(2j+1)}$ $(j = 0, 1, \ldots, N-1)$ are nonzero. It can be shown that $\pm\lambda_k$, $\pm\lambda_k^*$ $(k = 1, 2, \ldots, N)$ are the $4N$ roots of $\det T(\lambda) = 0$ in terms of Eqs. (11) and (12), i.e.,

$$\det T(\lambda) = \prod_{k=1}^{N} (\lambda^2 - \lambda_k^2)(\lambda^2 - \lambda_k^{*2}). \tag{13}$$

The substitution of Eq. (10) into Eqs. (8a) and (8b) with the conditions (11) and (12) yields the following theorem for the N-fold Darboux transformation of Eq. (1).

Theorem 1. Let $\varphi_1(\lambda_1), \varphi_2(\lambda_2), \ldots, \varphi_N(\lambda_N)$ be N distinct column vector solutions of the corresponding spectral problem (3) and (4) for the spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$ and the initial solution $q_0(x, t)$ of Eq. (1), respectively. The N-fold Darboux transformation of Eq. (1) then is given by

$$\widetilde{q}_N(x, t) = q_0(x, t) + \frac{\partial B^{(2N-1)}}{\partial x} - 2i\rho B^{(2N-1)}, \tag{14}$$

with $B^{(2N-1)}(x, t) = \frac{\Delta B^{(2N-1)}}{\Delta_N}$,

$$\Delta_N = \begin{vmatrix} \lambda_1^{2(N-1)}\phi_1 & \lambda_1^{2(N-2)}\phi_1 & \cdots & \phi_1 & \lambda_1^{2N-1}\psi_1 & \lambda_1^{2N-3}\psi_1 & \cdots & \psi_1 \\ \lambda_2^{2(N-1)}\phi_2 & \lambda_2^{2(N-2)}\phi_2 & \cdots & \phi_2 & \lambda_1^{2N-1}\psi_2 & \lambda_2^{2N-3}\psi_2 & \cdots & \psi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_N^{2(N-1)}\phi_N & \lambda_N^{2(N-2)}\phi_N & \cdots & \phi_N & \lambda_N^{2N-1}\psi_N & \lambda_N^{2N-3}\psi_N & \cdots & \psi_N \\ \lambda_1^{*2(N-1)}\psi_1^* & \lambda_1^{*2(N-2)}\psi_1^* & \cdots & \psi_1^* & -\lambda_1^{*(2N-1)}\phi_1 & -\lambda_1^{*(2N-3)}\phi_1^* & \cdots & -\phi_1^* \\ \lambda_2^{*2(N-1)}\psi_2^* & \lambda_2^{*2(N-2)}\psi_2^* & \cdots & \psi_2^* & -\lambda_2^{*(2N-1)}\phi_2 & -\lambda_2^{*(2N-3)}\phi_2^* & \cdots & -\phi_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_N^{*2(N-1)}\psi_N^* & \lambda_N^{*2(N-2)}\psi_N^* & \cdots & \psi_N^* & -\lambda_N^{*(2N-1)}\phi_N & -\lambda_N^{*(2N-3)}\phi_N^* & \cdots & -\phi_N^* \end{vmatrix},$$

and $\Delta B^{(2N-1)}$ is given by the determinant Δ_N by replacing its $(N+1)$-th column with the column vector $[-\lambda_1^{2N}\phi_1, -\lambda_2^{2N}\phi_2, \ldots, -\lambda_N^{2N}\phi_N, -\lambda_1^{*(2N)}\psi_1^*, -\lambda_2^{*(2N)}\psi_2^*, \ldots, -\lambda_N^{*(2N)}\psi_N^*]^{\mathrm{T}}$.

Notice that when $\rho = 0$, the DT (14) reduces to the DT of the derivative NLS equation, which differs from the known DT [37,38]. The N-fold Darboux transformation with the initial solution $q_0 = 0$ (or q_0 is an initial plane-wave solution) can be used to seek multisoliton solutions (or breather solutions) of Eq. (1). This is not the main aim of our paper. Our aim is to extend the N-fold DT to generate the generalized perturbation (n, M)-fold DT such that multi-rogue-wave solutions of Eq. (1) are found in terms of determinants.

In the next sections, we will discuss the generalized perturbation (n, M)-fold DT and multi-rogue-wave solutions of Eqs. (1) and (2) through the Taylor expansion and a limit procedure.

III. GENERALIZED PERTURBATION (n, M)-FOLD DARBOUX TRANSFORMATIONS AND MULTI-ROGUE-WAVE SOLUTIONS

In the following we first choose Eq. (1) as an example to investigate the "novel" generalized perturbation (n, M)-fold DTs in applications of nonlinear integrable equations. In fact,

this idea can also be extended to other nonlinear integrable equations and differs from the known ones [11,12].

Nowadays, to study other types of solutions of Eq. (1) such as multi-rogue-wave solutions, we need to change some functions $A^{(2j)}$ and $B^{(2j-1)}$ ($j = 0,1,\ldots,N-1$) in the above-mentioned Darboux matrix T given by Eq. (10) and initial solution q_0 such that we may obtain other types of solutions of Eq. (1) in terms of some generalized DTs.

A. Generalized perturbation $(1, N-1)$-fold Darboux transformation method

Here we still consider the Darboux matrix (10), but we only consider one spectral parameter $\lambda = \lambda_1$ and not N ($N > 1$) distinct spectral parameters $\lambda = \lambda_k$ ($k = 1,2,\ldots,N$), in which the condition $T(\lambda_1)\varphi(\lambda_1) = 0$ leads to the linear algebraic system with only two equations:

$$\left[\lambda_1^{2N} + \sum_{j=0}^{N-1} A^{(2j)}\lambda_1^{2j}\right]\phi(\lambda_1)$$
$$+ \sum_{j=0}^{N-1} B^{(2j+1)}\lambda_1^{2j+1}\psi(\lambda_1) = 0, \qquad (15)$$

$$\left[\lambda_1^{*(2N)} + \sum_{j=0}^{N-1} A^{(2j)}\lambda_1^{*(2j)}\right]\psi^*(\lambda_1)$$
$$- \sum_{j=0}^{N-1} B^{(2j+1)}\lambda_1^{*(2j+1)}\phi^*(\lambda_1) = 0, \qquad (16)$$

where $\varphi(\lambda_1) = (\phi(\lambda_1), \psi(\lambda_1))^{\mathrm{T}}$ is a solution of the linear spectral problem (3) and (4) with the one spectral parameter $\lambda = \lambda_1$ and the initial solution $q_0(x,t)$ of Eq. (1).

For the two linear algebraic equations (15) and (16) containing $2N$ unknown functions $A^{(2j)}$ and $B^{(2j+1)}$ ($j = 0,1,\ldots,N-1$), we have two cases for the parameter N:

(i) If $N = 1$, then we can determine only two complex functions $A^{(0)}$ and $B^{(1)}$ from Eqs. (15) and (16), in which we

can not obtain the different functions $A^{(0)}$ and $B^{(1)}$ comparing from the above-mentioned onefold DT such that the "new" solutions cannot be found; and

(ii) If $N > 1$, then we have $2(N-1) > 0$ free functions for $A^{(2j)}$ and $B^{(2j+1)}$ ($j = 0,1,\ldots,N-1$). This means that the number of the unknown variables $A^{(2j)}$ and $B^{(2j+1)}$ is larger than one of equations such that we have some free functions, which seems to be useful for the Darboux matrix, but it may be difficult to show the invariant conditions (8a) and (8b).

For the case $N > 1$, we only have two above-given algebraic constraints (15) and (16) for $2N$ functions $A^{(2j)}$ and $B^{(2j+1)}$ ($j = 0,1,\ldots,N-1$). To determine these $2N$ unknown functions $A^{(2j)}$ and $B^{(2j+1)}$, we need to find additional $2(N-1)$ equations about $2N$ functions $A^{(2j)}$ and $B^{(2j+1)}$ such that we have $2N$ equations about $2N$ functions $A^{(2j)}$ and $B^{(2j+1)}$, in which we may determine them. If we can determine these $2N$ functions $A^{(2j)}$ and $B^{(2j+1)}$, then we may obtain "new" solutions of Eq. (1) in terms of the Draboux transformation.

Now we start from Eqs. (15) and (16), i.e., $T(\lambda_1)\varphi(\lambda_1) = 0$. It follows from Eq. (10) that T_{ii} ($i = 1,2$) are two polynomials of degree $2N$ for λ_1 with the coefficients being $A^{(2j)}$ and T_{ij} ($i \neq j, i,j = 1,2$) are two polynomials of degree $2N - 1$ for λ_1 with the coefficients being $B^{(2j+1)}$. Since $\varphi(\lambda_1)$ is a column vector solution of the spectral problem (3) and (4) for the given spectral parameters λ_k and the initial solution q_0, thus $\varphi(\lambda_1)$ is a vector function with every component being a function of parameter λ_1. To generate "new" additional $2(N-1)$ equations from $T(\lambda_1)\varphi(\lambda_1) = 0$, we consider the Taylor expansion of $T(\lambda_1)\varphi(\lambda_1)|_{\{\lambda_1 \to \lambda_1 + \varepsilon\}} = T(\lambda_1 + \varepsilon)\varphi(\lambda_1 + \varepsilon)$ at $\varepsilon = 0$. We know that

$$\varphi(\lambda_1 + \varepsilon) = \varphi^{(0)}(\lambda_1) + \varphi^{(1)}(\lambda_1)\varepsilon + \varphi^{(2)}(\lambda_1)\varepsilon^2 + \cdots, \quad (17)$$

where $\varphi^{(k)}(\lambda_1) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\varphi(\lambda_1) = (\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\phi(\lambda_1), \frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\psi(\lambda_1))^{\mathrm{T}}$ with $\varphi^{(0)}(\lambda_1) = \varphi(\lambda_1) = (\phi(\lambda_1), \psi(\lambda_1))^{\mathrm{T}}$, ($k = 0,1,2,\ldots$), and

$$T(\lambda_1 + \varepsilon) = T(\lambda_1) + \sum_{k=1}^{2N} T^{(k)}(\lambda_1)\varepsilon^k = T(\lambda_1) + \sum_{k=1}^{N} T^{(2k-1)}(\lambda_1)\varepsilon^{2k-1} + \sum_{k=1}^{N} T^{(2k)}(\lambda_1)\varepsilon^{2k}, \qquad (18)$$

where $T^{(2k-1)}(\lambda_1)$ and $T^{(2k)}(\lambda_1)$ are given by

$$T^{(2k-1)}(\lambda_1) = \begin{pmatrix} C_{2N}^{2k-1}\lambda_1^{2N-2k+1} + \sum_{j=k}^{N-1} C_{2j}^{2k-1}A^{(2j)}\lambda_1^{2j-2k+1} & \sum_{j=k-1}^{N-1} C_{2j+1}^{2k-1}B^{(2j+1)}\lambda_1^{2j-2k+2} \\ -\sum_{j=k-1}^{N-1} C_{2j+1}^{2k-1}B^{(2j+1)*}\lambda_1^{2j-2k+2} & C_{2N}^{2k-1}\lambda_1^{2N-2k+1} + \sum_{j=k}^{N-1} C_{2j}^{2k-1}A^{(2j)*}\lambda_1^{2j-2k+1} \end{pmatrix}, \qquad (19)$$

$$T^{(2k)}(\lambda_1) = \begin{pmatrix} C_{2N}^{2k}\lambda_1^{2N-2k} + \sum_{j=k}^{N-1} C_{2j}^{2k}A^{(2j)}\lambda_1^{2j-2k} & \sum_{j=k}^{N-1} C_{2j+1}^{2k}B^{(2j+1)}\lambda_1^{2j-2k+1} \\ -\sum_{j=k}^{N-1} C_{2j+1}^{2k}B^{(2j+1)*}\lambda_1^{2j-2k+1} & C_{2N}^{2k}\lambda_1^{2N-2k} + \sum_{j=k}^{N-1} C_{2j}^{2k}A^{(2j)*}\lambda_1^{2j-2k} \end{pmatrix} \qquad (20)$$

with $C_j^k = \frac{j(j-1)\ldots(j-k+1)}{k!}$.

Therefore, it follows from Eqs. (17) and (18) that we obtain

$$T(\lambda_1)\varphi(\lambda_1)|_{\{\lambda_1 = \lambda_1 + \varepsilon\}} = T(\lambda_1 + \varepsilon)\varphi(\lambda_1 + \varepsilon) = \sum_{k=0}^{+\infty}\sum_{j=0}^{k} T^{(j)}(\lambda_1)\varphi^{(k-j)}(\lambda_1)\varepsilon^k. \qquad (21)$$

Let

$$\lim_{\varepsilon \to 0} \frac{T(\lambda_1 + \varepsilon)\varphi(\lambda_1 + \varepsilon)}{\varepsilon^s}$$

$$= \lim_{\varepsilon \to 0} \frac{\sum_{k=0}^{+\infty} \sum_{j=0}^{k} T^{(j)}(\lambda_1)\varphi^{(k-j)}(\lambda_1)\varepsilon^k}{\varepsilon^s}$$

$$= 0 \quad (s = 0, 1, 2, \dots). \tag{22}$$

Since we know that $T(\lambda_1)\varphi(\lambda_1) = T^{(0)}(\lambda_1)\varphi^{(0)}(\lambda_1) = 0$ [i.e., Eqs. (15) and (16)], we have $\sum_{k=0}^{1} \sum_{j=0}^{k} T^{(j)}(\lambda_1)\varphi^{(k-j)}(\lambda_1) = 0$ for $s = 1$. Similarly, we can also get $\sum_{k=0}^{s} \sum_{j=0}^{k} T^{(j)}(\lambda_1)\varphi^{(k-j)}(\lambda_1) = 0$ for any $s > 1$. Since for every $s \in \{0, 1, 2, \dots\}$, we can have two algebraic equations for unknown functions $A^{(2j)}$ and $B^{(2j+1)}$, thus we choose $s = 0, 1, \dots, N-1$ to generate $2N$ algebraic equations for these $2N$ unknown functions $A^{(2j)}$ and $B^{(2j+1)}$ $(j = 0, 1, \dots, N-1)$, i.e.,

$$T^{(0)}(\lambda_1)\varphi^{(0)}(\lambda_1) = 0,$$

$$T^{(0)}(\lambda_1)\varphi^{(1)}(\lambda_1) + T^{(1)}(\lambda_1)\varphi^{(0)}(\lambda_1) = 0,$$

$$T^{(0)}(\lambda_1)\varphi^{(2)}(\lambda_1) + T^{(1)}(\lambda_1)\varphi^{(1)}(\lambda_1) + T^{(2)}(\lambda_1)\varphi^{(0)}(\lambda_1) = 0,$$

$$\cdots \cdots,$$

$$\sum_{j=0}^{N-1} T^{(j)}(\lambda_1)\varphi^{(N-1-j)}(\lambda_1) = 0,$$

$$\tag{23}$$

in which the first vector system, i.e., $T^{(0)}(\lambda_1)\varphi^{(0)}(\lambda_1) = T(\lambda_1)\varphi(\lambda_1) = 0$, are just Eqs. (15) and (16), which are required.

Therefore we have obtained the system (23) containing $2N$ algebraic equations with the $2N$ unknowns functions $A^{(2j)}$ and $B^{(2j+1)}$ $(j = 0, 1, \dots, N-1)$. When the eigenvalue λ_1 is suitably chosen so the determinant of the coefficients for system (23) is nonzero, the transformation matrix T is uniquely determined by system (23). It can be shown that Theorem 1 still holds for the Darboux matrix (10) with $A^{(2j)}, B^{(2j+1)}(j = 0, 1, \dots, N-1)$ being determined by system (23). Due to new distinct functions $A^{(2j)}, B^{(2j+1)}$ obtained in the N-order Darboux matrix T, we can derive the "new" DT with the same eigenvalue $\lambda = \lambda_1$. Here we refer to Eqs. (14) and (5) associated with new functions $A^{(2j)}, B^{(2j+1)}$ determined by system (23) as a generalized perturbation $(1, N-1)$-fold DT.

Theorem 2. Let $\varphi(\lambda_1) = (\phi(\lambda_1), \psi(\lambda_1))^{\mathrm{T}}$ be a column vector solution of the spectral problem (3) and (4) for the spectral parameter λ_1 and initial solution $q_0(x,t)$ of Eq. (1). The generalized perturbation $(1, N-1)$-fold Darboux transformation of Eq. (1) then is given by

$$\widetilde{q}_N(x,t) = q_0(x,t) + \frac{\partial B^{(2N-1)}}{\partial x} - 2i\rho B^{(2N-1)}, \tag{24}$$

where $B^{(2N-1)}$ is by solving the linear algebraic system (23) in terms of the Cramer's rule, i.e., $B^{(2N-1)} = \frac{\Delta^{\epsilon} B^{(2N-1)}}{\Delta_N^{\epsilon}}$, with

$$\Delta_N^{\epsilon} = \begin{vmatrix} \lambda^{2N-2}\phi^{(0)} & \lambda^{2N-4}\phi^{(0)} & \dots & \phi^{(0)} & \lambda^{2N-1}\psi^{(0)} & \lambda^{2N-3}\psi^{(0)} & \dots & \lambda\psi^{(0)} \\ \Delta_{2,1} & \Delta_{2,2} & \dots & \phi^{(1)} & \Delta_{2,N+1} & \Delta_{2,N+2} & \dots & \lambda\psi^{(1)} + \psi^{(0)} \\ \dots & \dots & \dots \dots & \dots & \dots & \dots & \dots \dots \\ \Delta_{N,1} & \Delta_{N,2} & \dots & \phi^{(N-1)} & \Delta_{N,N+1} & \Delta_{N,N+2} & \dots & \lambda\psi^{(N-1)} + \psi^{(N-2)} \\ \lambda^{*(2N-2)}\psi^{(0)*} & \lambda^{*(2N-4)}\psi^{(0)*} & \dots & \psi^{(0)*} & -\lambda^{*(2N-1)}\phi^{(0)*} & -\lambda^{*(2N-3)}\phi^{(0)*} & \dots & -\lambda^*\phi^{(0)*} \\ \Delta_{N+2,1} & \Delta_{N+2,2} & \dots & \psi^{(1)*} & \Delta_{N+2,N+1} & \Delta_{N+2,N+2} & \dots & -\lambda^*\phi^{(1)*} - \phi^{(0)*} \\ \dots & \dots & \dots \dots & \dots & \dots & \dots \dots \\ \Delta_{2N,1} & \Delta_{2N,2} & \dots & \psi^{(N-1)*} & \Delta_{2N,N+1} & \Delta_{2N,N+2} & \dots & -\lambda^*\phi^{(N-1)*} - \phi^{(N-2)*} \end{vmatrix}, \tag{25}$$

$$\Delta_{j,s} = \begin{cases} \sum_{k=0}^{j-1} C_{2N-2s}^k \lambda_1^{(2N-2s-k)} \phi^{(j-1-k)} & \text{for} \quad 1 \leqslant j, s \leqslant N, \\ \sum_{k=0}^{j-1} C_{4N-2s+1}^k \lambda_1^{(4N-2s+1-k)} \psi^{(j-1-k)} & \text{for} \quad 1 \leqslant j \leqslant N, N+1 \leqslant s \leqslant 2N, \\ \sum_{k=0}^{j-(N+1)} C_{2N-2s}^k \lambda_1^{*(2N-2s-k)} \psi^{(j-N-1-k)*} & \text{for} \quad N+1 \leqslant j \leqslant 2N, 1 \leqslant s \leqslant N, \\ -\sum_{k=0}^{j-(N+1)} C_{4N-2s+1}^k \lambda_1^{*(4N-2s+1-k)} \phi^{(j-N-1-k)*} & \text{for} \quad N+1 \leqslant j, s \leqslant 2N \end{cases} \tag{26}$$

and $\Delta^{\epsilon} B^{(2N-1)}$ is formed from Δ_N^{ϵ} by replacing its $(N+1)$-th column by the column vector $b = (b_j)_{2N \times 1}$ with

$$b_j = \begin{cases} -\sum_{k=0}^{j-1} C_{2N}^k \lambda_1^{(2N-k)} \phi^{(j-1-k)} & \text{for} \quad 1 \leqslant j \leqslant N \\ -\sum_{k=0}^{j-(N+1)} C_{2N}^k \lambda_1^{*(2N-k)} \psi^{(j-N-1-k)*} & \text{for} \quad N+1 \leqslant j \leqslant 2N. \end{cases} \tag{27}$$

Notice that in the name of the generalized perturbation $(1, N-1)$-fold DT, the number 1 means that we use the number of the distinct spectral parameters and $N-1$ means that the sum of the orders of the highest derivative of the Darboux matrix T in system (23) or the vector eigenfunction φ for the used distinct spectral parameters in Eq. (22).

B. Generalized perturbation (n, M)-fold Darboux transformation method

In fact, we can also further extend the above-found generalized perturbation $(1, N-1)$-fold DT, in which we only use one spectral parameter $\lambda = \lambda_1$ and its m_1th − order perturbation derivatives of $T(\lambda_1)$ and $\varphi(\lambda_1)$ with $m_1 = 1, 2, \dots, N-1$.

Here we use n distinct spectral parameters λ_i $(i = 1,2,\ldots,n)$ and their corresponding highest-order m_i $(m_i = 0,1,2,\ldots)$ perturbation derivatives, where the non-negative integers n, m_i are required to satisfy $N = n + \sum_{i=1}^{n} m_i = n + M$ with $M = \sum_{i=1}^{n} m_i$, where N is the same as the one in the Darboux matrix T (10).

We consider the Darboux matrix (10) and the eigenfunctions $\varphi_i(\lambda_i)$ $(i = 1,2,\ldots,n)$ are the solutions of the linear spectral problem (3) and (4) for the spectral parameter λ_i and initial solution q_0 of Eq. (1). Thus we have

$$T(\lambda_i + \varepsilon)\varphi_i(\lambda_i + \varepsilon) = \sum_{k=0}^{N-1}\sum_{j=0}^{k} T^{(j)}(\lambda_1)\varphi_i^{(k-j)}(\lambda_i)\varepsilon^k, \quad (28)$$

where $\varphi_i^{(k)}(\lambda_i) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda_i^k}\varphi_i(\lambda)|_{\lambda=\lambda_i}$, and ε is a small parameter.

It follows from Eq. (28) and

$$\lim_{\varepsilon\to 0}\frac{T(\lambda_i + \varepsilon)\varphi_i(\lambda_i + \varepsilon)}{\varepsilon^{k_i}} = 0 \quad (29)$$

with $i = 1,2,\ldots,n$ and $k_i = 0,1,\ldots,m_i$ that we obtain the linear algebraic system with the $2N$ equations ($N = n + \sum_{i=1}^{n} m_i = n + M$):

$$T^{(0)}(\lambda_i)\varphi_i^{(0)}(\lambda_i) = 0,$$
$$T^{(0)}(\lambda_i)\varphi_i^{(1)}(\lambda_i) + T^{(1)}(\lambda_i)\varphi_i^{(0)}(\lambda_i) = 0,$$
$$\cdots\cdots, \quad (30)$$
$$\sum_{j=0}^{m_i} T^{(j)}(\lambda_i)\varphi_i^{(m_i-j)}(\lambda_i) = 0,$$

$i = 1,2,\ldots,n$, in which we have some first systems for every index i, i.e., $T^{(0)}(\lambda_i)\varphi_i^{(0)}(\lambda_i) = T(\lambda_i)\varphi_i(\lambda_i) = 0$ are just some ones in system (23), but they differ if there exists at least one index $m_i \neq 0$.

Theorem 3. Let $\varphi_i(\lambda_i) = (\phi_i(\lambda_i), \psi_i(\lambda_i))^{\mathrm{T}}$ $(i = 1,2,\ldots,n)$ be the column vector solutions of the spectral problem (3) and (4) for the spectral parameters λ_i $(i = 1,2,\ldots,n)$ and the same initial solution $q_0(x,t)$ of Eq. (1), respectively. Then the generalized perturbation (n,M)-fold DT of Eq. (1) is given by

$$\widetilde{q}_N(x,t) = q_0(x,t) + \frac{\partial B^{(2N-1)}}{\partial x} - 2i\rho B^{(2N-1)}, \quad (31)$$

with $B^{(2N-1)} = \frac{\Delta^{\epsilon(n)} B^{(2N-1)}}{\Delta_N^{\epsilon(n)}}$ being defined by solving the linear algebraic system (30) in terms of the Cramer's rule, where $\Delta_N^{\epsilon(n)} = \det([\Delta^{(1)}\cdots\Delta^{(n)}]^{\mathrm{T}})$ with

$$\Delta^{(i)} = \begin{bmatrix} \lambda_i^{N-1}\phi_i^{(0)} & \lambda_i^{N-2}\phi_i^{(0)} & \cdots & \phi_i^{(0)} & \lambda_i^{N-1}\psi_i^{(0)} & \lambda_i^{N-2}\psi_i^{(0)} & \cdots & \psi_i^{(0)} \\ \Delta_{2,1}^{(i)} & \Delta_{2,2}^{(i)} & \cdots & \phi_i^{(1)} & \Delta_{2,N+1}^{(i)} & \Delta_{2,N+2}^{(i)} & \cdots & \psi_i^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{m_i+1,1}^{(i)} & \Delta_{m_i+1,2}^{(i)} & \cdots & \phi_i^{(m_i)} & \Delta_{m_i+1,N+1}^{(i)} & \Delta_{m_i+1,N+2}^{(i)} & \cdots & \psi_i^{(m_i)} \\ \lambda_i^{*(N-1)}\psi_i^{(0)*} & \lambda_i^{*(N-2)}\psi_i^{(0)*} & \cdots & \psi_i^{(0)*} & -\lambda_i^{*(N-1)}\phi_i^{(0)*} & -\lambda_i^{*(N-2)}\phi_i^{(0)*} & \cdots & -\phi_i^{(0)*} \\ \Delta_{m_i+3,1}^{(i)} & \Delta_{m_i+3,2}^{(i)} & \cdots & \psi_i^{(1)*} & \Delta_{m_i+3,N+1}^{(i)} & \Delta_{m_i+3,N+2}^{(i)} & \cdots & -\phi_i^{(1)*} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{2(m_i+1),1}^{(i)} & \Delta_{2(m_i+1),2}^{(i)} & \cdots & \psi_i^{(m_i)*} & \Delta_{2(m_i+1),N+1}^{(i)} & \Delta_{2(m_i+1),N+2}^{(i)} & \cdots & -\phi_i^{(m_i)*} \end{bmatrix}, \quad (32)$$

and $\Delta_{j,s}^{(i)}$ $(1 \leqslant j \leqslant 2(m_i + 1),\ 1 \leqslant s \leqslant N\ i = 1,2,\ldots,n)$ being given by the following formulas:

$$\Delta_{j,s}^{(i)} = \begin{cases} \sum_{k=0}^{j-1} C_{2N-2s}^k \lambda_i^{2N-2s-k}\phi_i^{(j-1-k)} & \text{for } 1 \leqslant j \leqslant m_i + 1, 1 \leqslant s \leqslant N, \\ \sum_{k=0}^{j-1} C_{4N-2s+1}^k \lambda_i^{4N-2s+1-k}\psi_i^{(j-1-k)} & \text{for } 1 \leqslant j \leqslant m_i + 1, N + 1 \leqslant s \leqslant 2N, \\ \sum_{k=0}^{j-(N+1)} C_{2N-2s}^k \lambda_i^{*(2N-2s-k)}\psi_i^{(j-N-1-k)*} & \text{for } m_i + 2 \leqslant j \leqslant 2(m_i + 1), 1 \leqslant s \leqslant N, \\ -\sum_{k=0}^{j-(N+1)} C_{4N-2s+1}^k \lambda_i^{*(4N-2s+1+k)}\phi_i^{(j-N-1-k)*} & \text{for } m_i + 2 \leqslant j \leqslant 2(m_i + 1), N + 1 \leqslant s \leqslant 2N \end{cases} \quad (33)$$

and $\Delta^\epsilon B^{(2N-1)}$ is formed from the determinant $\Delta_N^{\epsilon(n)}$ by replacing its $(N+1)$-th column by the column vector $(b^{(1)}\cdots b^{(n)})^{\mathrm{T}}$ with $b^{(i)} = (b_j^{(i)})_{2(m_i+1)\times 1}$ and

$$b_j^{(i)} = \begin{cases} -\sum_{k=0}^{j-1} C_{2N}^k \lambda_i^{2N-k}\phi_i^{(j-1-k)} & \text{for } 1 \leqslant j \leqslant m_i + 1, \\ -\sum_{k=0}^{j-(N+1)} C_{2N}^k \lambda_i^{*(2N-k)}\psi_i^{(j-N-1-k)*} & \text{for } m_i + 2 \leqslant j \leqslant 2N \end{cases} \quad (34)$$

Notice that when $n = 1$ and $m_1 = N - 1$, Theorem 3 reduces to Theorem 2; when $n = N$ and $m_i = 0$, $1 \leqslant i \leqslant N$, Theorem 3 reduces to Theorem 1. In the following we will use the gener-

alized perturbation $(1, N - 1)$-fold Darboux transformation to investigate multi-rogue-wave solutions of the MNLS equation (1) from the initial plane-wave solution.

C. Multi-rogue-wave solutions and parameters controlling

In this section, we give some multi-rogue-wave solutions in terms of determinants, which differs from Ref. [12], of Eq. (1) by means of the generalized perturbation $(1, N-1)$-fold DT. We now consider the "seed" plane-wave solution

$$q_0(x,t) = ce^{i[ax+(2\rho c^2 - ac^2 - a^2)t]},$$

where a and $c \neq 0$ are real-valued constants, a is the wave number, and c is the amplitude of the plane wave. It is known

that the phase velocity is $(a + c^2 - 2\rho c^2/a)$, the group velocity is $2a + c^2$, and $|q_0(x,t)| \to |c| \neq 0$ as $|x|, |t| \to \infty$.

The substitution of the plane-wave solution $q_0(x,t)$ into the Lax pair (3) and (4) yields their solution for the spectral parameter λ as follows:

$$\varphi(\lambda) = \begin{bmatrix} (C_1 e^{-A} + C_2 e^A)e^B \\ i(C_1 e^A - C_2 e^{-A})e^{-B} \end{bmatrix}, \tag{35}$$

with

$$C_1 = C_+, \quad C_2 = C_-,$$

$$C_\pm = \frac{\sqrt{\pm[2 + \lambda^2(a - 2\rho)] + \sqrt{\lambda^4(a-2\rho)^2 + 4\lambda^2(a+c^2-2\rho)+4}}}{\sqrt{2}c\lambda},$$

$$A = \frac{i\sqrt{\lambda^4(a-2\rho)^2 + 4\lambda^2(a+c^2-2\rho)+4}}{2\lambda^2}\left[x + \left(\frac{2}{\lambda^2} - 2\rho - c^2 - a\right)t + \Theta(\varepsilon)\right],$$

$$B = -\frac{i}{2}[ax + (2\rho c^2 - ac^2 - a^2)t],$$

$$\Theta(\varepsilon) = \sum_{k=1}^{N}(b_k + d_k i)\varepsilon^{2k},$$

where $b_k, d_k (k = 1, 2, \ldots, N)$ are real free parameters and ε is a small parameter.

Nowadays, we fix the spectral parameter $\lambda = \lambda_1 + \varepsilon^2$ with

$$\lambda_1 = \frac{\sqrt{2(a-2\rho)^2[2\rho - a - c^2 + \sqrt{c^2(2a - 4\rho + c^2)}]}}{(a-2\rho)^2},$$

in Eq. (35), and then we can expand the vector function φ in Eq. (35) as two Taylor series at $\varepsilon = 0$, because the expansion expressions of $\varphi(\varepsilon^2)$ are so complicated. Here we may choose $a = -1$, $\rho = 2$, and $c = 1$, in which $\lambda_1 = \frac{3}{5} + \frac{1}{5}i$, to simplify our calculation process. Therefore, we obtain

$$\varphi(\varepsilon^2) = \varphi^{(0)} + \varphi^{(1)}\varepsilon^2 + \varphi^{(2)}\varepsilon^2 + \varphi^{(3)}\varepsilon^2 + \cdots, \tag{36}$$

where

$$\varphi^{(0)} = \begin{pmatrix} \phi^{(0)} \\ \psi^{(0)} \end{pmatrix} = \begin{pmatrix} \sqrt{2}e^{\frac{i}{2}(-x+4t)} \\ \sqrt{2}e^{-\frac{i}{2}(-x+4t)} \end{pmatrix}, \tag{37}$$

$$\varphi^{(1)} = \begin{bmatrix} \frac{3\sqrt{2}}{8}e^{\frac{i}{2}(-x+4t)}[234t^2 - 26x^2 - 108xt + 12x + 48t - 1 + i(156xt - 18x^2 + 162t^2 - 36t + 16x - 3)] \\ \frac{3\sqrt{2}}{8}e^{-\frac{i}{2}(-x+4t)}[162t^2 - 18x^2 + 156xt - 16x + 36t - 3 + i(26x^2 - 234t^2 + 108xt + 12x + 48t + 1)] \end{bmatrix}, \tag{38}$$

$$\varphi^{(2)} = \begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix}, \quad \varphi^{(3)} = \begin{pmatrix} \phi^{(3)} \\ \psi^{(3)} \end{pmatrix}, \ldots, \tag{39}$$

and $(\phi^{(i)}, \psi^{(i)})^{\mathrm{T}}(i = 2, 3)$ are listed in Appendix A.

By means of Eqs. (5), (14), and (37), we can derive new solutions of Eq. (1) as follows:

$$\widetilde{q}_N(x,t) = q_0 + \frac{\partial B^{(2N-1)}}{\partial x} - 2i\rho B^{(2N-1)}. \tag{40}$$

It is worth pointing out that we rederive the seed solution $\widetilde{q}_1(x,t) = -ce^{i[ax+(2\rho c^2 - ac^2 - a^2)t]}$ for $N = 1$. To understand the wave propagation of nontrivial solution (40) of Eq. (1) for different parameters, we study their wave structures as shown in Figs. 1–7 for $N = 2, 3, 4$.

Case I. When $N = 2$, according to Theorem 2, we have the first-order rogue-wave solution of Eq. (1):

$$\widetilde{q}_2(x,t) = q_0(x,t) + \frac{\partial B^{(3)}}{\partial x} - 2i\rho B^{(3)}, \tag{41}$$

with $B^{(3)} = \frac{\Delta^\epsilon B^{(3)}}{\Delta_2^\epsilon}$ and

$$
\Delta_2^\epsilon = \begin{vmatrix}
\lambda^2\phi^{(0)} & \phi^{(0)} & \lambda^3\psi^{(0)} & \psi^{(0)} \\
\lambda^2\phi^{(1)} + 2\lambda\phi^{(0)} & \phi^{(1)} & \lambda^3\psi^{(1)} + 3\lambda^2\psi^{(0)} & \lambda\psi^{(1)} + \psi^{(0)} \\
\lambda^{*2}\psi^{(0)*} & \psi^{(0)*} & -\lambda^{*3}\phi^{(0)*} & -\lambda^*\phi^{(0)*} \\
\lambda^{*2}\psi^{(1)*} + 2\lambda^*\psi^{(0)*} & \psi^{(1)*} & -\lambda^{*3}\phi^{(1)*} - 3\lambda^{*2}\phi^{(0)*} & -\lambda^*\phi^{(1)*} - \phi^{(0)*}
\end{vmatrix},
$$

$$
\Delta^\epsilon B^{(3)} = \begin{vmatrix}
\lambda^2\phi^{(0)} & \phi^{(0)} & -\lambda^4\phi^{(0)} & \psi^{(0)} \\
\lambda^2\phi^{(1)} + 2\lambda\phi^{(0)} & \phi^{(1)} & -\lambda^4\phi^{(1)} - 4\lambda^3\phi^{(0)} & \lambda\psi^{(1)} + \psi^{(0)} \\
\lambda^{*2}\psi^{(0)*} & \psi^{(0)*} & -\lambda^{*4}\psi^{(0)*} & -\lambda^*\phi^{(0)*} \\
\lambda^{*2}\psi^{(1)*} + 2\lambda^*\psi^{(0)*} & \psi^{(1)*} & -\lambda^{*4}\psi^{(1)*} - 4\lambda^{*3}\psi^{(0)*} & -\lambda^*\phi^{(1)*} - \phi^{(0)*}
\end{vmatrix}.
$$

For example, we give the simplification forms of solution (41).

Case Ia. For $a = -1, c = 1, \rho = 2$, we have the solution

$$
\widetilde{q}_{21}(x,t) = \frac{[10(x^2 + 9t^2) + 1 - 2i(x - t)][10(x^2 + 9t^2) - 3 - 2i(x + 19t)]}{[10(x^2 + 9t^2) + 1 + 2i(x - t)]^2} e^{i(4t-x)}. \tag{42}
$$

Case Ib. For $a = -1, c = 1, \rho = 0$, we have the solution

$$
\widetilde{q}_{22}(x,t) = \frac{[2(x^2 + t^2) + 1 - 2i(x - t)][2(x^2 + t^2) - 3 - 2i(x + 3t)]}{[2(x^2 + t^2) + 1 + 2i(x - t)]^2} e^{-ix}. \tag{43}
$$

whose wave profiles are exhibited in Fig. 1.

Case II. When $N = 3$, according to Theorem 2, we have the second-order rogue-wave solution of Eq. (1):

$$
\widetilde{q}_3(x,t) = q_0(x,t) + \frac{\partial B^{(5)}}{\partial x} - 2i\rho B^{(5)}, \tag{44}
$$

with $B^{(5)} = \frac{\Delta^\epsilon B^{(5)}}{\Delta_3^\epsilon}$ and

$$
\Delta_3^\epsilon = \begin{vmatrix}
\lambda^4\phi^{(0)} & \lambda^2\phi^{(0)} & \phi^{(0)} & \lambda^5\psi^{(0)} & \lambda^3\psi^{(0)} & \psi^{(0)} \\
\Delta_{2,1} & \Delta_{2,2} & \phi^{(1)} & \Delta_{2,4} & \Delta_{2,5} & \lambda\psi^{(1)} + \psi^{(0)} \\
\Delta_{3,1} & \Delta_{3,2} & \phi^{(2)} & \Delta_{3,4} & \Delta_{3,5} & \lambda\psi^{(2)} + \psi^{(1)} \\
\lambda^{*4}\psi^{(0)*} & \lambda^{*2}\psi^{(0)*} & \psi^{(0)*} & -\lambda^{*5}\phi^{(0)*} & -\lambda^{*3}\phi^{(0)*} & -\lambda^*\phi^{(0)*} \\
\Delta_{5,1} & \Delta_{5,2} & \psi^{(1)*} & \Delta_{5,4} & \Delta_{5,5} & -\lambda^*\phi^{(1)*} - \phi^{(0)*} \\
\Delta_{6,1} & \Delta_{6,2} & \psi^{(2)*} & \Delta_{6,4} & \Delta_{6,5} & -\lambda^*\phi^{(2)*} - \phi^{(1)*}
\end{vmatrix}, \tag{45}
$$

$$
\Delta^\epsilon B^{(5)} = \begin{vmatrix}
\lambda^4\phi^{(0)} & \lambda^2\phi^{(0)} & \phi^{(0)} & -\lambda^6\phi^{(0)} & \lambda^3\psi^{(0)} & \psi^{(0)} \\
\Delta_{2,1} & \Delta_{2,2} & \phi^{(1)} & -\lambda^6\phi^{(1)} - 6\lambda^5\phi^{(0)} & \Delta_{2,5} & \lambda\psi^{(1)} + \psi^{(0)} \\
\Delta_{3,1} & \Delta_{3,2} & \phi^{(2)} & -\lambda^6\phi^{(2)} - 6\lambda^5\phi^{(1)} - 15\lambda^4\phi^{(0)} & \Delta_{3,5} & \lambda\psi^{(2)} + \psi^{(1)} \\
\lambda^{*4}\psi^{(0)*} & \lambda^{*2}\psi^{(0)*} & \psi^{(0)*} & -\lambda^{*6}\psi^{(0)*} & -\lambda^{*3}\phi^{(0)*} & -\lambda^*\phi^{(0)*} \\
\Delta_{5,1} & \Delta_{5,2} & \psi^{(1)*} & -\lambda^{*6}\psi^{(1)*} - 6\lambda^{*5}\psi^{(0)*} & \Delta_{5,5} & -\lambda^*\phi^{(1)*} - \phi^{(0)*} \\
\Delta_{6,1} & \Delta_{6,2} & \psi^{(2)*} & -\lambda^{*6}\psi^{(2)*} - 6\lambda^{*5}\psi^{(1)*} - 15\lambda^{*4}\phi^{(0)*} & \Delta_{6,5} & -\lambda^*\phi^{(2)*} - \phi^{(1)*}
\end{vmatrix}, \tag{46}
$$

where

$$
\Delta_{2,1} = \lambda^4\phi^{(1)} + 4\lambda^3\phi^{(0)}, \quad \Delta_{2,2} = \lambda^2\phi^{(1)} + 2\lambda\phi^{(0)}, \quad \Delta_{2,4} = \lambda^5\psi^{(1)} + 5\lambda^4\psi^{(0)}, \quad \Delta_{2,5} = \lambda^3\psi^{(1)} + 3\lambda^2\psi^{(0)},
$$
$$
\Delta_{3,1} = \lambda^4\phi^{(2)} + 4\lambda^3\phi^{(1)} + 6\lambda^2\phi^{(0)}, \quad \Delta_{3,2} = \lambda^2\phi^{(2)} + 2\lambda\phi^{(1)} + \phi^{(0)}, \quad \Delta_{3,4} = \lambda^5\psi^{(2)} + 5\lambda^4\psi^{(1)} + 10\lambda^3\psi^{(0)},
$$
$$
\Delta_{3,5} = \lambda^3\psi^{(2)} + 3\lambda^2\psi^{(1)} + 3\lambda\psi^{(0)}, \quad \Delta_{5,1} = \lambda^{*4}\psi^{(1)*} + 4\lambda^{*3}\psi^{(0)*}, \quad \Delta_{5,2} = \lambda^{*2}\psi^{(1)*} + 2\lambda^*\psi^{(0)*}, \tag{47}
$$
$$
\Delta_{5,4} = -\lambda^{*5}\phi^{(1)*} - 5\lambda^{*4}\phi^{(0)*}, \quad \Delta_{5,5} = -\lambda^{*3}\phi^{(1)*} - 3\lambda^{*2}\phi^{(0)*}, \quad \Delta_{6,1} = \lambda^{*4}\psi^{(2)*} + 4\lambda^{*3}\psi^{(1)*} + 6\lambda^{*2}\psi^{(0)*},
$$
$$
\Delta_{6,2} = \lambda^{*2}\psi^{(2)*} + 2\lambda^*\psi^{(1)*} + \psi^{(0)*}, \quad \Delta_{6,4} = -\lambda^{*5}\phi^{(2)*} - 5\lambda^{*4}\phi^{(1)*} - 10\lambda^{*3}\phi^{(0)*},
$$
$$
\Delta_{6,5} = -\lambda^{*3}\phi^{(2)*} - 3\lambda^{*2}\phi^{(1)*} - 3\lambda^*\phi^{(0)*}.
$$

With the aid of symbolic computation, we know that the second-order solution (44) can explicitly be given from Eqs. (44) and (39) with Eqs. (45)–(47), but it is of the long expression about x, t and parameters a, c, ρ, b_1, and d_1. Here we give its explicit expressions for some special parameters:

Case IIa. For the parameters $a = -1$, $c = 1$, $\rho = 0$, and $b_1 = d_1 = 0$, we have the second-order rogue wave of Eq. (1),

$$\widetilde{q}_{31}(x,t) = \frac{G_{11}G_{12}}{(F_{R1} + i\,F_{I1})^2}e^{-ix}, \tag{48}$$

with

$$F_{R1} = 8x^6 + 666t^2 - 12x^4 - 216x^2t^2 + 8t^6 + 180t^4 - 72xt + 24x^4t^2 + 48x^3t + 24\,x^2t^4 + 48xt^3 + 90x^2 + 9, \tag{49}$$

$$F_{I1} = 24\,x^5 - 24\,x^4t + 48\,x^3t^2 + 54\,x - 336\,t^3 - 198\,t + 48\,x^3 - 24\,t^5 + 24\,t^4x - 288\,t^2x - 48\,x^2t^3, \tag{50}$$

$$G_{11} = 8x^6 - 12x^4 + 180t^4 + 8t^6 + 666t^2 + 90x^2 + 24x^4t^2 + 48t^3x + 48x^3t - 216x^2t^2 - 72xt + 24t^4x^2 + 9$$
$$+ i(24t^5 - 48x^3t^2 + 198t - 24x^5 + 336t^3 + 48x^2t^3 + 288xt^2 + 24x^4t - 48x^3 - 54x - 24xt^4),$$

$$G_{12} = -8x^6 + 60x^4 + 60t^4 - 8t^6 + 486t^2 + 198x^2 - 24x^4t^2 + 144t^3x + 144x^3t + 504x^2t^2 - 504xt - 24t^4x^2 - 45$$
$$+ i(24x^5 - 48x^3 + 72t^5 + 528t^3 - 414t - 90x - 288x^2t + 72x^4t - 576xt^2 + 144x^2t^3 + 24xt^4 + 48x^3t^2). \tag{51}$$

The second-order rogue-wave solution profile is displayed in Figs. 2(a) and 2(c).

Case IIb. For another parameters $a = -1$, $c = 1$, $\rho = 2$, and $b_1 = d_1 = 0$, we have the second-order rogue wave of Eq. (1),

$$\widetilde{q}_{32}(x,t) = \frac{G_{21}G_{22}}{(F_{R2} + i\,F_{I2})^2}e^{i(4t-x)}, \tag{52}$$

with

$$F_{R2} = 306\,x^2 + 11106\,t^2 + 2160\,xt^3 + 240\,x^3t - 19800\,x^2t^2 - 72\,xt + 180\,x^4 + 239220\,t^4 + 27000\,x^4t^2$$
$$+ 729000\,t^6 + 1000\,x^6 + 243000\,x^2t^4,$$

$$F_{I2} = 600\,x^5 + 144\,x^3 - 198\,t - 10224\,t^3 + 54\,x + 48600\,t^5 + 288\,x^2t + 48600\,t^4x - 10800\,x^2t^3$$
$$+ 10800\,x^3t^2 - 4608\,t^2x - 600\,x^4t, \tag{53}$$

$$G_{21} = 1000x^6 + 180x^4 + 239220t^4 + 729000t^6 + 11106t^2 + 306x^2 + 27000x^4t^2 + 2160t^3x + 240x^3t$$
$$- 19800x^2t^2 - 72x + 243000t^4x^2 + 9 + i(48600t^5 + 10224t^3 + 198t - 600x^5 - 144x^3 - 54x$$
$$+ 10800x^2t^3 - 48600xt^4 - 10800x^3t^2 - 288x^2t + 600x^4t + 4608xt^2), \tag{54}$$

$$G_{22} = -1000x^6 + 1020x^4 + 246780t^4 - 729000t^6 + 11358t^2 + 558x^2 - 27000x^4t^2 + 41040t^3x$$
$$+ 4560x^3t + 84600x^2t^2 - 1656xt - 243000t^4x^2 - 45 + i(205200x^2t^3 + 11400x^4t + 48600xt^4$$
$$+ 10800x^3t^2 - 17568xt^2 - 6912x^2t + 923400t^5 + 34416t^3 + 600x^5 - 1854t - 336x^3 - 90x).$$

The second-order rogue-wave solution profile is displayed in Figs. 2(b) and 2(d).

In fact, the parameters b_1 and d_1 in solution (44) can be used to split the second-order rogue wave (44) into three first-order rogue waves, whose center points make the triangle exhibited in Fig. 3. In fact, we find that the sides of this triangle become

bigger and bigger as $|b_1|$ and $|d_1|$ increase from zero and the parameter d_1 can also control the rotation of the rogue-wave profile [see Figs. 3(b) and 3(d)].

Case III. When $N = 4$, according to Theorem 2, we have the third-order rogue-wave solution of Eq. (1)

$$\widetilde{q}_4(x,t) = q_0(x,t) + \frac{\partial B^{(7)}}{\partial x} - 2i\rho B^{(7)}, \tag{55}$$
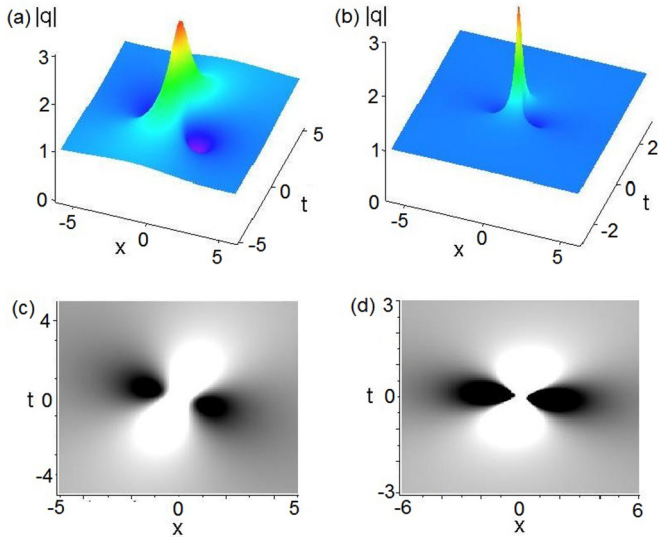

FIG. 1. (Color online) The first-order rogue-wave solution $\widetilde{q}_2(x,t)$ given by Eqs. (42) and (43) with $a = -1$, $c = 1$. [(a) and (c)] $\rho = 0$; [(b) and (d)] $\rho = 2$.
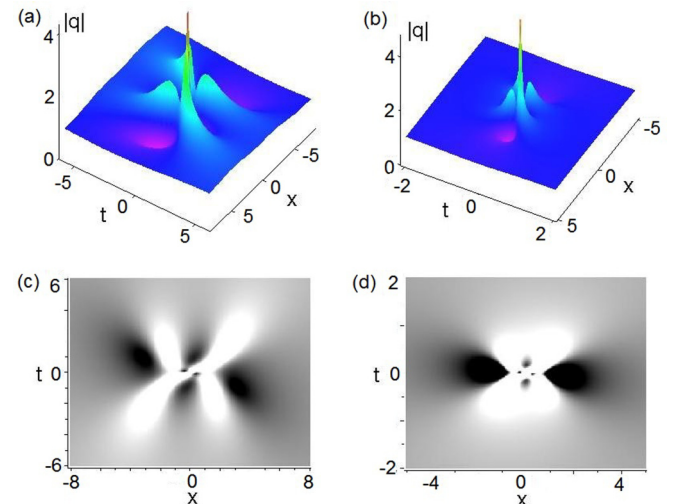
FIG. 2. (Color online) The second-order rogue-wave solution given by Eqs. (48) and (52) with $a = -1$, $c = 1$, $b_1 = d_1 = 0$. [(a) and (c)] $\rho = 0$; [(b) and (d)] $\rho = 2$.

with $B^{(7)} = \frac{\Delta^\epsilon B^{(7)}}{\Delta_5^\epsilon}$ and

$$\Delta_5^\epsilon = \begin{vmatrix} \lambda^6\phi^{(0)} & \lambda^4\phi^{(0)} & \lambda^2\phi^{(0)} & \phi^{(0)} & \lambda^7\psi^{(0)} & \lambda^5\psi^{(0)} & \lambda^3\psi^{(0)} & \lambda\psi^{(0)} \\ \Delta_{2,1} & \Delta_{2,2} & \Delta_{2,3} & \phi^{(1)} & \Delta_{2,5} & \Delta_{2,6} & \Delta_{2,7} & \lambda\psi^{(1)}+\psi^{(0)} \\ \Delta_{3,1} & \Delta_{3,2} & \Delta_{3,3} & \phi^{(2)} & \Delta_{3,5} & \Delta_{3,6} & \Delta_{3,7} & \lambda\psi^{(2)}+\psi^{(1)} \\ \Delta_{4,1} & \Delta_{4,2} & \Delta_{4,3} & \phi^{(3)} & \Delta_{4,5} & \Delta_{4,6} & \Delta_{4,7} & \lambda\psi^{(3)}+\psi^{(2)} \\ \lambda^{*6}\psi^{(0)*} & \lambda^{*4}\psi^{(0)*} & \lambda^{*2}\psi^{(0)*} & \psi^{(0)*} & -\lambda^{*7}\phi^{(0)*} & -\lambda^{*5}\phi^{(0)*} & -\lambda^{*3}\phi^{(0)*} & -\lambda^*\phi^{(0)*} \\ \Delta_{6,1} & \Delta_{6,2} & \Delta_{6,3} & \psi^{(1)*} & \Delta_{6,5} & \Delta_{6,6} & \Delta_{6,7} & -\lambda^*\phi^{(1)*}-\phi^{(0)*} \\ \Delta_{7,1} & \Delta_{7,2} & \Delta_{7,3} & \psi^{(2)*} & \Delta_{7,5} & \Delta_{7,6} & \Delta_{7,7} & -\lambda^*\phi^{(2)*}-\phi^{(1)*} \\ \Delta_{8,1} & \Delta_{8,2} & \Delta_{8,3} & \psi^{(3)*} & \Delta_{8,5} & \Delta_{8,6} & \Delta_{8,7} & -\lambda^*\phi^{(3)*}-\phi^{(2)*} \end{vmatrix}, \tag{56}$$

where

$$\Delta_{2,1} = \lambda^6\phi^{(1)} + 6\lambda^5\phi^{(0)}, \quad \Delta_{2,2} = \lambda^4\phi^{(1)} + 4\lambda^3\phi^{(0)}, \quad \Delta_{2,3} = \lambda^2\phi^{(1)} + 2\lambda\phi^{(0)}, \quad \Delta_{2,5} = \lambda^7\psi^{(1)} + 7\lambda^6\psi^{(0)},$$

$$\Delta_{2,6} = \lambda^5\psi^{(1)} + 5\lambda^4\psi^{(0)}, \quad \Delta_{2,7} = \lambda^3\psi^{(1)} + 3\lambda^2\psi^{(0)}, \quad \Delta_{3,1} = \lambda^6\phi^{(2)} + 6\lambda^5\phi^{(1)} + 15\lambda^4\phi^{(0)},$$

$$\Delta_{3,2} = \lambda^4\phi^{(2)} + 4\lambda^3\phi^{(1)} + 6\lambda^2\phi^{(0)}, \quad \Delta_{3,3} = \lambda^2\phi^{(2)} + 2\lambda\phi^{(1)} + \phi^{(0)}, \quad \Delta_{3,5} = \lambda^7\psi^{(2)} + 7\lambda^6\psi^{(1)} + 21\lambda^5\psi^{(0)},$$

$$\Delta_{3,6} = \lambda^5\psi^{(2)} + 5\lambda^4\psi^{(1)} + 10\lambda^3\psi^{(0)}, \quad \Delta_{3,7} = \lambda^3\psi^{(2)} + 3\lambda^2\psi^{(1)} + 3\lambda\psi^{(0)},$$

$$\Delta_{4,1} = \lambda^6\phi^{(3)} + 6\lambda^5\phi^{(2)} + 15\lambda^4\phi^{(1)} + 20\lambda^3\phi^{(0)}, \quad \Delta_{4,2} = \lambda^4\phi^{(3)} + 4\lambda^3\phi^{(2)} + 6\lambda^2\phi^{(1)} + 4\lambda\phi^{(0)},$$

$$\Delta_{4,3} = \lambda^2\phi^{(3)} + 2\lambda\phi^{(2)} + \phi^{(1)}, \quad \Delta_{4,5} = \lambda^7\psi^{(3)} + 7\lambda^6\psi^{(2)} + 21\lambda^5\psi^{(1)} + 35\lambda^4\psi^{(0)},$$

$$\Delta_{4,6} = \lambda^5\psi^{(3)} + 5\lambda^4\psi^{(2)} + 10\lambda^3\psi^{(1)} + 10\lambda^2\psi^{(0)}, \quad \Delta_{4,7} = \lambda^3\psi^{(3)} + 3\lambda^2\psi^{(2)} + 3\lambda\psi^{(1)} + \psi^{(0)},$$

$$\Delta_{6,1} = \lambda^{*6}\psi^{(1)*} + 6\lambda^{*5}\psi^{(0)*}, \quad \Delta_{6,2} = \lambda^{*4}\psi^{(1)*} + 4\lambda^{*3}\psi^{(0)*}, \quad \Delta_{6,3} = \lambda^{*2}\psi^{(1)*} + 2\lambda^*\psi^{(0)*},$$

$$\Delta_{6,5} = -\lambda^{*7}\phi^{(1)*} - 7\lambda^{*6}\phi^{(0)*}, \quad \Delta_{6,6} = -\lambda^{*5}\phi^{(1)*} - 5\lambda^{*4}\phi^{(0)*}, \quad \Delta_{6,7} = -\lambda^{*3}\phi^{(1)*} - 3\lambda^{*2}\phi^{(0)*},$$

$$\Delta_{7,1} = \lambda^{*6}\psi^{(2)*} + 6\lambda^{*5}\psi^{(1)*} + 15\lambda^{*4}\psi^{(0)*}, \quad \Delta_{7,2} = \lambda^{*4}\psi^{(2)*} + 4\lambda^{*3}\psi^{(1)*} + 6\lambda^{*2}\psi^{(0)*},$$

$$\Delta_{7,3} = \lambda^{*2}\psi^{(2)*} + 2\lambda^*\psi^{(1)*} + \psi^{(0)*}, \quad \Delta_{7,5} = -\lambda^{*7}\phi^{(2)*} - 7\lambda^{*6}\phi^{(1)*} - 21\lambda^{*5}\phi^{(0)*},$$

$$\Delta_{7,6} = -\lambda^{*5}\phi^{(2)*} - 5\lambda^{*4}\phi^{(1)*} - 10\lambda^{*3}\phi^{(0)*}, \quad \Delta_{7,7} = -\lambda^{*3}\phi^{(2)*} - 3\lambda^{*2}\phi^{(1)*} - 3\lambda^*\phi^{(0)*},$$

$$\Delta_{8,1} = \lambda^{*6}\psi^{(3)*} + 6\lambda^{*5}\psi^{(2)*} + 15\lambda^{*4}\psi^{(1)*} + 20\lambda^{*3}\psi^{(0)*}, \quad \Delta_{8,2} = \lambda^{*4}\psi^{(3)*} + 4\lambda^{*3}\psi^{(2)*} + 6\lambda^{*2}\psi^{(1)*} + 4\lambda^*\psi^{(0)*},$$

$$\Delta_{8,3} = \lambda^{*2}\psi^{(3)*} + 2\lambda^*\psi^{(2)*} + \psi^{(1)*}, \quad \Delta_{8,5} = -\lambda^{*7}\phi^{(3)*} - 7\lambda^{*6}\phi^{(2)*} - 21\lambda^{*5}\phi^{(1)*} - 35\lambda^{*4}\phi^{(0)*},$$

$$\Delta_{8,6} = -\lambda^{*5}\phi^{(3)*} - 5\lambda^{*4}\phi^{(2)*} - 10\lambda^{*3}\phi^{(1)*} - 10\lambda^{*2}\phi^{(0)*}, \quad \Delta_{8,7} = -\lambda^{*3}\phi^{(3)*} - 3\lambda^{*2}\phi^{(2)*} - 3\lambda^*\phi^{(1)*} - \phi^{(0)*}. \tag{57}$$

Here $\Delta^\epsilon B^{(7)}$ is produced from Δ_5^ξ by replacing its first column with $(-\lambda^8\phi^{(0)}, \; -\lambda^8\phi^{(1)} - 8\lambda^7\phi^{(0)}, \; -\lambda^8\phi^{(2)} - 8\lambda^7\phi^{(1)} - 28\lambda^6\phi^{(0)}, \; -\lambda^8\phi^{(3)} - 8\lambda^7\phi^{(2)} - 28\lambda^6\phi^{(1)} - 56\lambda^5\phi^{(0)},$

(a) |q|

(b) |q|

(c)

(d)

FIG. 3. (Color online) The second-order rogue-wave solution $\tilde{q}_3(x,t)$ given by Eq. (44) with $a = -1, c = 1, \rho = 2$. [(a) and (c)] $b_1 = -10^3, d_1 = 0$; [(b) and (d)] $b_1 = 0, d_1 = 10^3$.

$-\lambda^{*8}\psi^{(0)*}, -\lambda^{*8}\psi^{(1)*} - 8\lambda^{*7}\psi^{(0)*}, -\lambda^{*8}\psi^{(2)*} - 8\lambda^{*7}\psi^{(1)*} - 28\lambda^{*6}\psi^{(0)*}, -\lambda^{*8}\psi^{(3)*} - 8\lambda^{*7}\psi^{(2)*} - 28\lambda^{*6}\psi^{(1)*} - 56\lambda^{*5}\psi^{(0)*})^{\mathrm{T}}$.

With the aid of symbolic computation, we know that the second-order solution (55) can explicitly be given from Eqs. (55) and (39) with Eqs. (56) and (57), but it is of the long expression about x, t and parameters $a, c, \rho, b_1, b_2, d_1$, and d_2.

For the given parameters $a = -1, c = 1, \rho = 0, 2$, other parameters b_1, b_2, d_1, d_2 can make the third-order rogue wave become the different structures.

(a) When the parameters $b_1 = b_2 = d_1 = d_2 = 0$, the strong interaction of the third-order rogue wave and their corresponding density graphs are shown in Fig. 4.

(b) When the parameters $b_1 = 10^3, -10^3, d_1 = b_2 = d_2 = 0$, the weak interaction of the third-order rogue wave is split into six first-order rogue waves, and they array a triangle structure (see Fig. 5).

(c) When the parameters $b_2 = 10^4, -10^4, d_1 = b_2 = d_2 = 0$, the weak interaction of the third-order rogue wave is also split into six first-order rogue waves, but they array a pentagon structure with a first-order rogue wave being almost located in the center of the pentagon structure (see Fig. 6).

(d) If we choose one nonzero parameter from two families $\{b_1, d_1\}$ and $\{b_2, d_2\}$, respectively, then the third-order rogue-wave solution (55) displays the different structures (see Fig. 7).
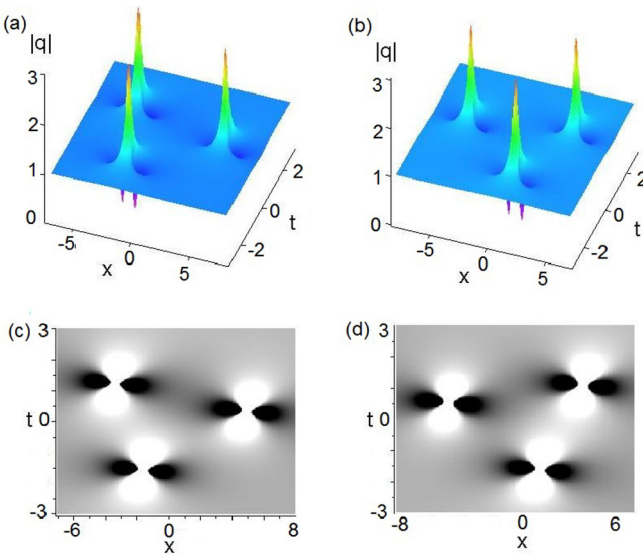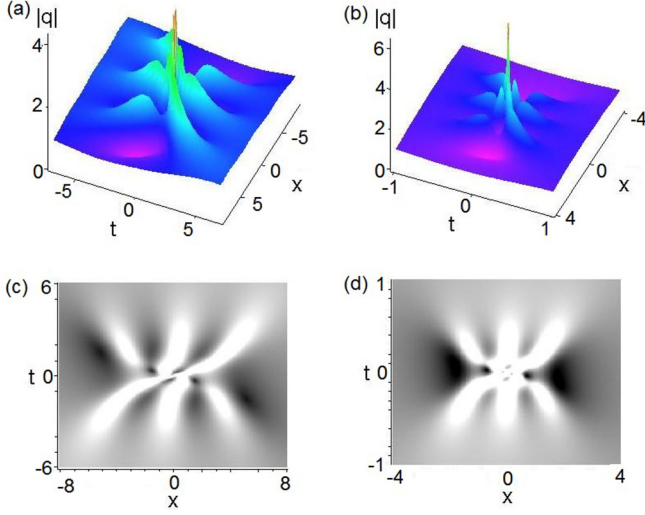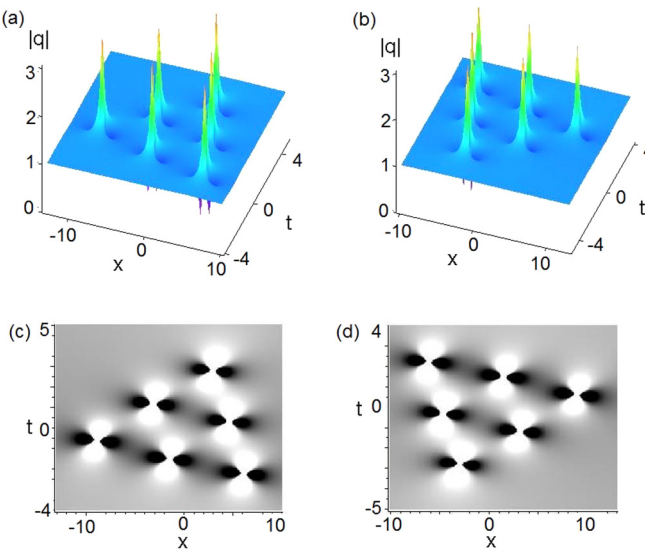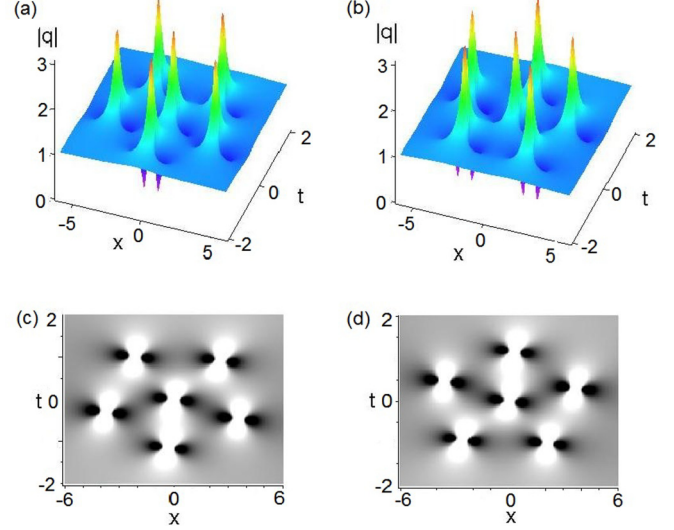
FIG. 4. (Color online) The third-order rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eq. (55) with $a = -1$, $c = 1$ and $b_1 = b_2 = d_1 = d_2 = 0$. [(a) and (c)] $\rho = 0$; [(b) and (d)] $\rho = 2$.

D. Dynamical behaviors of multi-rogue-wave solutions

To further illustrate the wave propagations of some above-obtained rogue-wave solutions, we here consider the dynamical behaviors of these rogue-wave solutions of Eq. (1) by comparing these obtained exact multi-rogue-wave solutions (e.g., first-order, second-order, and third-order rogue-wave solutions) of Eq. (1) with their time evolutions using them as initial conditions with or without a small noise via numerical simulations.

Case 1: First-order rogue waves. For two families of parameters $\{a = -1, c = 1, \rho = 0\}$ and $\{a = -1, c = 1, \rho = 2\}$, Figs. 8 and 9 exhibit the exact first-order rogue-wave solution (42) of Eq. (1), time evolutions of the rogue wave

FIG. 5. (Color online) The third-order rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eq. (55) with $a = -1$, $c = 1$, $\rho = 2$. [(a) and (c)] $b_1 = 10^3$, $d_1 = b_2 = d_2 = 0$; [(b) and (d)] $b_1 = -10^3$, $d_1 = b_2 = d_2 = 0$.

FIG. 6. (Color online) The third-order rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eq. (55) with $a = -1$, $c = 1$, $\rho = 2$. [(a) and (c)] $b_2 = 10^4$, $d_1 = b_1 = d_2 = 0$; [(b) and (d)] $b_2 = -10^4$, $d_1 = b_1 = d_2 = 0$.

of Eq. (1) using exact solution (42) and exact solution (42) perturbed by a small noise [2% and 1% for Figs. 8(c) and 9(c), respectively] as the initial conditions, respectively. It follows from Figs. 8(a), 8(b), 9(a), and 9(b) that the profiles of the time evolutions of the rogue waves of Eq. (1) without noise agree with ones of the corresponding exact rogue-wave solutions. Figure 8(c) shows that the wave profile exhibits an almost-stable propagation, except for some oscillations when time approaches 3. Figure 9(c) illustrates no collapse-instead stable wave propagation, except for some oscillations in the wings of waves when time approaches to 0.

Case 2: Second-order rogue waves. For three families of parameters $\{a = -1, c = 1, \rho = 0, b_1 = d_1 = 0\}$, $\{a = -1, c = 1, \rho = 2, b_1 = d_1 = 0\}$, and $\{a = -1, c = 1, \rho = 2, b_1 = 0, d_1 = 10^3\}$, Figures 10–12 illustrate the exact second-order rogue-wave solution (52) of Eq. (1), the time evolution of the rogue wave of Eq. (1) using exact solution (43) and exact solution (52) perturbed by a small noise [e.g., 2%, 1%, and 0.5% for Figs. 10(c), 11(c), and 12(c), respectively] as the initial conditions, respectively. It follows from Figs. 10(a), 10(b), 11(a), 11(b), 12(a), and 12(b) that the profiles of the time evolutions of the rogue waves of Eq. (1) without a noise agree with ones of the corresponding exact rogue-wave solutions. Figure 10(c) displays the almost-stable wave propagation; however, Figs. 11(c) and 12(c) exhibit the no collapse-instead stable wave propagation, except for some oscillations in the wings of waves when time approaches 0.

Case 3: Third-order rogue waves. For three families of parameters $\{a = -1, c = 1, \rho = 2, b_1 = b_2 = d_1 = d_2 = 0\}$, $\{a = -1, c = 1, \rho = 2, b_1 = 10^3, b_2 = d_1 = d_2 = 0\}$, and $\{a = -1, c = 1, \rho = 2, b_1 = 10^4, b_2 = d_1 = d_2 = 0\}$, Figs. 13–15 illustrate the exact third-order rogue-wave solution (55) of Eq. (1) and the time evolution of the rogue wave of Eq. (1) using exact solution (55) and exact solution (55) perturbed by a small noise [e.g., 1.5% for Figs. 13(c) and 0.1% for Figs. 14(c) and 15(c)] as the initial conditions,
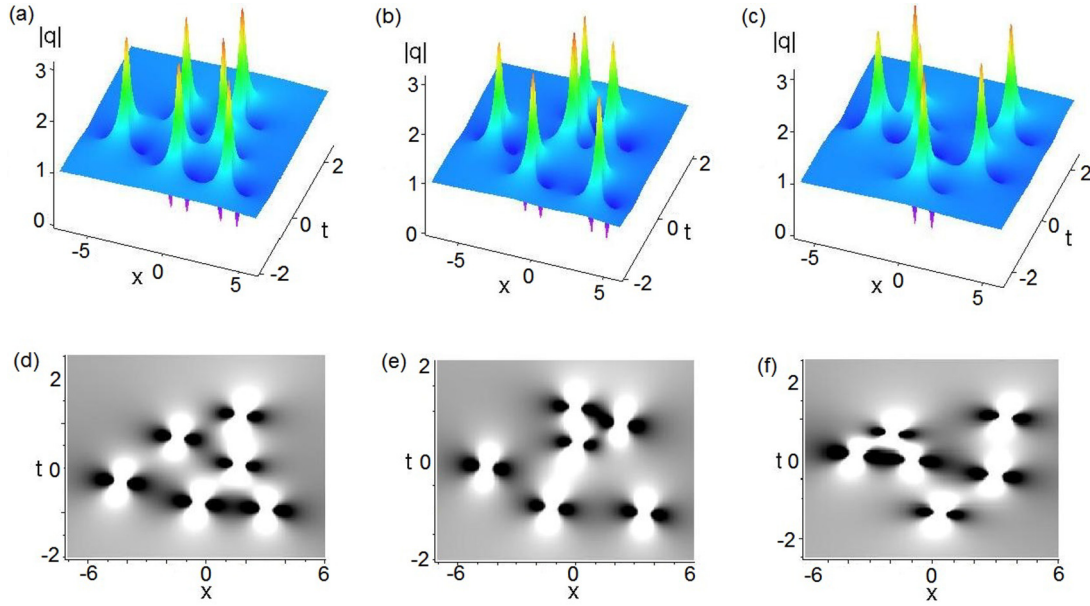
FIG. 7. (Color online) The third-order rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eqs. (55) and (43) with $a = -1, c = 1, \rho = 2$. [(a) and (c)] $b_1 = 100, b_2 = 3000, d_1 = d_2 = 0$; [(b) and (e)] $b_1 = 100, d_1 = 0, b_2 = 10, d_2 = 10^4$; [(c) and (f)] $b_1 = d_2 = 0, d_1 = 100, b_2 = 10^4$.
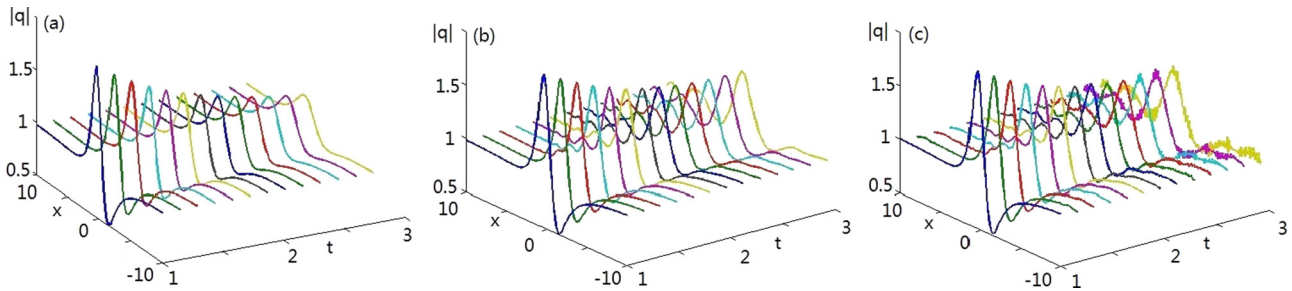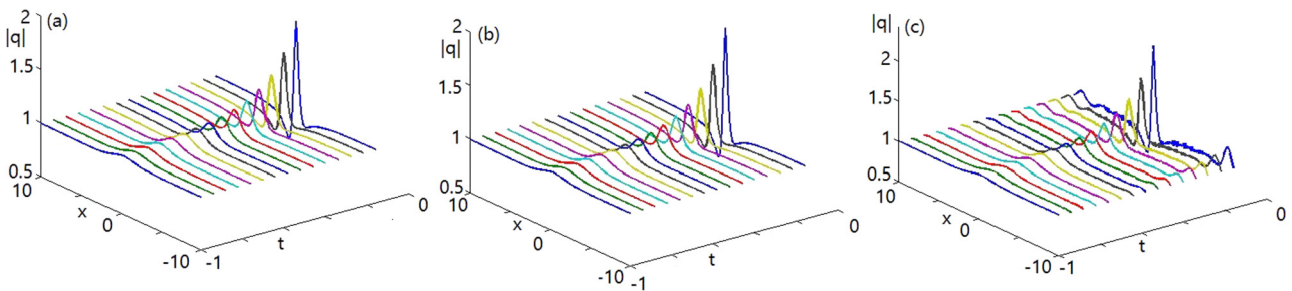

FIG. 8. (Color online) The first-order rogue-wave solution $\widetilde{q}_{21}(x,t)$ given by Eq. (42) with $a = -1, c = 1, \rho = 0$. (a) Exact solution, (b) time evolution using exact solution (42) as the initial condition, and (c) time evolution using exact solution (42) perturbated by a 2% noise as the initial condition.

FIG. 9. (Color online) The first-order rogue-wave solution $\widetilde{q}_{22}(x,t)$ given by Eq. (43) with $a = -1, c = 1, \rho = 2$. (a) Exact solution, (b) time evolution of the wave using exact solution (43) as the initial condition, and (c) time evolution of the wave using exact solution (43) perturbated by a 1% noise as the initial condition.
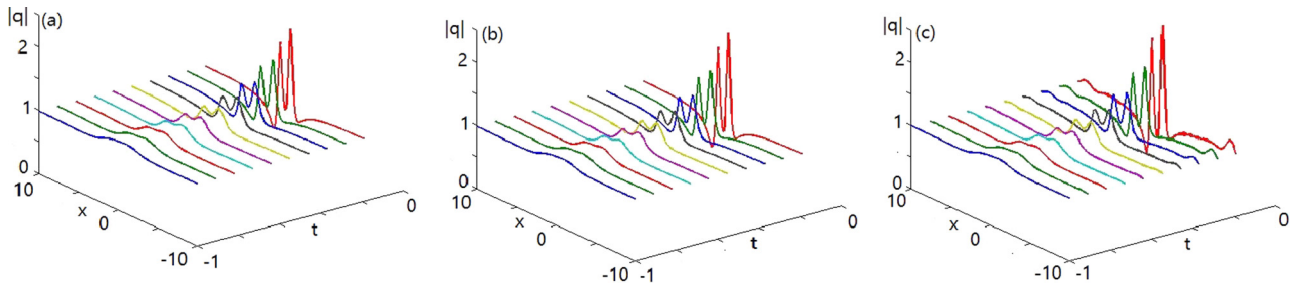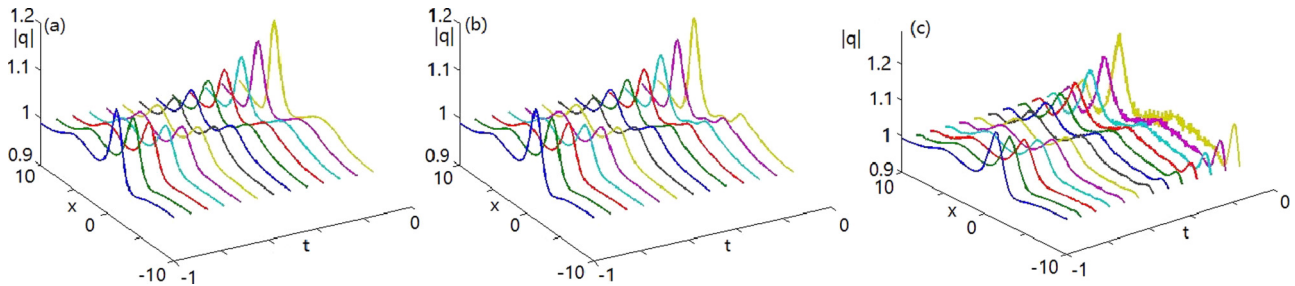
FIG. 10. (Color online) The second-order interactive rogue-wave solution $\tilde{q}_{31}(x,t)$ given by Eq. (48) with $a = -1, c = 1, \rho = 0, b_1 = d_1 = 0$. (a) Exact solution, (b) time evolution of the wave using exact solution (48) as the initial condition, and (c) time evolution of the wave using exact solution (48) perturbed by a 2% noise as the initial condition.

FIG. 11. (Color online) The second-order interactive rogue-wave solution $\tilde{q}_{32}(x,t)$ given by Eq. (52) with $a = -1, c = 1, \rho = 2, b_1 = d_1 = 0$. (a) Exact solution, (b) time evolution of the wave using exact solution (52) as the initial condition, and (c) time evolution of the wave using exact solution (52) perturbed by a 1% noise as the initial condition.

FIG. 12. (Color online) The second-order separatable rogue-wave solution $\tilde{q}_3(x,t)$ given by Eq. (44) with $a = -1, c = 1, \rho = 2, b_1 = 0, d_1 = 10^3$. (a) Exact solution, (b) time evolution of the wave using exact solution (44) as the initial condition, and (c) time evolution of the wave using exact solution (44) perturbed by a 0.5% noise as the initial condition.
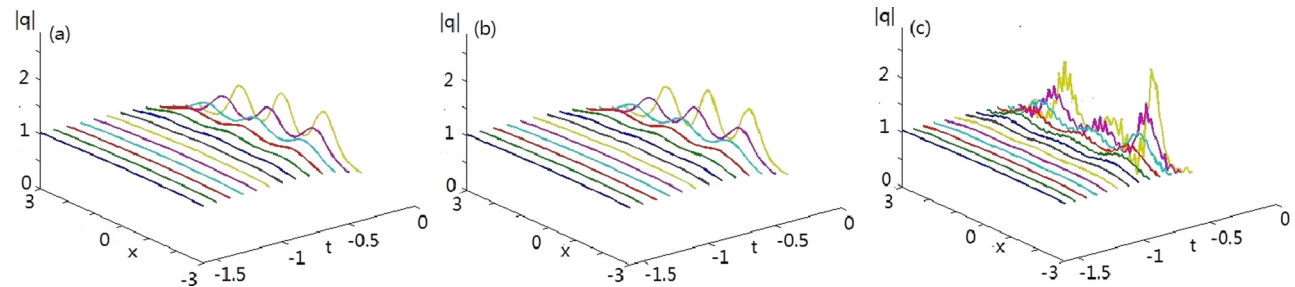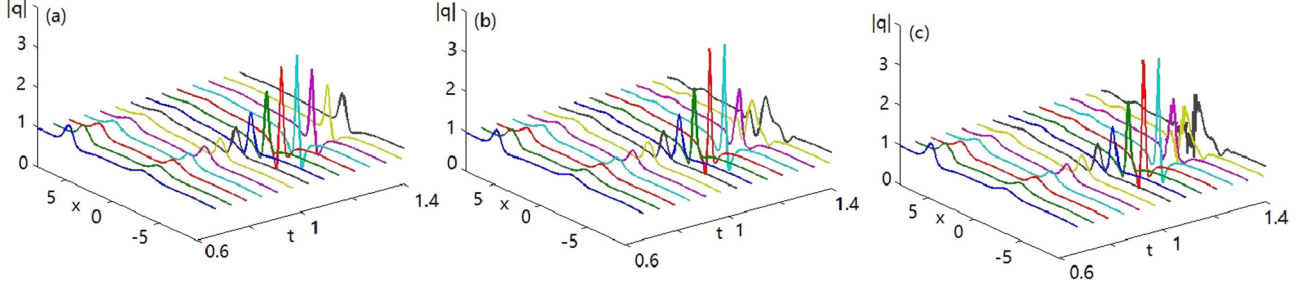

FIG. 13. (Color online) The third-order interactive rogue-wave solution $\tilde{q}_4(x,t)$ given by Eq. (55) with $a = -1, c = 1, \rho = 2, b_1 = b_2 = d_1 = d_2 = 0$. (a) Exact solution, (b) time evolution of the wave using exact solution (55) as the initial condition, and (c) time evolution of the wave using exact solution (55) perturbed by a 1.5% noise as the initial condition.

FIG. 14. (Color online) The third-order separatable rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eq. (55) with $a = -1, c = 1, \rho = 2, b_1 = 10^3, b_2 = d_1 = d_2 = 0$. (a) Exact solution, (b) time evolution of the wave using exact solution (55) as the initial condition, (c) time evolution of the wave using exact solution (55) perturbated by a 0.1% noise as the initial condition.

respectively. It follows from Figs. 13(a), 13(b), 14(a), 14(b), 15(a), and 15(b) that the profiles of time evolutions of rogue waves of Eq. (1) without a noise are agree with ones of the corresponding exact rogue wave solutions. Figure 13(c) exhibits the almost unstable wave propagation from the beginning of about $t = 0.5$; however, Figs. 14(c) and 15(c) exhibit no collapse-instead stable wave propagation, except for some oscillations when time approaches to 1.4 and 0, respectively.

$$\varphi_x = U\varphi, U = \begin{pmatrix} -i\lambda^2 + \frac{i}{2}|q|^2 & \lambda q \\ -\lambda q^* & i\lambda^2 - \frac{i}{2}|q|^2 \end{pmatrix}, \tag{58}$$

$$\varphi_t = V\varphi, V = \begin{bmatrix} -2i\lambda^4 + i\lambda^2|q|^2 + \frac{1}{2}(qq_x^* - q^*q_x) + \frac{1}{4}|q|^4 & 2\lambda^3 q + i\lambda q_x \\ -2\lambda^3 q^* + i\lambda q_x^* & 2i\lambda^4 - i\lambda^2|q|^2 - \frac{1}{2}(qq_x^* - q^*q_x) - \frac{1}{4}|q|^4 \end{bmatrix}, \tag{59}$$

where $\varphi = (\phi,\psi)^{\mathrm{T}}$ is the complex eigenfunction, $\lambda \in \mathbb{C}$ is the spectral parameter, $q = q(x,t)$ denotes the complex potential and is also the solution of Eq. (2), the subscript denotes the partial derivative with respect to the variables x, t, and the star denotes the complex conjugate of the corresponding variables.

Similarly to the MNLS equation (1), we choose the same Darboux matrix T given by Eq. (10) to consider the Darboux transformation of Eq. (2) such that we have the following theorem for the multisoliton solutions and multi-rogue-wave solutions of GI equation (2).

Theorem 4. Let $\varphi_i(\lambda_i) = (\phi_i(\lambda_i), \psi_i(\lambda_i))^{\mathrm{T}}$ $(i = 1,2,\ldots,n)$ be column vector solutions of the spectral problem (58) and (59) for the spectral parameters λ_i $(i = 1,2,\ldots,n)$ and the same initial solution $q_0(x,t)$ of Eq. (2), respectively; then the generalized perturbation (n,M)-fold DT of Eq. (2) is given by

$$\widetilde{q}_N(x,t) = q_0(x,t) + 2i\,B^{(2N-1)}, \tag{60}$$

where $B^{(2N-1)} = \frac{\Delta^{\epsilon(n)} B^{(2N-1)}}{\Delta_N^{\epsilon(n)}}$ and $\Delta_N^{\epsilon(n)} = \det([\Delta^{(1)} \ldots \Delta^{(n)}]^{\mathrm{T}})$ with $\Delta^{(i)}$ being given by Eq. (32) and $\Delta^{\epsilon} B^{(2N-1)}$ is formed from the determinant $\Delta_N^{\epsilon(n)}$ by replacing its $(N+1)$-th column by the column vector $(b^{(1)} \cdots b^{(n)})^{\mathrm{T}}$ with $b^{(i)} = (b_j^{(i)})_{2(m_i+1)\times1}$ and $b_j^{(i)}$ being given by Eq. (34).

IV. THE MULTI-ROGUE-WAVE SOLUTIONS OF THE GERJIKOV-IVANOV EQUATION

A. The generalized perturbation (n,M)-fold Darboux transformation method

The Gerjikov-Ivanov equation (2) is just a zero-curvature equation $U_t - V_x + [U, V] = 0$ with $[U, V] \equiv UV - VU$ and two 2×2 matrices U and V satisfying the linear isospectral problem (Lax pair) [28]

B. The multi-rogue-wave solutions

In the following we give some multi-rogue-wave solutions of Eq. (2) in terms of determinants by use of generalized perturbation $(1, N-1)$-fold DT in Theorem 4. We consider the seed solution of Eq. (2) in the plane-wave form

$$q_0 = ce^{i[ax+(\frac{1}{2}c^4-ac^2-a^2)t]}, \tag{61}$$

where a and $c \neq 0$ are real-valued constants, a is the wave number, and c is the amplitude of the plane wave. It is known that the phase velocity is $[a + c^2 - c^4/(2a)]$, the group velocity is $2a + c^2$, and $|q_0(x,t)| \to |c| \neq 0$ as $|x|,|t| \to \infty$.

Substituting Eq. (61) into Eqs. (58) and (59), we can give the solution of Lax pair (58) and (59) with the spectral parameter λ as follows:

$$\varphi(\lambda) = \begin{bmatrix} i(C_1 e^{-A} - C_2 e^A)e^{-B} \\ (C_2 e^A + C_1 e^{-A})e^B \end{bmatrix}, \tag{62}$$

with

$$C_1 = C_+, \ C_2 = C_-,$$

$$C_\pm = \frac{\sqrt{\pm(2\lambda^2 + a - c^2) + \sqrt{(a+2\lambda^2)^2 + c^2(c^2 - 2a)}}}{\sqrt{2c\lambda}},$$

$$A = i\sqrt{(a+2\lambda^2)^2 + c^2(c^2 - 2a)}$$

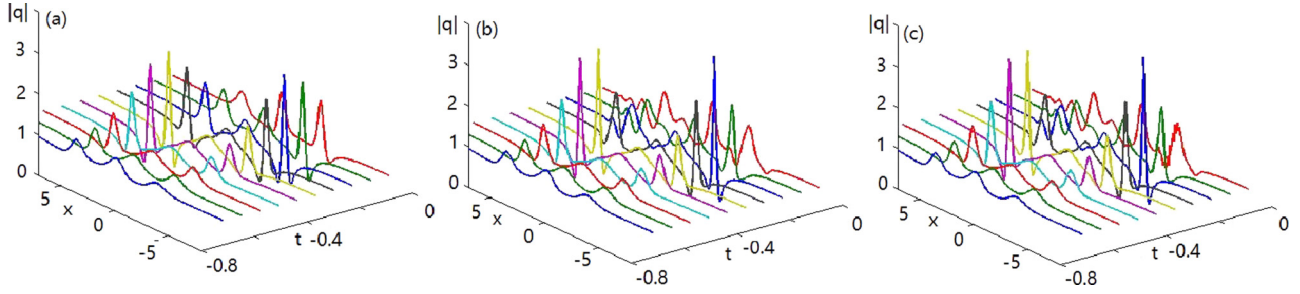$$\times \left[\frac{1}{2}x + \left(\lambda^2 - \frac{a}{2}\right)t + \Theta(\varepsilon)\right],$$

FIG. 15. (Color online) The third-order separatable rogue-wave solution $\widetilde{q}_4(x,t)$ given by Eq. (55) with $a = -1, c = 1, \rho = 2, b_1 = 10^4, b_2 = d_1 = d_2 = 0$. (a) Exact solution, (b) time evolution of the wave using exact solution (55) as the initial condition, (c) time evolution of the wave using exact solution (55) perturbated by a 0.1% noise as the initial condition.

$$B = -\frac{i}{2}\left[ax + \left(\frac{1}{2}c^4 - ac^2 - a^2\right)t\right],$$

$$\Theta(\varepsilon) = \sum_{k=1}^{N}(b_k + d_k i)\varepsilon^{2k},$$

where $b_k, d_k (k = 1, 2, \ldots, N)$ are real free parameters and ε is a small parameter. Next,

we fix

$$\lambda = \tfrac{1}{2}\sqrt{2\sqrt{2c^2 a - c^4} - 2a + \epsilon^2}, \quad (63)$$

for the special case $a = 0, c = 2$, we have $\lambda = 1 + i + \varepsilon^2$ for simplification, expanding the vector function $\varphi(\varepsilon^2)$ in Eq. (62) at $\varepsilon = 0$, we obtain

$$\varphi(\varepsilon^2) = \varphi^{(0)} + \varphi^{(1)}\varepsilon^2 + \varphi^{(2)}\varepsilon^4 + \varphi^{(3)}\varepsilon^6 + \cdots, \quad (64)$$

where

$$\varphi^{(0)} = \begin{pmatrix} \phi^{(0)} \\ \psi^{(0)} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}e^{4it} \\ \sqrt{2}e^{-4it} \end{pmatrix}, \quad (65)$$

$$\varphi^{(1)} = \begin{bmatrix} -\frac{\sqrt{2}}{8}e^{4it}[32x^2 - 512t^2 + 256xt + 64t - 1 + i(512t^2 - 32x^2 - 16x + 256xt - 1)] \\ \frac{\sqrt{2}}{8}e^{-4it}[32x^2 - 512t^2 + 256xt - 64t - 1 + i(512t^2 - 32x^2 + 16x + 256xt - 1)] \end{bmatrix}, \quad (66)$$

$$\varphi^{(2)} = \begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix}, \quad \varphi^{(3)} = \begin{pmatrix} \phi^{(3)} \\ \psi^{(3)} \end{pmatrix}, \quad \ldots \quad (67)$$

and $(\phi^{(i)}, \psi^{(i)})^{\mathrm{T}} (i = 2, 3)$ are listed in Appendix B.

For $N = 1$ we only deduce the trivial plane-wave solution of Eq. (2). In the following we consider the multi-rogue-wave solutions of Eq. (2) for $N = 2, 3, 4$.

Case I. For $N = 2$, we have the first-order rogue-wave solution (60) of Eq. (2) with $a = 0, c = 2$:

$$\widetilde{q}_2(x,t) = 2\left[1 - \frac{4(1 + 32it)}{1 + 32(x^2 + 16t^2) - 8i(x - 4t)}\right]e^{8it}, \quad (68)$$

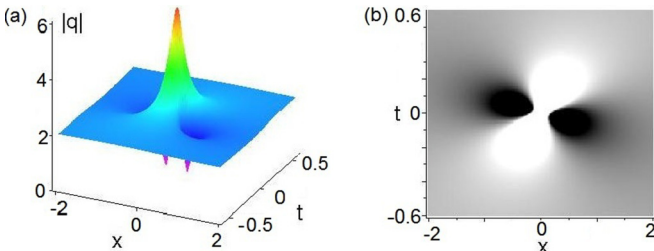whose wave profile is shown in Fig. 16. This solution is the same as the one in Ref. [39].

FIG. 16. (Color online) [(a) and (b)] The first-order rogue-wave solution $\widetilde{q}_2(x,t)$ given by Eq. (68) with $a = 0, c = 2$.

Case II. For $N = 3$ we have the second-order rogue-wave solution of Eq. (2), which is complicated and are omitted here, but we give its wave profiles for different parameters. In fact, the parameters b_1 and d_1 in the second-order rogue-wave solution $\widetilde{q}_3(x,t)$ can be used to split the second-order rogue wave into three first-order rogue waves, whose center points make the triangle exhibited in Figs. 17(b) and 17(d). In fact, we find that the sides of this triangle become bigger and bigger as $|b_1|$ and $|d_1|$ increase from zero and the parameter d_1 can also control the rotation of the rogue-wave profile [see Figs. 17(b) and 17(d)].

Case III. For $N = 4$ and the given parameters $a = 0, c = 2$, other parameters b_1, b_2, d_1, d_2 can make the third-order rogue wave become the different structures. Figure 18 displays the the interaction of three-order rogue waves.

(i) When the parameters $b_1 = b_2 = d_1 = d_2 = 0$, the interaction of the third-order rogue wave and their corresponding density graphs are shown in Figs. 18(a) and 18(d).

(ii) When the parameters $b_1 = 10^3, d_1 = b_2 = d_2 = 0$, the interaction of the third-order rogue wave is split into six first-order rogue waves, and they form a triangle structure [see Figs. 18(b) and 18(e)].
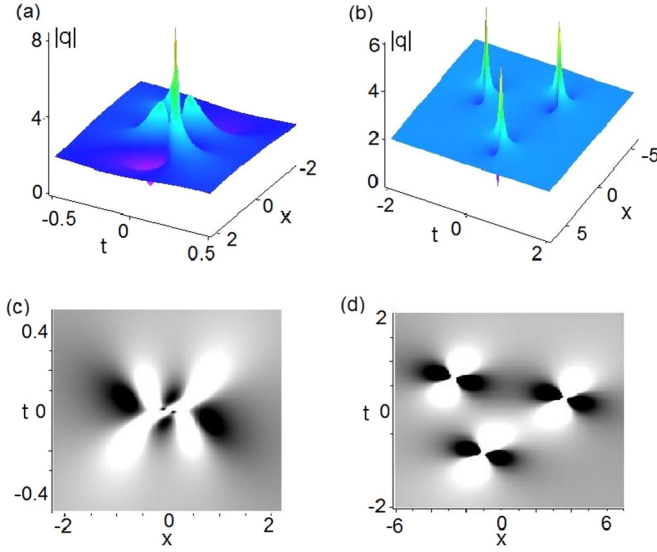
FIG. 17. (Color online) The second-order rogue-wave solution $\tilde{q}_3(x,t)$ with $a = 0, c = 2$. (a), (c): $b_1 = d_1 = 0$; (b), (d): $b_1 = 100, d_1 = 0$.

(iii) When the parameters $b_2 = 10^3, b_1 = d_1 = d_2 = 0$, the interaction of the third-order rogue wave is also split into six first-order rogue waves, but they form a pentagon structure with a first-order rogue wave being almost located in the center of the pentagon structure [see Figs. 18(c) and 18(f)].

For other cases $N > 4$, we can also obtain the higher-order rogue-wave solutions of Eq. (2), which display the abundant structures. Similarly to Eq. (1), we can also illustrate the time evolutions of these solutions using numerical simulations, which are omitted here.

V. CONCLUSIONS

In conclusion, we have presented a simple and constructive method to find the generalized perturbation (n, M)-fold Darboux transformations (DTs) of the modified nonlinear Schrödinger equation and the Gerjikov-Ivanov equation in terms of fractional forms of determinants. In particular, we apply the generalized perturbation $(1, N - 1)$-fold DTs to find their explicit higher-order rogue-wave solutions. The dynamics behaviors of these rogue waves are discussed in detail for the different parameters, which display abundant interesting wave structures, including the triangle and pentagon, etc., and may be useful to study the physical mechanism of multirogue waves in optics. Moreover, we study the time evolutions of these obtained multi-rogue-wave solutions using numerical simulations.

For the MNLS equation, if we choose two different spectral parameters $\lambda_1 = (3 + i)/5$ and $\lambda_2 = 2i$, then the higher-order rogue waves can be degraded to lower-order rogue waves. It is still a problem to generate abundant wave structures by choosing more spectral parameters. In fact, this method can also be extended to seek multi-rogue-wave solutions of other nonlinear integrable equations such as the NLS equation, KP equation, AB system, AKNS hierarchy, which will be studied in a future work.
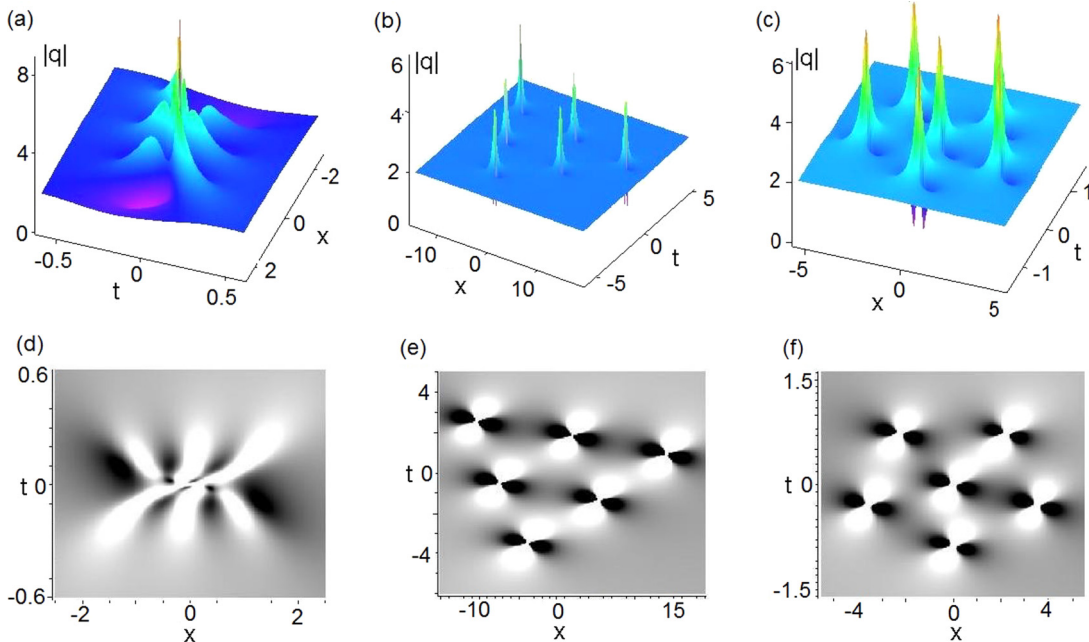
FIG. 18. (Color online) The third-order rogue-wave solution $\tilde{q}_4(x,t)$ with $a = 0, c = 2$. [(a) and (d)] $b_1 = b_2 = d_1 = d_2 = 0$; [(b) and (e)] $b_1 = 1000, d_1 = b_2 = d_2 = 0$; [(c) and (f)] $b_2 = 1000, b_1 = d_1 = d_2 = 0$.

APPENDIX A

Some components of eigenfunctions in Eq. (36) are listed below

$$\phi^{(2)} = \frac{1}{64}\sqrt{2}e^{\frac{i}{2}(-x+4t)}(88 + 384ib_1 - 684x - 10620t + 51192t^3 + 1944xt^2 + 57024it^3x + 648ix^2t + 288b_1 - 384d_1 - 1896ix^3$$
$$- 72x^3 - 1012ix - 2592itd_1 - 864ixb_1 + 1404ix^4 + 2180x^2 - 44964t^2 + 2340it - 28512x^2t^2 - 151632t^3x$$
$$+ 864d_1x + 16848x^3t + 41904xt - 1248xb_1 - 3744td_1 - 2592tb_1 + 42768t^4 + 528x^4 + 3744itb_1 - 17064x^2t - 21528ixt$$
$$+ 51192ixt^2 - 1248id_1x + 3240ix^2 - 96552it^2 + 113724it^4 + 288id_1 + 9i - 6336ix^3t - 1944it^3 - 75816ix^2t^2),$$

$$\psi^{(2)} = -\frac{i}{64}\sqrt{2}e^{-\frac{i}{2}(-x+4t)}(-9 - 288ib_1 - 1012x + 2340t + 528ix^4 - 51192it^3 + 88i - 1944t^3 + 51192xt^2 - 2592itb_1$$
$$+ 16848ix^3t + 10620it + 384b_1 + 288d_1 - 151632it^3x - 1896x^3 - 3240x^2 + 96552t^2 + 75816x^2t^2 - 57024t^3x + 1248d_1x$$
$$+ 6336x^3t + 21528xt + 864xb_1 + 2592td_1 - 3744tb_1 - 113724t^4 - 1404x^4 + 648x^2t + 684ix - 28512ix^2t^2 - 1944ixt^2$$
$$+ 17064ix^2t + 2180ix^2 + 864id_1x - 3744itd_1 - 1248ixb_1 + 41904ixt + 42768it^4 + 72ix^3 - 44964it^2 + 384id_1),$$

$$\phi^{(3)} = \frac{1}{2560}\sqrt{2}e^{\frac{i}{2}(-x+4t)}(-2925 + 1676160itd_1 + 2021760ix^2td_1 + 100960x + 1709740t - 41636160t^3$$
$$+ 6065280xt^2d_1 + 12862800xt^2 - 2280960ixt^2d_1 + 11757312it^5 + 3481660ixt - 27360b_1 + 40480d_1 + 84480id_1x^3$$
$$+ 11520b_2 - 15360d_2 - 6065280ixt^2b_1 - 572580ix^4 + 19852560it^3x - 46320x^3 - 49920id_1b_1 - 34560ixb_2 - 49920id_2x$$
$$+ 2975i + 51840ixtb_1 + 64045080ix^2t^2 + 174400id_1x - 352170x^2 + 33539832it^6 + 5428410t^2 - 5770440x^2t^2$$
$$+ 161215920t^3x - 259200d_1x - 10551600x^3t - 9475380xt + 174400xb_1 + 861120td_1 + 1676160tb_1 + 21123180t^4$$
$$- 47060x^4 + 7807680x^2t - 8415576t^6 + 24960d_1^2 + 11544x^6 - 24960b_1^2 - 67079664t^5x - 6065280t^3b_1 - 828144x^5t$$
$$- 224640d_1x^3 + 34560d_1b_1 + 14025960t^4x^2 - 1558440x^4t^2 + 34560d_2x + 24844320t^3x^3 + 84480x^3b_1 - 49920xb_2$$
$$- 103680tb_2 - 149760td_2 + 760320x^2td_1 + 2021760x^2tb_1 - 2280960xt^2b_1 - 2280960t^3d_1 - 1365120ixtd_1 + 259200ixb_1$$
$$+ 15360ib_2 - 358800ix^3t - 145756260it^4 + 530240ix^3 + 17280id_1^2 - 6065280it^3d_1 - 46008ix^6 - 17280ib_1^2 + 75888ix^5$$
$$- 461310ix^2 - 8640ix^2d_1 + 2280960it^3b_1 + 6233760it^3x^3 + 87780ix + 18440784t^5 - 48384x^5 - 207792ix^5t$$
$$- 13063680ix^2t^3 + 725760ix^4t - 760320ix^2tb_1 + 6211080ix^4t^2 + 149760itb_2 - 18668880it^3 + 2352240ix^2t - 547680it$$
$$- 103680itd_2 - 1365120xtb_1 - 8640x^2b_1 + 227520x^2d_1 - 19595520t^4x + 4354560x^3t^2 + 77760t^2b_1 - 2047680t^2d_1$$
$$- 20489760x^2t^3 + 1138320x^4t - 51840xtd_1 - 40480ib_1 + 11520id_2 + 30734640it^4x - 27360id_1 - 16831152it^5x$$
$$+ 31635630it^2 - 32529600ixt^2 + 224640ix^3b_1 - 55899720it^4x^2 + 77760it^2d_1 - 227520ix^2b_1 - 6829920ix^3t^2$$
$$- 861120itb_1 + 2047680it^2b_1),$$

$$\psi^{(3)} = \frac{i}{2560}\sqrt{2}e^{-\frac{i}{2}(-x+4t)}(2975 - 21123180it^4 - 41636160it^3 - 46320ix^3 - 27360ib_1 + 11520ib_2 - 15360id_2$$
$$- 48384ix^5 + 18440784it^5 + 40480id_1 - 2021760ix^2tb_1 - 760320ix^2td_1 + 1709740it - 51840ixtd_1 - 87780x$$
$$- 5428410it^2 + 100960ix + 352170ix^2 + 4354560ix^3t^2 + 47060ix^4 + 547680t + 18668880t^3 - 2280960xt^2d_1$$
$$+ 32529600xt^2 + 67079664it^5x - 24960id_1^2 - 11544ix^6 - 14025960it^4x^2 + 40480b_1 + 27360d_1 - 15360b_2 - 11520d_2$$
$$+ 828144ix^5t + 24960ib_1^2 - 530240x^3 + 259200id_1x - 34560id_2x + 6065280it^3b_1 - 84480ix^3b_1 + 1558440ix^4t^2$$
$$+ 7807680ix^2t - 461310x^2 + 31635630t^2 + 64045080x^2t^2 + 19852560t^3x + 174400d_1x - 358800x^3t + 3481660xt$$
$$+ 259200xb_1 + 1676160td_1 - 861120tb_1 - 145756260t^4 - 572580x^4 - 24844320it^3x^3 - 2352240x^2t - 861120itd_1$$
$$+ 103680itb_2 + 33539832t^6 + 17280d_1^2 - 46008x^6 - 17280b_1^2 - 16831152t^5x + 2280960t^3b_1 - 207792x^5t + 84480d_1x^3$$
$$- 49920d_1b_1 - 55899720t^4x^2 + 6211080x^4t^2 - 49920d_2x + 6233760t^3x^3 + 224640x^3b_1 - 34560xb_2 + 149760tb_2$$
$$- 103680td_2 + 2021760x^2td_1 - 760320x^2tb_1 - 6065280xt^2b_1 - 6065280t^3d_1 + 49920ixb_2 - 34560id_1b_1 + 5770440ix^2t^2$$
$$+ 8415576it^6 + 2925i - 19595520it^4x - 2047680it^2d_1 + 224640id_1x^3 - 6065280ixt^2d_1 - 161215920it^3x + 149760itd_2$$
$$- 11757312t^5 - 75888x^5 - 1676160itb_1 - 1365120ixtb_1 - 51840xtb_1 + 77760it^2b_1 + 227520x^2b_1 + 8640x^2d_1$$
$$- 30734640t^4x + 6829920x^3t^2 - 2047680t^2b_1 - 77760t^2d_1 + 13063680x^2t^3 - 725760x^4t + 1365120xtd_1 - 174400ixb_1$$
$$+ 227520ix^2d_1 - 20489760ix^2t^3 - 8640ix^2b_1 + 10551600ix^3t + 2280960ixt^2b_1 + 1138320ix^4t + 2280960it^3d_1$$
$$+ 12862800ixt^2 + 9475380ixt).$$

APPENDIX B

Some components of eigenfunctions in Eq. (64) are listed below

$$\phi^{(2)} = \frac{1}{192}e^{4it}\sqrt{2}(-12 + 48x - 2112t - 768d_1 + 192x^2 + 32768t^3 + 512x^3 - 24576xt^2 - 6144x^2t - 3072t^2$$
$$- 21504xt - 3072xb_1 - 3072xd_1 - 12288tb_1 + 12288td_1 - 16384x^3t + 262144t^3x + 262144it^4 - 24576ixt^2$$
$$- 98304ix^2t^2 + 1728it + 768ib_1 - 12288itd_1 - 3072ixd_1 + 3072ixb_1 - 12288itb_1 + 6144ix^2t + 1152ix^2 + 15i$$
$$+ 144ix - 67584it^2 - 32768it^3 + 512ix^3 + 1536ixt + 1024ix^4),$$

$$\psi^{(2)} = -\frac{1}{192}e^{-4it}\sqrt{2}(-12 - 768ib_1 - 48x + 2112t - 1728it + 15i - 67584it^2 + 768d_1 + 192x^2 - 32768t^3 - 512x^3$$
$$+ 24576xt^2 + 6144x^2t - 3072t^2 - 21504xt - 3072xb_1 - 3072xd_1 - 12288tb_1 + 12288td_1 - 16384x^3t + 262144t^3x$$
$$+ 1024ix^4 + 1152ix^2 + 32768it^3 + 262144it^4 - 144ix + 3072ixb_1 - 12288itb_1 - 3072ixd_1 - 12288itd_1 - 512ix^3$$
$$- 6144ix^2t + 24576ixt^2 - 98304ix^2t^2 + 1536ixt),$$

$$\phi^{(3)} = -\frac{1}{23040}e^{4it}\sqrt{2}(405 - 368640itb_1 + 2949120itd_1x - 5160960ix^3t + 1474560itb_2 - 125829120it^4x^2$$
$$+ 47185920it^2b_1x - 2949120itb_1x + 11796480ix^2td_1 - 50331648it^5 - 92160ib_2 - 6480x + 1474560itd_2 + 5160960itd_1$$
$$- 368640ib_1^2 - 210240it - 41760ix^2 - 168960ix^4 + 8778240it^2 - 169082880it^4 - 34560ib_1 + 31457280it^3 - 15360ix^3$$
$$- 11520id_1 + 46080t + 134217728it^6 - 32768ix^6 - 983040ix^4t + 368640id_1^2 + 5898240it^2b_1 - 368640ix^2b_1$$
$$- 368640ix^2d_1 + 5898240it^2d_1 + 11796480x^2tb_1 + 31457280ix^2t^3 + 145489920it^3x + 368640ixd_2 + 7864320ix^4t^2$$
$$- 201326592it^5x - 786432ix^5t + 41943040it^3x^3 + 737280ib_1d_1 - 62914560it^3d_1 - 983040ix^3b_1 - 92160ixd_1$$
$$- 552960ixb_1 - 2177280ixt - 2949120ix^2t + 737280ixt^2 + 2949120td_1x + 2949120tb_1x - 47185920xt^2d_1$$
$$- 368640ixb_2 + 39813120ix^2t^2 - 5898240t^2d_1 + 368640x^2d_1 - 11520b_1 + 34560d_1 + 92160d_2 - 64800x^2 - 983040t^3$$
$$- 122880x^3 + 17694720xt^2 + 184320x^2t + 10621440t^2 - 199680x^4 - 176947200t^4 + 1624320xt - 92160xb_1$$
$$+ 552960xd_1 + 5160960tb_1 + 368640td_1 + 4669440x^3t - 137625600t^3x + 42762240x^2t^2 + 368640xb_2 + 368640xd_2$$
$$+ 1474560tb_2 - 1474560td_2 + 737280b_1d_1 + 983040x^3d_1 - 62914560t^3b_1 + 786432x^5t + 7864320x^4t^2 - 41943040t^3x^3$$
$$- 125829120t^4x^2 + 201326592t^5x + 368640b_1^2 - 368640d_1^2 - 32768x^6 + 134217728t^6 + 1395i + 7864320x^3t^2$$
$$- 62914560t^4x - 49152x^5 - 368640x^2b_1 + 5898240t^2b_1),$$

$$\psi^{(3)} = \frac{1}{23040}e^{-4it}\sqrt{2}(405 - 2177280ixt + 145489920it^3x + 6480x - 737280ixt^2 - 46080t - 368640ixb_2$$
$$+ 39813120ix^2t^2 - 368640ib_1^2 + 11796480x^2tb_1 - 125829120it^4x^2 - 2949120td_1x - 2949120tb_1x - 47185920xt^2d_1$$
$$+ 5898240t^2d_1 - 368640x^2d_1 + 983040ix^4t + 11520b_1 - 34560d_1 - 92160d_2 + 47185920it^2b_1x - 552960ixb_1$$
$$- 368640itb_1 - 64800x^2 + 983040t^3 + 122880x^3 - 17694720xt^2 - 184320x^2t + 10621440t^2 - 199680x^4 - 176947200t^4$$
$$+ 1624320xt - 92160xb_1 + 552960xd_1 + 5160960tb_1 + 368640td_1 + 4669440x^3t - 137625600t^3x + 42762240x^2t^2$$
$$- 5898240it^2b_1 + 368640xb_2 + 368640xd_2 + 1474560tb_2 - 1474560td_2 + 737280b_1d_1 + 983040x^3d_1 - 62914560t^3b_1$$
$$+ 786432x^5t + 7864320x^4t^2 - 41943040t^3x^3 - 125829120t^4x^2 + 201326592t^5x + 368640b_1^2 - 368640d_1^2 - 32768x^6$$
$$+ 134217728t^6 - 5898240it^2d_1 + 1474560itd_2 + 1474560itb_2 + 210240it + 11796480ix^2td_1 - 201326592it^5x$$
$$- 7864320x^3t^2 + 62914560t^4x - 31457280ix^2t^3 + 49152x^5 - 31457280it^3 - 5160960ix^3t + 50331648it^5 + 737280ib_1d_1$$
$$+ 368640x^2b_1 - 5898240t^2b_1 - 41760ix^2 + 2949120ix^2t + 15360ix^3 + 34560ib_1 + 5160960itd_1 - 168960ix^4$$
$$+ 368640id_1^2 - 32768ix^6 + 92160ib_2 - 2949120itd_1x + 134217728it^6 + 11520id_1 + 368640ix^2d_1 + 41943040it^3x^3$$
$$- 983040ix^3b_1 - 62914560it^3d_1 - 169082880it^4 + 2949120itb_1x + 1395i + 368640ixd_2 + 368640ix^2b_1 - 786432ix^5t$$
$$+ 8778240it^2 - 92160ixd_1 + 7864320ix^4t^2).$$

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[1] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Muria, Phys. Rev. Lett. **19**, 1095 (1967).

[3] M. J. Ablowitz and H. Segur, *Solitons and Inverse Scattering Transformation* (SIAM, Philadelphia, 1981).

[4] M. J. Ablowitz and P. A. Clarkson, Solitons, *Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[5] P. Deift and E. Trubowitz, Commun. Pure Appl. Math. **32**, 121 (1979).

[6] V. B. Matveev and M. A. Salle, *Darboux Transformation and Solitons* (Springer-Verlag, Berlin, 1991).

[7] C. H. Gu (ed.), *Soliton Theory and Its Applications* (Srpinger-Verlag, Berlin, 1995), pp. 122–151.

[8] C. H. Gu, A. N. Hu, and Z. X. Zhou, *Darboux Transformations in Integrable Systems: Theory and Their Applications to Geometry* (Springer, Berlin, 2005).

[9] G. Darboux, C. R. Acad. Sci. Paris **94**, 1456 (1882).

[10] N. N. Akhmediev, V. I. Korneev, and N. V. Mitskevich, Zh. Eksp. Teor. Fiz. **74**, 159 (1988).

[11] N. Akhmediew, A. Ankiewicz, and J. M. Soto-Crespo, Phys. Rev. E **80**, 026601 (2009).

[12] B. L. Guo, L. M. Ling, and Q. P. Liu, Phys. Rev. E **85**, 026607 (2012).

[13] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, and J. M. Dudley, Nature Phys. **6**, 790 (2010); A. Chabchoub, N. Hoffmann, M. Onorato, and N. Akhmediev, Phys. Rev. X **2**, 011015 (2012).

[14] Y. Ohta and J. Yang, Proc. Roy. Soc. A. **468**, 1716 (2012).

[15] Z. Y. Yan, V. V. Konotop, and N. Akhmediev, Phys. Rev. E **82**, 036610 (2010); Z. Y. Yan, Phys. Lett. A **374**, 672 (2010); Commun. Theor. Phys. **54**, 947 (2010); Phys. Lett. A **375**, 4274 (2011); J. Math. Anal. Appl. **380**, 689 (2011); Z. Y. Yan and C. Dai, J. Opt. **15**, 064012 (2013); Z. Y. Yan, Nonlinear Dyn. **79**, 2515 (2015).

[16] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, Phys. Rev. Lett. **109**, 044102 (2012); F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, and S. Wabnitz, *ibid.* **113**, 034101 (2014).

[17] N. V. Priya, M. Senthilvelan, and M. Lakshmanan, Phys. Rev. E **89**, 062901 (2014).

[18] G. P. Agrawal, *Nonlinear Fiber Optics*, 4th ed. (Academic Press, Boston, 2007).

[19] J. K. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).

[20] D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).

[21] D. Mihalache, N. Truta, N.-C. Panoiu, and D.-M. Baboiu, Phys. Rev. A **47**, 3190 (1993).

[22] H. Nakatsuka, D. Grischkowsky, and A. C. Balant, Phys. Rev. Lett. **47**, 910 (1981); N. Tzoar and M. Jain, Phys. Rev. A **23**, 1266 (1981).

[23] K. Mio, T. Ogino, K. Minami, and S. Takeda, J. Phys. Soc. Jpn. **41**, 265 (1976).

[24] M. Stiassnie, Wave Motion **6**, 431 (1984).

[25] A. I. Maimistov, J. Exp. Theor. Phys. **77**, 727 (1993).

[26] M. Wadati, K. Konno, and Y. H. Ichikawa, J. Phys. Soc. Jpn. **46**, 1965 (1979).

[27] V. V. Konotop and V. E. Vekslerchik, Phys. Lett. A **131**, 357 (1988).

[28] Y. Xiao, Commun. Theor. Phys. **15**, 365 (1991).

[29] Z. Y. Chen and N. N. Huang, Phys. Rev. A **41**, 4066 (1990).

[30] S. L. Liu and W. Z. Wang, Phys. Rev. E **48**, 3054 (1993).

[31] J. P. Liu, Commun. Theor. Phys. **20**, 65 (1993).

[32] Y. Kodama and A. Hasegawa, IEEE J. Quant. Elect. **QE-23**, 510 (1987).

[33] L. Bergé, J. J. Rasmussen, and J. Wyller, J. Phys. A: Math. Gen. **29**, 3581 (1996).

[34] J. S. Hesthaven *et al.*, J. Phys. A: Math. Gen. **30**, 8207 (1997).

[35] L. Berge and S. Skupin, Phys. Rev. E **71**, 065601R (2005).

[36] B. L. Guo, L. M. Ling, and Q. P. Liu, Stud. Appl. Math. **130**, 317 (2013).

[37] E. G. Fan, J. Math. Phys. **42**, 4327 (2001); **41**, 7769 (2000).

[38] K. Imai, J. Phys. Soc. Jpn. **68**, 355 (1999).

[39] L. J. Guo *et al.*, Phys. Scr. **89**, 035501 (2014).