Scaling of the dynamics of a homogeneous one-dimensional anisotropic classical Heisenberg model with long-range interactions

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The dynamics of quasistationary states of long-range interacting systems with N particles can be described by kinetic equations such as the Balescu-Lenard and Landau equations. In the case of one-dimensional homogeneous systems, two-body contributions vanish as two-body collisions in one dimension only exchange momentum and thus cannot change the one-particle distribution. Using a Kac factor in the interparticle potential implies a scaling of the dynamics proportional to N^δ with $\delta = 1$ except for one-dimensional homogeneous systems. For the latter different values for δ were reported for a few models. Recently it was shown by Rocha Filho and collaborators [Phys. Rev. E **90**, 032133 (2014)] for the Hamiltonian mean-field model that $\delta = 2$ provided that N is sufficiently large, while small N effects lead to $\delta \approx 1.7$. More recently, Gupta and Mukamel [J. Stat. Mech. (2011) P03015] introduced a classical spin model with an anisotropic interaction with a scaling in the dynamics proportional to $N^{1.7}$ for a homogeneous state. We show here that this model reduces to a one-dimensional Hamiltonian system and that the scaling of the dynamics approaches N^2 with increasing N. We also explain from theoretical consideration why usual kinetic theory fails for small N values, which ultimately is the origin of noninteger exponents in the scaling.

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I. INTRODUCTION

Systems with long-range interactions may present unusual properties such as as nonergodicity, anomalous diffusion, aging, non-Gaussian quasistationary states (QSS), negative microcanonical heat capacity, ensemble inequivalence, and, more importantly for the present work, very long relaxation time to thermodynamic equilibrium of a QSS, diverging with the number of particles N [1–9]. A pair interaction potential is long ranged in d spatial dimensions if it decays at long distances as $r^{-\alpha}$ with $\alpha \leqslant d$. The dynamics of such systems can be decomposed in three stages: a violent relaxation into a QSS in a short time, a slow relaxation of the QSS, and the final thermodynamic equilibrium. In some cases after the violent relaxation the system may also oscillate for a very long or even infinite time around a QSS [10]. By introducing a Kac factor proportional to $1/N$ in the pair-interaction potential the fluid (Vlasov) limit is well defined and given by $N \to \infty$ [11–14]. The dynamics is exactly described by the Vlasov equation for the one-particle distribution function, while for finite N it is valid only for short times. Collisional terms must be considered for a more accurate description of the dynamics for finite N, leading to kinetic equations such as the Landau or Balescu-Lenard equations [13,15,16].

As already noted, the dynamics of relaxation to equilibrium depends on the number of particles in the system and has been extensively studied in the recent literature [1–5,14,17–24]. Its dependence on N can be obtained from collisional corrections to the Vlasov equation, i.e., by determining the relevant kinetic equation. Two-body collisions lead to a collisional integral in the kinetic equations of order $1/N$, and thus relaxation occurs in a time scale proportional to N, an exception being three-dimensional gravity with a relaxation time of order $N/\log N$ [25,26]. For one-dimensional homogeneous systems two-body terms in the kinetic equation vanish identically as collisions between two particles result only in momentum exchange [27–29]. For instance, the Balescu-Lenard and Boltzmann equations for a homogeneous one-dimensional system with a pair interaction potential are respectively written as [16]:

$$\frac{\partial}{\partial t} f_1(p_1;t) = \frac{2\pi^2 n}{N} \frac{\partial}{\partial p_1} \int dp_2 \int dk \frac{k^2 \tilde{V}(k)^2}{|\varepsilon(k,kp_1)|^2} \times \delta[k(p_1-p_2)] \left(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) f_1(p_1;t) f_1(p_2;t) \tag{1}$$

and

$$\frac{\partial}{\partial t} f_1(p_1;t) = \frac{1}{N} \int dp_2 |p_1 - p_2| \times [f(p_1';t) f(p_2';t) - f(p_1;t) f(p_2;t)], \tag{2}$$

where p is the one-dimensional momentum variable, n the particle density, $\varepsilon(k,kp_1)$ the dielectric function, and p_1' and p_2' the postcollisional momenta for incoming particles with momenta p_1 and p_2. Setting $\varepsilon(k,kp_1) = 1$ is equivalent to neglegible collective effects and yields the Landau equation. In both cases the right hand is identically zero due to the Dirac δ function in the collisional integral in Eq. (1), while for the Boltzmann equation in Eq. (2) we have $p_1' = p_2$ and $p_2' = p_1$.

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In both cases we obtain $\partial f/\partial t = 0$ if only two-body collisions are considered, and the dominant term comes then from three-body or higher-order terms. This has been considered recently for the Hamiltonian mean-field (HMF) model, resulting in a dynamics of the homogeneous states scaling with N^2 [30], at variance with previous results with scalings proportional to $N^{1.7}$ and $\exp(N)$ which are due to small N effects [31–33]. The $N^{1.7}$ was also reported for a classical anisotropic Heisenberg model by Gupta and Mukamel in Ref. [21] and the question remains if it is due also to small N effects. In this paper we investigate this issue for small and large N. We reobtain a N^2 scaling for large N as predicted from kinetic theory, while noninteger exponents in the scaling are due to finite N effects, as a result of the failure of basic approximations usually considered for the determination of kinetic equation in closed form, as shown below.

The structure of the paper is as follows: In Sec. II we present the model and discuss some of its properties. The scaling of the dynamics of a QSS is determine numerically in Sec. III. We address the limits of applicability of kinetic theory in Sec. III and close the paper with some concluding remarks in Sec. V.

II. THE MODEL

The mean-field classical anisotropic Heisenberg model consists of N classical spins $\vec{S}_i = (S_{ix}, S_{ij}, S_{iz})$, $i = 1,2,\ldots,N$, of unit length globally coupled, with the Hamiltonian [21]:

$$H = -\frac{J}{2N}\sum_{i,j=1}^{N}\vec{S}_i\cdot\vec{S}_j + D\sum_{i=1}^{N}S_{iz}^2. \tag{3}$$

The first term in the right-hand side with $J > 0$ describes a ferromagnetic mean-field coupling and the second term a local anisotropy. Following Gupta and Mukamel, we take $J = 1$ and $D = 15$. Note that the coefficient $1/N$ in the coupling term in the Hamiltonian is a Kac factor that makes the energy extensive. The magnetization of the system is defined by:

$$\vec{m} = \langle\vec{S}\rangle = \frac{1}{N}\sum_{i=1}^{N}\vec{S}_i. \tag{4}$$

Using spherical coordinates the spin components are written as $S_{ix} = \sin(\theta_i)\cos(\phi_i)$, $S_{iy} = \sin(\theta_i)\sin(\phi_i)$, and $S_{iz} = \cos(\theta_i)$, and the equations of motion are given by:

$$\frac{d\vec{S}_i}{dt} = \{\vec{S}_i, H\}, \tag{5}$$

with $i = 1,2,\ldots,N$ and the Poisson bracket:

$$\{A,B\} = \sum_{i=1}^{N}\left\{\frac{\partial A}{\partial\phi_i}\frac{\partial B}{\partial S_{iz}}, \frac{\partial A}{\partial S_{iz}}\frac{\partial B}{\partial\phi_i}\right\}. \tag{6}$$

Thus

$$\begin{aligned}\dot{S}_{ix} &= S_{iy}m_z - S_{iz}m_y - 2DS_{iy}S_{iz},\\ \dot{S}_{iy} &= S_{iz}m_x - S_{ix}m_z + 2DS_{ix}S_{iz},\\ \dot{S}_{iz} &= S_{ix}m_y - S_{iy}m_x.\end{aligned} \tag{7}$$

These equations of motion admit as first integrals the z component m_z of $\vec{m}$, the total energy, and the the length of each spin. This allows us to rewrite the equations of motion as

$$\begin{aligned}\dot{\theta}_i &= m_x\sin(\phi_i) - m_y\cos(\phi_i),\\ \dot{\phi}_i &= m_x\cot(\theta_i)\cos(\phi_i) + m_y\cot(\theta_i)\sin(\phi_i) - m_z + 2D\cos(\theta_i).\end{aligned} \tag{8}$$

As a first result we observe that these equations are canonical and derive from the Hamiltonian:

$$\begin{aligned}H = -\sum_{i=1}^{N}\Big[&m_x\cos(\phi_i)\sqrt{1-S_{iz}^2} + m_y\sin(\phi_i)\sqrt{1-S_{iz}^2}\\ &+ m_zS_{iz} - DS_{iz}^2\Big].\end{aligned} \tag{9}$$

where $p_i \equiv \cos\theta_i = S_{iz}$ and $q_i \equiv \phi_i$ are canonically conjugate and correspond to the momentum and position variables, respectively. As a consequence, the model is effectively one-dimensional and thus explains why a scaling proportional to N^δ with $\delta \neq 1$ is observed. As the model is effectively one dimensional and Hamiltonian, the tools of kinetic theory can be used to derive a kinetic equation, as described, for instance, in Ref. [13]. The first consequence of this fact is that for a homogeneous state in ϕ, the collisional integral proportion to $1/N$ of the Balescu-Lenard equation vanishes, and one must go to the next order in an expansion in powers of $1/N$ (see Ref. [30] and references therein).

III. SCALING OF THE DYNAMICS

In order to study the dynamics of a homogeneous state, and for comparison purposes, we use here the same initial condition as in Ref. [21], a waterbag state (uniform distribution) in the intervals $\phi \in [0,2\pi)$ and $\theta \in [\pi/2 - a, \pi/2 + a]$, with energy per particle:

$$e = \frac{D}{3}\sin^2 a, \tag{10}$$

and a chosen such that $e = 0.24$. The state is spatially homogeneous and stable for this energy. From Ref. [30] the expected scaling of the dynamics of this QSS is N^2. On the other hand, Gupta and Mukamel obtained from numerical simulations a different scaling in $N^{1.7}$. We argue that, similarly to what occurs in the HMF model, the $N^{1.7}$ scaling only occurs for a sufficiently small number of particles, while for larger N the scaling becomes proportional to N^2.

In a homogeneous stable state the spatial distribution for variable ϕ is always uniform up to small fluctuations, but the distribution for variable θ slowly varies with time towards thermodynamic equilibrium [30]. As a consequence, the dynamics can be probed by the statistical moments $M_n = \langle(\theta - \langle\theta\rangle)^n\rangle$. Odd moments of θ vanish for an even distribution in θ as is the case here. Figure 1 shows the second moment M_2 as a function of time. It varies very slowly for the states considered here (it is almost a constant of motion) so we consider the time evolution of the fourth moment M_4, which is more responsive to small changes in the statistical state of the system. In Ref. [21] Gupta and Mukamel considered the average $\langle\cos\theta\rangle$, which is more difficult to characterize the small variations in the distribution function of θ (compare, for instance, Fig. 3 of their paper to our Figs. 1 and 2 below).

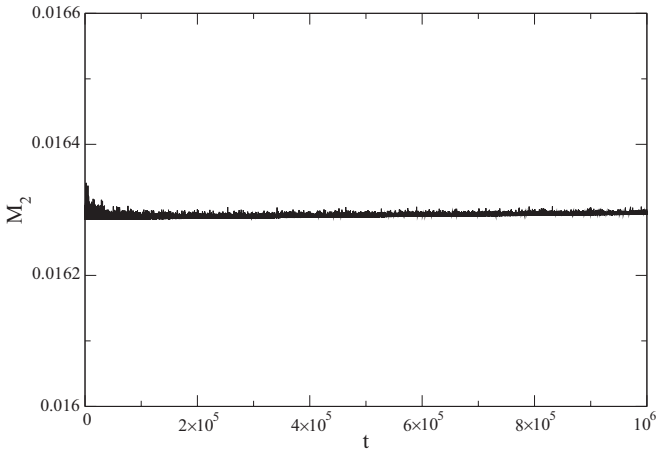

FIG. 1. Second statistical moment M_2 of variable θ for $N = 100\,000$.

The equations of motion in Eq. (8) are solved using a parallel implementation of a fourth-order Runge-Kutta algorithm in a graphics processing unit using the CUDA extension to the C language [34,35]. This allows us to perform simulations with a much greater number of particles than considered in Ref. [21]. The time step used is $\delta t = 0.01$ and ensures a maximum relative error in the energy or order 10^{-4}. Figure 2 shows the time evolution of M_4 for different number of particles up to $N = 100\,000$. Figures 3 and 4 show the same results but with $1/N^{1.7}$ and $1/N^2$ time rescalings, respectively. A better data collapse is obtained for the N^2 scaling.

In order to compare quantitatively ours with previous results, we performed a series of simulations with the same number of particles as in Ref. [21] but also considering values of N up to 60 000. By averaging over many realizations we compare the time evolution of M_4 for a given value of N with the previous smaller number of N in the simulations and perform a least-squares fit for the difference between both time series rescaled by $1/N^{\delta}$. The results are shown in Table I and corroborate, up to some small deviations, a scaling in N^2. Figure 5 shows the statistical moment M_4 for the same number

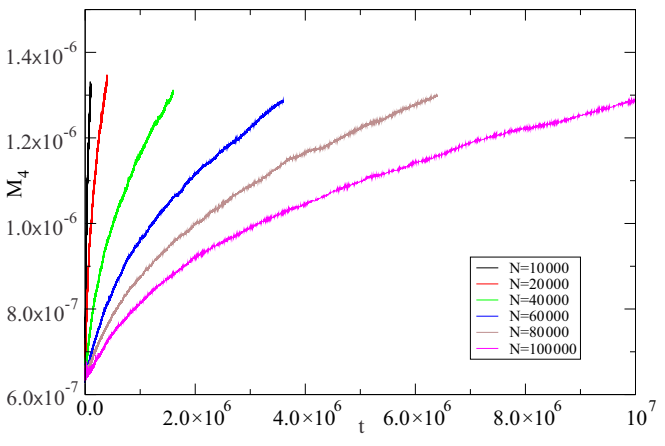

FIG. 2. (Color online) Moment $\langle M_4 \rangle$ of variable θ_i as a function of time for different numbers of particles $N = 10\,000$, 20 000, 40 000, 60 000, 80 000, 100 000.

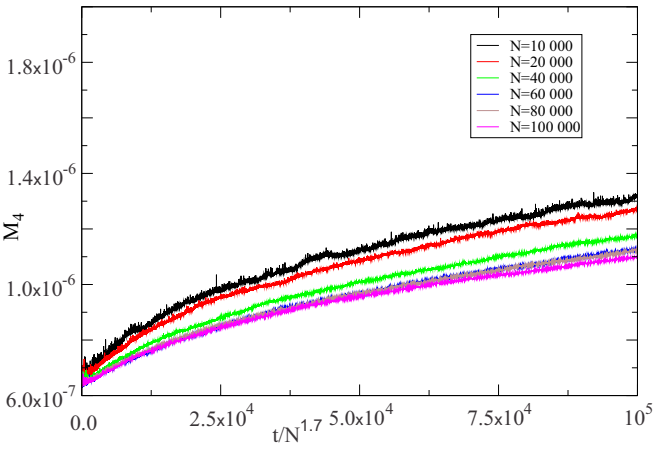

FIG. 3. (Color online) Same as in Fig. 2 but with a time rescaled by $(N - 10\,000)^{-1,7}$.

of particles as in Table I with time scaled as $1/N^2$ with a very good data collapse for $N \geqslant 5000$.

We note that, in Ref. [21], Gupta and Mukamel determined the scaling behavior considering the values $N = 300, 1000, 3000, 5000$. The difference of theirs and our results for the case $N = 3000$ and 5000 comes from the fact that considering the magnetization as a relevant variable yields more imprecise results than when considering the statistical moments of the momenta variables (see also the discussion of this in Ref. [35]).

IV. LIMITATIONS OF KINETIC THEORY

Our results are in agreement with what is expected from a kinetic theory derived from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in a series expansion in power of $1/N$. Two- and three-particle correlation functions contribute with terms proportional to $1/N$ and $1/N^2$, respectively. As two-particle contributions to the kinetic equation vanish in the present case, one must retain the contributions from three-particle collisions which are proportional to $1/N^2$.

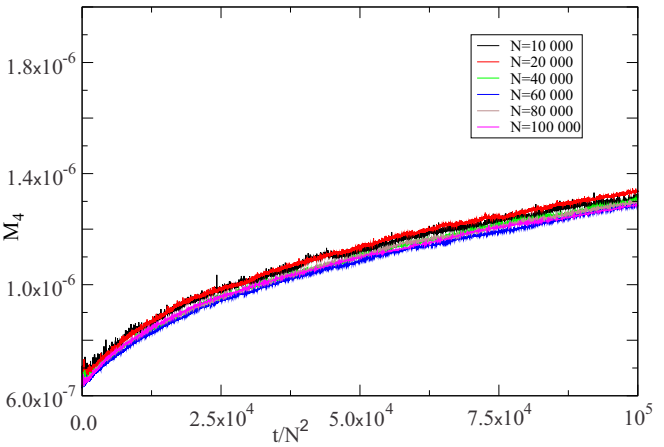

FIG. 4. (Color online) Same as in Fig. 2 but with a time rescaled by $(N - 10\,000)^{-2}$.

TABLE I. Best scaling in N^{δ} for the moment M_4 between a pair of simulated data with N_1 and N_2 particles.

N_1	N_2	δ
300	1000	1.767
1000	3000	1.797
3000	5000	2.015
5000	10 000	2.056
10 000	20 000	2.072
20 000	40 000	2.066
40 000	60 000	2.096

These considerations are based on the introduction of the Kac factor in the Hamiltonian and the scaling proportional to $N^{-1.7}$ reported by Gupta and Mukamel is reobtained here for smaller values of N. This unusual scaling stems on the failure for small N of the Markovianization hypothesis used in the determination of the Balescu-Lenard and Landau equations, which requires that the force autocorrelation function (for homogeneous systems) differs significantly from zero only for very short times if compared to the dynamical time scale over which the one-particle distribution function varies significantly. Let us show this explicitly for the simpler case of the Landau equation, i.e., for weak coupling, as the same kind of approximations are used in the deduction of the Balescu-Lenard equation (see Ref. [13] for a thorough discussion on these assumptions).

The N-particle distribution function $f_N(\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_N,\mathbf{v}_N;t)$ is the probability density in the N-particle phase space for a particle at time t to have position $\mathbf{r}_i$ and momentum $\mathbf{p}_i$. We define the s-particle distribution function by

$$f_s \equiv f_s(\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_s,\mathbf{v}_s;t) = \int d\mathbf{r}_{s+1}d\mathbf{v}_{s+1}\cdots d\mathbf{r}_N d\mathbf{v}_N\ f_N(\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_N,\mathbf{v}_N;t), \tag{11}$$

where $\mathbf{r}_i$ and $\mathbf{p}_i$ are the position and momentum vectors of particle i in d dimensions. The Liouville equation then implies

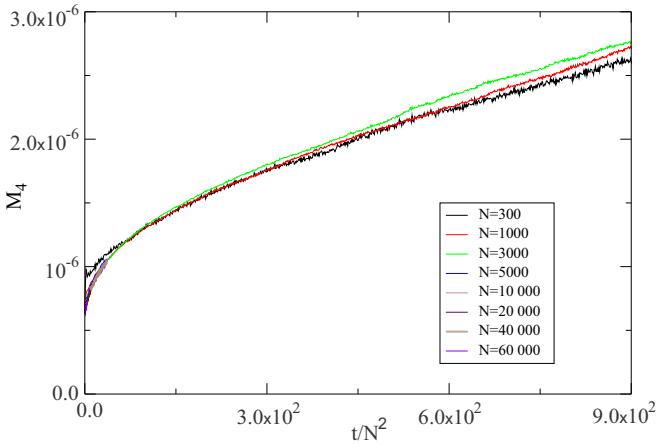

FIG. 5. (Color online) Moment M_4 of the θ_i as a function of time for $N = 300,\ 1000,\ 3000,\ 5000,\ 10\,000,\ 20\,000,\ 40\,0000,\ 60\,000$ with a time rescaling $(N-300)^{-2}$. The number of realization varies from 300 for $N = 300$ to 25 for $N = 60\,000$.

that the reduced distribution functions satisfy the BBGKY hierarchy [13,16]:

$$\frac{\partial}{\partial t}f_s = \sum_{j=1}^{s}\hat{K}_j f_s + \frac{1}{N}\sum_{j<k=1}^{s}\hat{\Theta}_{jk}f_s + \frac{N-s}{N}\sum_{j=1}^{s}\int d\mathbf{r}_{s+1}d\mathbf{v}_{s+1}\hat{\Theta}_{j,s+1}f_{s+1}, \tag{12}$$

where

$$\hat{K}_j = -\mathbf{v}_j\cdot\nabla_j, \qquad \hat{\Theta}_{jk} = -\nabla_j V(\mathbf{r}_j-\mathbf{r}_k)\partial_{jk},$$
$$\partial_{jk} \equiv \frac{\partial}{\partial\mathbf{v}_j} - \frac{\partial}{\partial\mathbf{v}_k}, \tag{13}$$

and ∇_j is the gradient operator for the position of particle j. In order to obtain a closed kinetic equation for the one-particle distribution function f_1 we have to determine the functional dependence of f_2 on f_1 (Bogoliubov hypothesis [16]). This can be accomplished in the present framework by writing the reduced distribution functions in the form of a cluster expansion, which for a homogeneous system is given by:

$$f_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r}_1-\mathbf{r}_2) = f_1(\mathbf{v}_1)f_1(\mathbf{v}_2) + C_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r}_1-\mathbf{r}_2), \tag{14}$$

$$f_3(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{r}_1-\mathbf{r}_2,\mathbf{r}_2-\mathbf{r}_3) = f_1(\mathbf{v}_1)f_1(\mathbf{v}_2)f_1(\mathbf{v}_3) + \sum_{P(1,2,3)} f_1(\mathbf{v}_1)C_2(\mathbf{v}_2,\mathbf{v}_3,\mathbf{r}_2-\mathbf{r}_3) + C_3(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{r}_1-\mathbf{r}_2,\mathbf{r}_2-\mathbf{r}_3), \tag{15}$$

and so on, where the time dependence is kept implicit; $P(1,2,3)$ stands for permutations of particles 1, 2, and 3; and C_s is the s-particle correlation function. Let us consider a parameter $\lambda \ll 1$ characterizing the strength of the interaction, i.e., $V = \mathcal{O}(\lambda)$. A two-particle correlation requires the interaction of two particles to be created and therefore C_2 is of order λ. A three-particle correlation requires the interaction between two pairs of particles and thus C_3 is of order λ^2, and so on. By considering the case $s = 1$ in Eq. (12) and using Eq. (14) we have:

$$\frac{\partial}{\partial t}f_1(\mathbf{v}_1;t) = \frac{N-1}{N}\int d\mathbf{v}_2 d\mathbf{r}_2\ \hat{\Theta}_{12}[f_1(\mathbf{v}_1;t)f_1(\mathbf{v}_2;t) + C_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r}_1-\mathbf{r}_2;t)]. \tag{16}$$

The two-particle correlation function is the solution of the equation obtained by replacing Eq. (15) into Eq. (12) for $s = 2$ and discarding higher-order terms containing three-particle correlations:

$$\left(\frac{\partial}{\partial t} - \hat{K}_1 - \hat{K}_2\right) C_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{r}_1 - \mathbf{r}_2; t) = \hat{\Theta}_{12} f_1(\mathbf{v}_1; t) f_1(\mathbf{v}_2; t). \tag{17}$$

Its solution can be written as:

$$\begin{aligned} &C_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{r}_1 - \mathbf{r}_2; t) \\ &\quad = e^{(\hat{K}_1 + \hat{K}_2)t} C_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{r}_1 - \mathbf{r}_2; 0) \\ &\qquad + \int_0^t dt\, e^{(\hat{K}_1 + \hat{K}_2)\tau} \hat{\Theta}_{12} f_1(\mathbf{v}_1; t - \tau) f_1(\mathbf{v}_2; t - \tau). \end{aligned} \tag{18}$$

The first term in the right-hand side of Eq. (18) is a transient term due to correlation at $t = 0$ and dies out rapidly [13]. By replacing Eq. (18) into Eq. (16) and noting that the mean-field force vanishes in a homogeneous state, we obtain (using $N - s \to N$ for large N):

$$\begin{aligned} &\frac{\partial}{\partial t} f_1(\mathbf{v}_1; t) \\ &\quad = \int_0^t dt\tau \int d\mathbf{v}_2 d\mathbf{r_2}\, \hat{\Theta}_{12} e^{(\hat{K}_1 + \hat{K}_2)\tau} \hat{\Theta}_{12} f_1(\mathbf{v}_1; t - \tau) \\ &\qquad \times f_1(\mathbf{v}_2; t - \tau) \\ &\quad = \int_0^t d\tau \int d\mathbf{v}_2 d\mathbf{r_2}\, \partial_{12} \nabla_1 V(\mathbf{r}_{12}) e^{(\hat{K}_1 + \hat{K}_2)\tau} \nabla_1 V(\mathbf{r}_{12}) \partial_{12} \\ &\qquad \times f_1(\mathbf{v}_1; t - \tau) f_1(\mathbf{v}_2; t - \tau) \\ &\quad = \int_0^t d\tau \int d\mathbf{v}_2 d\mathbf{r_2}\, \partial_{12} \nabla_1 V(\mathbf{r}_{12}) \nabla_1 V(\mathbf{r}_{12} - \mathbf{v}_{12}\tau) \partial_{12} \\ &\qquad \times f_1(\mathbf{v}_1; t - \tau) f_1(\mathbf{v}_2; t - \tau) \end{aligned} \tag{19}$$

with $\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$. The force autocorrelation of the $\mathbf{F}(\mathbf{r}, t)$ at position $\mathbf{r}$ is defined by

$$\begin{aligned} \mathcal{C}(t) \equiv \langle F(t) F(0) \rangle &= \int d\mathbf{r}\, \mathbf{F}(\mathbf{r}, 0) \mathbf{F}(\mathbf{r}, t) \\ &= \int d\mathbf{r}\, \nabla V(\mathbf{r} - \mathbf{v}_{12} t)\, \nabla V(\mathbf{r}). \end{aligned} \tag{20}$$

Thence we have:

$$\begin{aligned} \frac{\partial}{\partial t} f_1(\mathbf{v}_1; t) = \int_0^t d\tau \int d\mathbf{v}_2\, \partial_{12} \langle F(\tau) F(0) \rangle\, \partial_{12} f_1(\mathbf{v}_1; t - \tau) \\ \times f_1(\mathbf{v}_2; t - \tau). \end{aligned} \tag{21}$$

This is a master equation which is non-Markovian as it depends on f_1 at previous times form 0 to t. To obtain a true (Markovian) kinetic equation the usual procedure is to assume that the dynamic time scale t_d over which the one-particle distribution function f_1 varies significantly is much greater than the time scale t_c such that the force autocorrelation is sufficiently small. In this case, one can replace $f_1(\mathbf{v}; t - \tau)$ in the integrand in Eq. (21) by $f_1(\mathbf{v}_1; t)$, which corresponds to the ballistic approximation (free motion for a homogeneous system), and extend the time integration to infinity. We then finally obtain the Landau equation:

$$\begin{aligned} \frac{\partial}{\partial t} f_1(\mathbf{v}_1; t) = \int_0^\infty d\tau \int d\mathbf{v}_2\, \partial_{12} \langle F(\tau) F(0) \rangle\, \partial_{12} \\ \times f_1(\mathbf{v}_1; t) f_1(\mathbf{v}_2; t). \end{aligned} \tag{22}$$

This form will suffice for the present discussion. The same type of considerations are also necessary in the determination of the Balescu-Lenard equation [13,15]. As discussed above, for a one-dimensional homogeneous system these corrections vanish and one must go one order further in the $1/N$ expansion. Usually one always considers a Markovianization procedure taking into account the time scales such that $t_d \gg t_c$. A failure of this condition implies, among other things, that the collisional integral does not vanish exactly for a one-dimensional homogeneous system, and one should expect that the scaling of the dynamics is therefore affected.

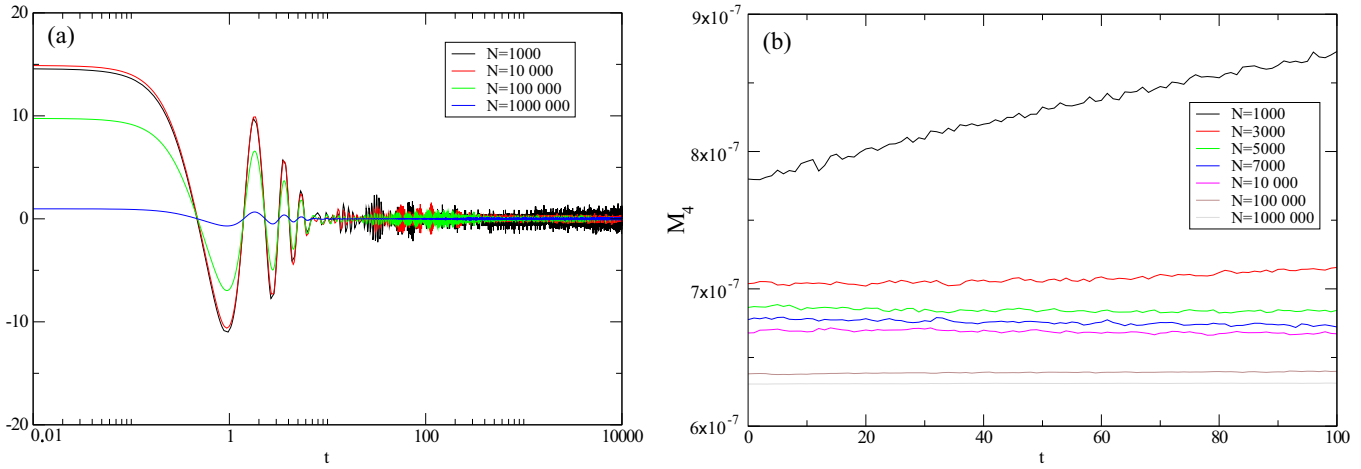

FIG. 6. (Color online) (a) Force autocorrelation $\mathcal{C}(t)$ as a function of time for different values of N. (b) Time evolution for the fourth moment $\langle M_4 \rangle$ of variable θ averaged over 1000 realizations except for $N = 100\,000$ and $N = 1\,000\,000$ with 300 and 200 realizations, respectively. The initial conditions are the same homogeneous state as in Fig. 2, thermalized up to $t = 20.0$ before starting the simulations shown here.

In order do address this point, we compute the force autocorrelation function from numeric simulations by:

$$\mathcal{C}(t) = \frac{1}{N}\sum_{i=1}^{N} F_i(t)F_i(0), \tag{23}$$

where $F_i(t)$ is the force on particle i at time t due to all other particles. Figure 6(a) shows $\mathcal{C}(t)$ for different numbers of particles for the present model and the time evolution of $\langle M_4\rangle$ for variable θ. We observe that the time required for a significant decrease of $\mathcal{C}(t)$, i.e., the correlation time t_c, is roughly the same for all values of N, while the dynamical time t_d is smaller for smaller N as shown in the right panel of Fig. 6. In this way, the correlation time can become of the same order of magnitude as the dynamical time, breaking down the Markovian condition, and therefore the usual derivation of Kinetic equations from the BBGKY hierarchy is no longer valid. Figure 6(b) shows the fourth moment M_4 of variable θ and it becomes evident that Markovianity is not valid for $N = 1000$ and $N = 3000$, while it is approximately valid for $N = 5000$. For $N \geqslant 10\,000$ the system is clearly Markovian, in agreement with the results in Table I. This explains why a different scaling in N^{δ} of the dynamics with $\delta \neq 2$ is observed for homogeneous one-dimensional systems for small N [30].

V. CONCLUDING REMARKS

We have shown in this paper that the mean-field anisotropic Heisenberg model introduced by Gupta and Mukamel in Ref. [21] is effectively a one-dimensional classical Hamiltonian system, and the dynamics of a QSS scales as N^2 for large N while the scaling in $N^{1.7}$ previously reported is due to small N non-Markovian effects in the dynamics. For large N a kinetic equation for a homogeneous one-dimensional long-range interacting system must consider three-body collisions, which are of order $1/N^2$. This approach is only valid if N is sufficiently large such that the contribution of strictly two-body collisions does vanish, while for small N the arguments leading to the N^2 scaling fail. The small N case can be tackled using an approach developed by Ettoumi and Firpo by determining the diffusion coefficient in terms of action variables who used a mean passage time approach [36] and obtained a $N^{1.7}$ scaling for the Hamiltonian mean-field model [37]. Based on time evolution of the autocorrelation of the force for the homogeneous case, one can consider if a similar behavior occurs for nonhomogeneous one-dimensional and for higher-dimensional systems and whether and how it influences the scaling for small N. This will be the subject of a separate publication. This also raises the question regarding whether similar effects might have a role in astrophysics. Indeed, smaller globular clusters can have a number of stars of the order of just a few thousand, as opposed to 10^{10} stars in a typical galaxy. Other long-range interacting systems may also have a similar behavior. More recently, Gupta and Mukamel introduced a different model of classical spins in a sphere described by a two-dimensional Hamiltonian [38] and also displaying a scaling of the dynamics of a homogeneous QSS in $N^{1.7}$. Taking into account the discussion in the present paper and in Ref. [30], this is a strong indication that this model is in fact effectively one dimensional, which is still to be shown.

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