Susceptibilities for the Müller-Hartmann-Zitartz countable infinity of phase transitions on a Cayley tree

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We obtain explicit susceptibilities for the countable infinity of phase transition temperatures of Müller-Hartmann-Zitartz on a Cayley tree. The susceptibilities are a product of the zeroth spin with the sum of an appropriate set of averages of spins on the outermost layer of the tree. A clear physical understanding for these strange phase transitions emerges naturally. In the thermodynamic limit, the susceptibilities tend to zero above the transition and to infinity below it.

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I. INTRODUCTION

Forty years ago, Müller-Hartmann and Zitartz (MHZ) [1–3] showed that the Ising ferromagnet on a Cayley tree has a countable infinity of phase transitions of an unusual type. Specifically, in the low temperature ordered phase, when one traverses vertically across the zero-field region in the magnetic field-temperature phase diagram, they showed that the *order* of the phase transition can be *anything* between 1 and ∞ depending on the value of the temperature at that point. Therefore, they dubbed them phase transitions of continuous order. By identifying the values of temperature at which the order acquires integer values, they also extracted a countable infinity of phase-transitions in the low-temperature phase. The MHZ work constitutes one of the classic examples of a strange phase transition. Most of their work is rather technical and mathematical, though, and a transparent physical understanding of the meaning of these strange phase transitions would be useful.

Although criticized as unphysical, the Cayley tree (or the close cousin called Bethe lattice) is a popular geometric structure on which innumerable studies continue to be carried out in current times [4–10]. One reason is, of course, the availability of exact solutions for some models. Another important, not often emphasized reason is that it simultaneously contains one-dimension-like properties and infinite-dimension-like properties, thus providing an interesting framework for theoretical explorations. For example, the partition function of the ferromagnet on a Cayley tree [11,12] is identical to that of the one-dimensional chain (barring an inconsequential constant factor), and yet a phase transition exists in the Cayley showing infinite-dimensional character.

MHZ extract the order of the phase transition at various points in the low-temperature phase by studying the leading-order singularities of the full free-energy as a function of field as it is taken to the zero limit. Here we show that a direct transparent understanding of these phase transitions may be obtained by the consideration of the pure zero-field model, building on a "memory approach" that we emphasized in recent work [13]. We were also partially propelled by the recent exact solution of a one-dimensional long-range ferromagnet, which admits an unusual phase transition of a different type, namely mixed-order, which can simultaneously show a discontinuous jump in magnetization (first-order-like) *and* a diverging correlation-length (second-order-like) [14,15].

II. THE COUNTABLE INFINITY OF PHASE TRANSITIONS OF MHZ

Figure 1 carries a schematic of the phase diagram of MHZ. The gist of their approach is to make a careful detailed study of the free energy $F(H,T)$. In a generic ferromagnetic system with a phase transition, $F(H,T)$ is analytic at all points except in the region shown in red: $[H=0, 0 \leqslant T \leqslant T_{\mathrm{BP}}]$. In this limit the free energy is given by [2]

$$F(H,T) = F(0,T) + f_{\mathrm{reg}}(H^2,T) + A(T)|H|^{\kappa}, \quad H \to 0, \quad (1)$$

where the regular part is a function of H^2 because of symmetry, and the leading singular part $A(T)$ shows a power law behavior. The unusual aspect of the Cayley tree lies in the fact that the critical exponent κ varies continuously from 1 at $T=0$ to ∞ [2] at the usual phase transition called the Bethe-Peierls transition T_{BP}. This behavior is in sharp contrast to most commonly encountered phase transitions where κ remains a constant (=1 for a first-order transition, 2 for a second-order transition, and so on). By studying the points at which κ takes integer values, they identify a countable infinity of phase transitions that fit into the Ehrenfest classification of phase transitions of integer power-order.

Here we point out that in fact a simple (albeit unusual) set of susceptibilities identify the countable infinity of phase transitions, and indeed a clear physical picture of the meaning behind the phase transitions comes naturally out of them. The susceptibilities turn out to be a product of the zeroth spin with the sum of averages of appropriately grouped spins in the outermost layer of the tree. Furthermore, we can work entirely with the zero-field model, with no requirement of complicated procedures or mathematical methods related to the application of a tiny field followed by taking the zero-field limit.

III. THE MODEL

Following the notation of a recent piece of work involving the author [13], we consider the following Hamiltonian:

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j + H\sum_j \sigma_j, \quad (2)$$

where the sum involves pairs of spins that are adjacent on the tree (Fig. 2) with coordination number z and depth n. σ_i are Ising variables that can take values ±1, and J is taken to be positive to make it a ferromagnet. The solution, when

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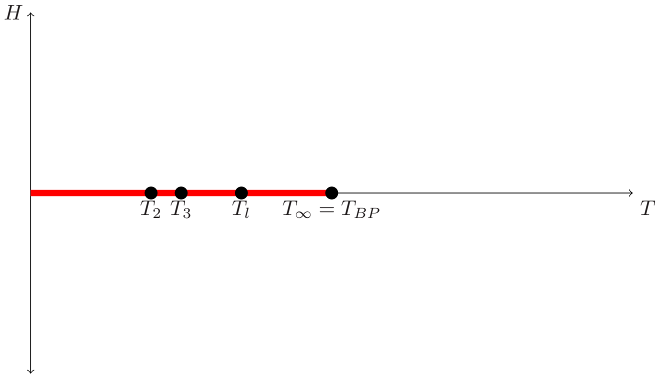

FIG. 1. (Color online) Schematic of the countable infinity of phase transitions of MHZ. The dots correspond to the countable infinity of phase transitions (as one traverses vertically by varying H through 0 at a given temperature in the ordered phase) of integer order.

the external field is $H = 0$, is trivial and we quickly recall it. We introduce new bond variables $\theta_{ij} = \sigma_i\sigma_j$. The θ_{ij} can take values ± 1, which make them effectively spin variables too. Specifying all the θ_{ij}, and the spin at the root of the tree σ_0, completely defines the system. The Hamiltonian then takes the following simple form:

$$\mathcal{H} = -J \sum_{\langle i,j\rangle} \theta_{ij}. \tag{3}$$

With the problem now rehashed into one with noninteracting spins under the influence of an external magnetic field, the partition function is readily written down [12]:

$$\mathcal{Z}(J)_{\text{Cayley}} = 2[2\cosh(\beta J)]^{N_b} = 2[2\cosh(\beta J)]^{N-1}, \tag{4}$$

where N_b is the number of bonds and N is the number of spins, and $\beta = 1/T$ is the inverse temperature as usual in equilibrium statistical mechanics. It follows directly [13,16–19] that the correlation function between any two spins σ_i, σ_j is given by

$$\langle \sigma_i\sigma_j\rangle = \tanh(\beta J)^{d_{ij}} = a^{d_{ij}}, \tag{5}$$

where we define $a \equiv \tanh(\beta J)$ for convenience, and d_{ij} is the distance of the (unique) shortest path between the points i, j.

When the external field H is nonzero, there is no simple closed-form expression; however, MHZ [1] write down an infinite-series expansion for the free energy $F(H,T)$, and by a careful, elaborate study of the order in field at which the leading singularity occurs as one crosses zero-field at low-

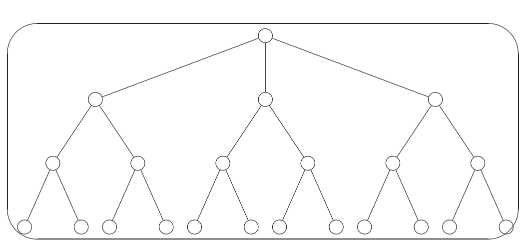

FIG. 2. A Cayley tree of depth $n = 3$ and coordination number $z = 3$.

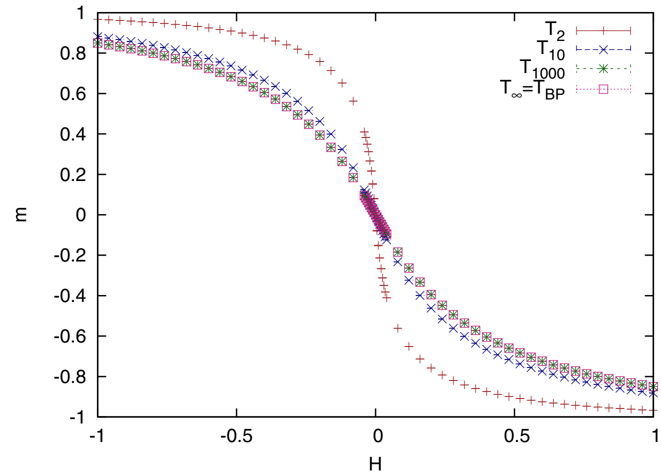

FIG. 3. (Color online) A plot of the magnetization $m = \frac{1}{N}\sum_i \langle\sigma_i\rangle$ versus magnetic field from Monte Carlo simulations of a finite-sized system of coordination number $z = 3$ and depth $n = 8$. Data are shown for different runs at the MHZ transition temperatures $T_2, T_{10}, T_{1000}, T_\infty \equiv T_{BP}$. At higher transition temperatures, the curve becomes smoother and smoother, indicating that in the thermodynamic limit the singularity would occur at a higher-order differentiation with respect to H. Since our system is finite, the data are practically indistinguishable for T_{1000} and T_∞.

temperature, they obtain the following countable infinity of transition temperatures:

$$T_l = \frac{J}{\tanh^{-1}\left[\frac{1}{\gamma^{(l-1)/l}}\right]}, \tag{6}$$

$l = 1,2,\ldots,\infty$, with l being identified as the order of the phase transition within the Ehrenfest scheme. $T_1 = 0$ has a first order phase transition and $T_\infty = T_{BP}$ with order infinity is the so-called "Bethe-Peierls" phase transition, which is the temperature at which the system first orders as it is cooled down from high temperature. Figure 3 shows a Monte Carlo simulation carried out on a finite-sized system of depth $n = 8$ and coordination number $z = 3$. We studied the dependence of magnetization as one crosses from negative to positive magnetic field at some of the transition temperatures of MHZ. It seems plausible that in the thermodynamic limit, at a higher transition temperature, a greater derivative of the magnetization with respect to field would diverge at $H = 0$. Although the simulations were run on a finite-system the data at T_2 display a considerably sharp drop near $H = 0$ indicative of the diverging derivative, since it is a second-order phase transition.

Here we show that the above countable infinity of transition temperatures may be directly obtained from the zero-field model bypassing the elaborate complicated methods of studying the infinite series and the order of divergence of the free energy in the limit of $H \to 0$. We do this by an explicit construction of a special set of "susceptibilities" for the transitions T_l.

IV. CONSTRUCTION OF THE SUSCEPTIBILITIES FOR THE MHZ TRANSITION TEMPERATURES

In order to construct the susceptibilities for the MHZ transition temperatures, we choose the depth of the lattice to be of the form $n = lp$, where l can take values $1, 2, 3, \ldots, \infty$. Since we are interested in the $p \to \infty$ limit, no loss of generality is incurred. The number of spins in the nth layer of the tree is $N_n = z(z-1)^{n-1} \equiv (1+\frac{1}{\gamma})\gamma^n$, where we have defined $\gamma \equiv (z-1)$ for convenience. Let us denote the nth layer spins by $\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,N_n}$ in order from left to right, as can be visualized in Fig. 2. Next, we group the first $M_p \equiv \gamma^p$ spins of the nth layer and call their average $\tilde{\sigma}_{n,1} = \frac{1}{M_p}\sum_{i=1}^{M_p}\sigma_{n,i}$; we then group the second M_p spins of the nth layer and call their average $\tilde{\sigma}_{n,2} = \frac{1}{M_p}\sum_{i=M_p+1}^{2M_p}\sigma_{n,i}$, and so on. By this procedure we form $K_{n,p} \equiv \frac{N_n}{M_p}$ spin averages:

$$\tilde{\sigma}_{n,j} = \frac{1}{M_p}\sum_{i=(j-1)M_p+1}^{jM_p}\sigma_{n,i}, \quad j = 1,2,3,\ldots,K_{n,p}. \tag{7}$$

Now we are ready to write down the susceptibilities. Recalling that σ_0 is the spin at the root node, the susceptibilities are simply given by

$$\mathcal{X}_l = \sigma_0 \sum_{j=1}^{K_{n,p}} \tilde{\sigma}_{n,j}, \tag{8}$$

which is our main result. To see that this leads to the MHZ transition temperatures, let us invoke the two-point correlation functions from the last section to compute the expectation value:

$$\begin{aligned}\langle \mathcal{X}_l \rangle &= \left\langle \sigma_0 \sum_{j=1}^{K_{n,p}} \tilde{\sigma}_{n,j} \right\rangle \qquad (9)\\ &= \sum_{j=1}^{K_{n,p}} \frac{1}{M_p} \sum_{i=(j-1)M_p+1}^{jM_p} \langle \sigma_0 \sigma_{n,i} \rangle \\ &= \sum_{j=1}^{K_{n,p}} \frac{1}{M_p} \sum_{i=(j-1)M_p+1}^{jM_p} a^n \\ &= K_{n,p} a^n \\ &= \left(1 + \frac{1}{\gamma}\right)[\gamma^{(l-1)} a^l]^p. \qquad (10)\end{aligned}$$

Therefore, we see that as $p \to \infty$,

$$\langle \mathcal{X}_l \rangle \to \begin{cases} 0 & \text{if } \ \gamma^{(l-1)} a^l < 1, \\ \infty & \text{if } \ \gamma^{(l-1)} a^l > 1. \end{cases} \tag{11}$$

$\gamma a^{\frac{l}{(l-1)}} = 1$ thus defines the lth transition temperature. These are precisely the transition temperatures of MHZ as given in Eq. (6) [1,20]. It is worth pointing out that for $l = 2$, this yields the so-called (because of the appearance of the same in the disordered version of the same Hamiltonian [13]) "spin-glass" transition temperature $T_{\mathrm{SG}} = \frac{J}{\tanh^{-1}(\frac{1}{\sqrt{\gamma}})}$, and the $l \to \infty$ limit yields the Bethe-Peierls transition temperature $T_{\mathrm{BP}} = \frac{J}{\tanh^{-1}(\frac{1}{\gamma})}$.

V. CONCLUSIONS

We have introduced a set of "susceptibilities" that help identify the mysterious MHZ transition temperatures on the Cayley tree in a transparent manner. They are given by the product of the zeroth spin with an appropriately averaged sum of spins from the outermost layer in a Cayley tree. A clear physical understanding of the phase-transitions emerges naturally. We observe that our susceptibilities have the feature that in the thermodynamic limit, they are zero above the phase transition, but tend to infinity below it. We are able also to identify the second-order phase transition T_2 as the phase transition known as the spin-glass phase transition in the literature.

Furthermore, although we have concentrated on the Ising ferromagnet here, the susceptibilites defined here are primarily attached to the geometry of the lattice. Therefore, they should be applicable much more generally; for example, with m-component vector spins and the bonds could be ferromagnetic or antiferromagnetic or disordered. Quantum models should display similar transitions as well; a detailed investigation of various models from this perspective would be desirable.

Finally, we remark that the statistical mechanics problem on the tree has been connected with the problem of reconstruction of information on trees in formal, extensive studies [21,22]. It would be interesting to understand if and how the MHZ countable infinity of phase transitions would fit into this generalized problem, which might be of interest to a broader community.

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