Class of perfect $1/f$ noise and the low-frequency cutoff paradox

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The low-frequency cutoff paradox occurring in $1/f$ processes has been revisited in a recent Letter [M. Niemann, H. Kantz, and E. Barkai, Phys. Rev. Lett. **110**, 140603 (2013)]. A model of independent pulses exhibiting an integrable $1/f^{\beta}$ power spectrum with $\beta > 1$ explains this paradox. In this paper we explore a complementary possibility based on the use of multiplicative models to generate integrable $1/f^{\beta}$ processes. Three distinct types of models are considered. One of the most used methods of generating $1/f$ processes based on correlated pulses is among these models. Consequently we find that, contrary to what is generally thought, the low-frequency cutoff is not necessary to avoid the postulated divergence in a wide variety of processes.

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I. INTRODUCTION

Parceval's identity states that the mean-squared fluctuation of a signal coincides with its total power spectrum. This implies that any physical process with finite mean-squared fluctuations is limited to having a $1/f^{\beta}$ spectrum, either with $\beta < 1$, or with $\beta \geqslant 1$ but with a cutoff at low frequencies. The so-called low-frequency cutoff paradox appears because there are many experiments, mainly with electronic devices [1], whose spectral density exhibits a $1/f^{\beta}$ shape with $\beta \geqslant 1$ but without any apparent cutoff, even at very low frequencies. In a recent Letter [2] a solution to this paradox has been postulated by including the possibility of having spectra whose amplitudes vary with the size of the time series as $A(T)/f^{\beta}$. A simple model of dichotomous noise with renewal times serves to support this possibility. The model is suggested by recent laser experiments where an intermittent behavior is detected.

In [3] the possibility of dealing with processes whose spectral power varies with the size of time series is considered in all detail by introducing an independent exponent η measuring this variation by using $A(T) \sim T^{\eta}$. In this work ensembles of time series $\{Y_T(t)\}$ with variable size T and size independent initial conditions are analyzed. The size of time series is an important variable that leads to the possibility of double scaling in statistical functions as correlation or spectral densities. The fact of having a double scaling in the spectrum is well known in the context of surface growth where the global or local character of the scaling is taken into account [4]. Using the same reasoning one can consider the local scaling by means of the so-called spectral exponent α_s which is directly related to β as $\beta = 2\alpha_s + 1$. Note that α_s coincides with the Hurst exponent when $\alpha_s \in (0,1/2)$ [3]. The global scaling is directly assumed by the scaling of the mean-squared fluctuations $\delta_T^2 = \overline{(Y_T(t) - \overline{Y_T(t)})^2}$ with the size as $\langle \delta_T^2 \rangle \sim T^{2\alpha}$. So, the condition of finite mean-squared fluctuations can be established, in terms of this exponent, as $\alpha \leqslant 0$. In our notation an overline $\overline{\#}$ means time averaging, whereas brackets $\langle \# \rangle$ indicate sample averaging. The scaling is better observed in sample averaged functions. The spectral density of each sample, defined as $S_T(f) = \frac{1}{T}\widetilde{Y_T}(f)\widetilde{Y_T}(-f)$ where $\widetilde{Y_T}(f)$ is the Fourier transform, is a very rough function. It is necessary to have a large enough number of samples to get an averaged spectrum $S(f,T) = \langle S_T(f) \rangle$ where the scaling is clearly observed. This scaling can be written in terms of the local and global exponents, by using the Parseval's identity in the form $\delta_T^2 \sim \int_{1/T}^{1/\Delta} S_T(f)df$ with $\Delta = \text{const}$ and $T \to \infty$, as

$$S(f,T) \sim \begin{cases} \frac{T^{2(\alpha-\alpha_s)}}{f^{2\alpha_s+1}} & \text{if } \alpha_s \geqslant 0, \quad \alpha \neq 0, \\ \frac{(\log(T))^{-1}}{f} & \text{if } \alpha_s = 0, \quad \alpha = 0, \\ \frac{T^{2\alpha}}{f^{2\alpha_s+1}} & \text{if } \alpha_s < 0. \end{cases} \quad (1)$$

This scaling clarifies the original suggestion of Mandelbrot [5] stating that a fractal curve ($\alpha_s > 0$) cannot be a strictly stationary process, which would imply the simultaneous conditions of finite variance ($\alpha \leqslant 0$) and a correlation only dependent on time differences ($\eta = 0$). Both conditions only hold for the class of stationary noise (SN) with $\alpha = 0$, $\alpha_s < 0$ [3]. But, however, for $\alpha_s \geqslant 0$ there exists the possibility of being stationary in a weak sense ($\alpha = 0$) as in the case of the model introduced in [2], which in [3] is generically called the class of stationary (weak sense) fractal curves (SF). Obviously, this class shows integrable spectra ($\alpha = 0$) and explains the cutoff paradox, although the spectra shift with size as $A(T) \sim T^{-2\alpha_s}$. In this paper we will show the existence of a wide variety of processes exhibiting integrable spectra ($\alpha \leqslant 0$) in the range of interest $\alpha_s \geqslant 0$. As in the previous case they are good candidates to explain the cutoff paradox, since cutoff at low frequencies is not necessary for the total power spectrum to be finite. This wide variety arises because, as we will show, it is enough to consider models derived from multiplicative stochastic equations with exponent $\mu > 1$. They are complementary to the model presented in [2]. In this sense it would be of interest from an experimental point of view to distinguish between both types of processes. We propose here three kinds of models: multiplicative stochastic equations, correlated pulses, and intermittent maps. For the sake of illustration we take the case of $\alpha_s = 0$ (called 1F class in [3]) as more representative throughout the whole range of parameters. Furthermore, as a guide for experimentalists, we also study techniques to identify the multiplicative character of the time series generated by these processes.

II. MODELS

Let us consider distinct models to generate processes with $1/f^{\beta}$ integrable spectra with $\beta \geqslant 1$. Our aim is not to be exhaustive but to show a significant variety of models that admit simple numerical realizations.

Renewal pulses. Series with $1/f^{\beta}$ spectra can be generated by using renewal point processes [6] although for $\beta > 1$ only the stationary case (with a cutoff in the inter-pulse generation) was considered. Here we take the same model used in [3], which is a generalization of the one used in [2]. It consists of a series of pulses with a renewal process for the interpulse time $\{\tau_i\}$ whose time probability follows a power law, $P(\tau) = (d-1)\tau^{-d}$, with $1 < d < 2$, $\tau \in (1,\infty)$, without cutoff at large times. The shape of pulses is irrelevant while pulse accumulation is insignificant. This happens when the pulse width is similar to its interpulse time. Simulations are done with exponential pulses in the form of $Y(t) = \sum_i y_i \exp(-\frac{|t-t_i|}{\tau_i})$, where $t_i = \sum_{j=1}^{i} \tau_j$. In these conditions the value of exponents is [3] $\alpha = 0$, $\alpha_s = 1 - d/2$. In order to study symmetry properties we take random amplitudes with uniform probability, $P(y) = \text{const}$, in the interval $(-b,1-b)$. With $b = 0$ the generated series is completely asymmetric, while for $b = 0.5$ it is symmetric. In previous works this model was limited to generating series with exponent $\alpha \in (0,1/2)$ forgetting the interesting case $\alpha = 0$, which is easily generated taking $d = 2$. As we will show, this case is especially important since it is a limit of all multiplicative processes. Simulations with $d = 2$ are presented along the paper with the label R1. For the sake of comparison the case with $d = 1.5$ is also shown (R2).

Multiplicative stochastic equations. The use of equations with multiplicative white noise as generators of $1/f^{\beta}$ noise is a well-known fact that has always been treated in its stationary version. That is, as the model is not stationary, some kind of cutoff is introduced to reach a stationary state [7,8]. Furthermore, the necessity of this cutoff is justified to keep an integrable spectral density. An important result that follows from these works is the possibility of relating parameters of the Langevin equation

$$\dot{Y}(t) = (\mu - \epsilon\kappa)Y^{(2\mu-1)} + \xi(t)Y^{\mu}, \qquad (2)$$

where $\epsilon = 1/2$ in the Ito prescription ($\epsilon = 1$ in the Stratonovich prescription) and $\xi(t)$ is a Gaussian white noise, with κ the exponent of the stationary probability distribution $P_{st}(Y) \sim Y^{-\kappa}$, α_s the spectral exponent, and ν a dynamical exponent. This last exponent comes from the scaling of the conditional probability [8]:

$$W(Y,\lambda t|Y',\lambda t') \sim \lambda^{-\nu} W(\lambda^{-\nu}Y,t|\lambda^{-\nu}Y',t'). \qquad (3)$$

Hence, in this version, this kind of model cannot be considered as a candidate to explain the long correlation observed in experiments. But when studying the case of free evolution from a given initial condition [3] one obtains a different conclusion. On the one hand, the global exponent α, that can be easily estimated by using a simple scaling counting in the Langevin equation, coincides with ν. On the other hand, the scaling of the stationary probability density can be substituted by $P(Y,t) \sim Y^{-\kappa}t^{-1}$. The resulting value of the exponents is $\alpha = \frac{1}{2(1-\mu)}$ and $\alpha_s = \frac{3-\kappa}{2}\alpha$. Here two facts are worth remarking. One is that simply with a multiplicative exponent $\mu > 1$ we obtain processes with $\alpha < 0$, that exhibit integrable $1/f^{\beta}$ spectra without necessity of cutoff. Another is that the case with $\alpha = 0$ is not attainable from a multiplicative process since it implies an infinite multiplicative exponent $\mu = \infty$. In this sense we say that multiplicative models are complementary to the renewal process. Although these results are rather general for multiplicative processes one can even achieve more generality by considering stationary correlated noise, $\xi_c(t)$ in the Langevin equation. $\xi_c(t)$ is a noise in the SN class, with $\alpha = 0$, $\alpha_s \in (-1,0)$ that is here numerically implemented as the derivative of a fractional Brownian motion (FBM) process of exponent (Hurst) $1 + c$ [3]. Now the exponent of the probability density κ' is not easily related with parameters of the Langevin equation, and the exponent α changes as a result of the new scaling of the noise $\xi_c(\lambda t) \sim \lambda^c \xi_c(t)$. Then a more general value of exponents $\alpha = \frac{1+c}{1-\mu}$ and $\alpha_s = \frac{3-\kappa'}{2}\alpha$ is obtained. Note that in the limit of noise memory $c \to -1$ and for any multiplicative process the exponents of the generated series tends to the value $\alpha_s = \alpha = 0$. In general, simulations of multiplicative equations with $\mu > 1$ are not trivial, since special techniques with varying time intervals are required [7], but there is a case where simulations are easily performed even with correlated noise. It consists of taking the q power, $Z(t) = |Y(t)|^q$, of a FBM process whose Langevin equation, $\dot{Y}(t) = \xi_c(t)$, is a multiplicative equation with $\mu = 0$. In the Stratonovich prescription, using the standard rules of calculus and reflective boundary conditions at $Z = 0$, we get the Langevin equation

$$\dot{Z}(t) = qZ^{1-1/q}\xi_c(t), \qquad (4)$$

that is, we obtain another multiplicative equation with a power exponent $\mu' = 1 - \frac{1}{q}$, and hence $\alpha = (1+c)q$. On the other hand, from simulations one sees that the exponent of the probability density holds, $\kappa' = \kappa = 1 - 1/q$ and consequently $\alpha_s = (q + 1/2)(1 + c)$. Then a band of pure $1/f$ processes in the 1F class can be generated taking $q = -1/2$ and varying $c \in (-1,0)$. For the sake of illustration we show in the figures two examples, one close to the stationary case ($c = -0.9, \alpha = -0.1$) labeled as M1 and the other with the standard white noise (M2, $c = -0.5, \alpha = -0.25$) which is clearly not stationary. To generate FBM processes with a normal initial distribution we use Levinson's algorithm implemented in Mathwork Matlab.

Correlated pulses. A standard method of generating $1/f^{\beta}$ processes consists of taking a series of pulses whose interpulse time process $\{\tau_i\}$ is correlated. The best-known case is when the time process is a random walk, $\tau_{i+1} - \tau_i = \xi(i)$, in which case the spectrum is a pure $1/f$ process [9]. In [3] the value of α was calculated showing that the series behaves asymptotically as a multiplicative process. The obtained value, $\alpha = -1/3$, which is corroborated numerically, indicates that the integrated spectrum is finite, the cutoff unnecessary, and the model generates a $1/f$ process without paradoxes. In this paper, as in the previous model, the effect of memory is investigated taking for the interpulse series an FBM process such as $\tau_{i+1} - \tau_i = \xi_c(i)$. The value of α can be calculated easily, as in the case without memory, obtaining the asymptotic multiplicative process. Note that the shape of pulses is irrelevant, so we can take a flat pulse with $Y \sim \tau^{-1}$, and that $\frac{d\tau}{dt} = \frac{1}{\tau^{1+c}}\xi_c(t)$, thus obtaining for the process Y a multiplicative equation such as

$$\dot{Y} \sim Y^{3+c}\xi_c(t). \qquad (5)$$

The value of α, obtained by using a simple power counting in this equation, is $\alpha = \frac{1+c}{2+c}$. As in the previous case the exponent $\kappa' = \kappa$ and $\alpha_s = 0$. Simulations presented in the figures are a series of exponential (E) and flat (F) pulses in the limit of noise memory with $c = -0.9$ (E1, F1) and in the white noise case $c = -0.5$ (E2).

Intermittent maps. The renewal model introduced in [2] was suggested by recent experiments where a kind of intermittent behavior was observed. Intermittent behavior can be easily modeled by chaotic maps [10]. Usually parameters of the map are fitted to have stationary behavior [11] and the generated time series belong to the SN class ($\alpha_s < 0, \alpha = 0$). But in general maps of this kind exhibit any possible dynamics. For the sake of illustration we have considered here a simple chaotic map given by

$$Y \to Y + Y^{\theta+1} \qquad \text{mod } 1. \tag{6}$$

The form of this map suggests a multiplicative process with exponent θ and complete dependent noise $\xi(t) = Y$. Simulations indicate that for $\theta \geqslant 3$ the generated time series are in our range of exponents $\alpha_s \geqslant 0$, $\alpha < 0$. We take random initial conditions with uniform probability in the interval (0.4,0.5). Representing this model we show the case $\theta = 3$ which generates pure $1/f$ noise with $\alpha = -1/3$. It is labeled as IM.

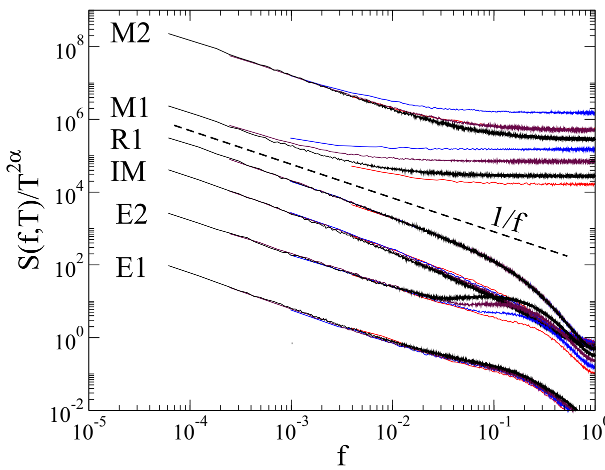

FIG. 1. (Color online) Plot of scaled spectra of $1/f$ processes ($\alpha_s = 0$) with sizes $T = 2^9$ (red), 2^{11} (blue), 2^{13} (brown), and 2^{15} (black). The values of α to obtain the best data collapse, shown in the figure, are -0.1 (E1), -0.3 (E2), -0.33 (IM), 0. (R1), -0.1 (M1), and -0.25 (M2). R1 is scaled in a logarithmic form as $A(T) \sim (\log(T))^{-1}$. Spectra of each model are shifted upwards by one decade with respect to the previous model. The number of samples used in the spectral average is 1000.

III. COMPLETE SPECTRAL ANALYSIS

In a first step a complete spectral analysis [3] is performed for the above introduced models to check the validity of the scaling theory. For the sake of illustration we have chosen the most representative case which is the pure $1/f$ noise ($\alpha_s = 0$). For the models with pulses only the complete asymmetric case ($b = 0$) is considered here. When $\alpha_s \geqslant 0$ the exponents α and α_s can be obtained from a log-log representation of $S(f,T)T^{-2(\alpha'-\alpha_s)}$ against f, as given by the scaling (1). The exponent α_s is obtained from the slope of each spectrum, which is $2\alpha_s + 1$. Then the exponent α' is varied until the best collapse of the spectral data with distinct sizes is found. This defines the true exponent α. We can see in Fig. 1 that the data collapse for the asymptotic low frequencies is almost perfect in all cases and the obtained exponents agree with the estimations. Note that the data collapse of spectra for $\alpha_s = \alpha = 0$ (R1) is in full agreement with the spectral scaling function where the shift of spectra follows a logarithmic variation with T (1). In Fig. 2 the probability density $P(Y,T)$ calculated as a normalized histogram of the values of series of the past ten times is shown for series of size $T = 2^{13}$ and 100 000 samples. The asymptotic behavior $P(Y,T) \sim \frac{1}{Y^3}$, corresponding to a $1/f$ process of multiplicative origin, is clearly observed in the multiplicative, correlated flat pulses and intermittent map models. As expected, the exponential decay of $P(Y,T)$, which is a sign of an additive process, is observed in the renewal pulse process but also in the case of correlated exponential pulses, which possess a multiplicative character. Hence not any process coming from a multiplicative process exhibits a power law in its probability density. In order to facilitate the inspection of figures a table (see Table I) with the main information of all simulated series is included in the paper.

IV. ANALYSIS OF SPECTRAL FLUCTUATIONS

The spectrum of a given sample $S_T(f)$ exhibits characteristic fluctuations that can be analyzed to give supplementary information useful in model identification. In our case we are interested in the identification of additive and multiplicative models. A first step in the analysis of $S_T(f)$ as a frequency series consists of transforming the nonuniform series into a uniform one. We can see that the series $Z_T = S_T(f) f^{2\alpha_s+1}$

TABLE I. Simulated time series.

Label	Symbol	Model	Parameters	α_s	α
R1	Triangle up	Renewal pulses	$d = 2$	0	0
R2	Line	Renewal pulses	$d = 1.5$	0.25	0
M1	Cross	Multi. stochastic eq.	$q = -1/2, c = -0.9$	0	-0.1
M2	Circle	Multi. stochastic eq.	$q = -1/2, c = -0.5$	0	-0.25
E1	Triangle down	Correlated pulses (Expt.)	$c = -0.9$	0	-0.1
E2	Diamond	Correlated pulses (Expt.)	$c = -0.5$	0	$-1/3$
F1	Star	Correlated pulses (Flat)	$c = -0.9$	0	-0.1
IM	Square	Intermittent map	$\theta = 3$	0	$-1/3$

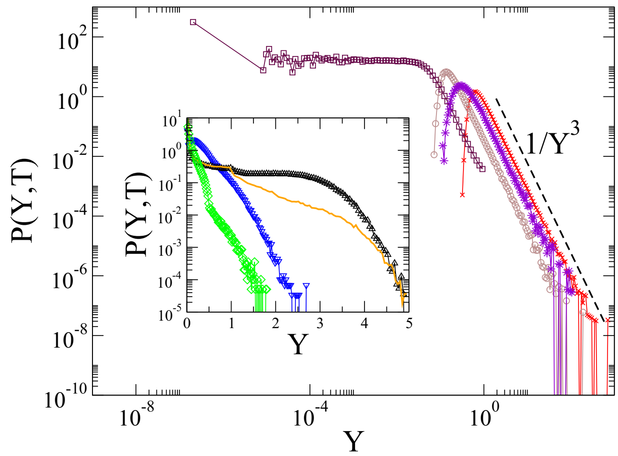

FIG. 2. (Color online) Probability density of Y at the end of the time interval for samples of size $T = 2^{13}$ and using 100 000 samples. Symbols and colors used for the distinct models are R1 (triangle up, black), E1 (triangle down, blue), E2 (diamond, green), IM (square, maroon), R2 (line, orange), M1 (cross, red), M2 (circle, brown), and F1 (star, violet). Note that the main figure is in log-log axes, while the inset is semilog.

is uniform in the sense that the sampling average, $\langle Z_T \rangle$, and variance, $\langle (Z_T - \langle Z_T \rangle)^2 \rangle$, are almost constant in frequency. Then we can treat this uniform series of frequencies, $\{Z_T\}$, as if they were time series $\{Y_T\}$. Correlations can be investigated with the spectral densities $\{S_{ZT}(f)\}$. Defining the amplitude of each spectrum $S_T(f)$ as $a_T = \overline{Z_T}$ we can investigate the statistics of the internal spectral fluctuations $\{S_T(f) - a_T/f^{2\alpha_s+1}\}$ and those of the amplitudes themselves $\{a_T\}$. In general, we have observed with our models two kinds of results associated with the multiplicative or additive character of the generated series. A pure multiplicative process exhibits a strong correlation in Z_T, with an averaged spectral density $S_Z(f,T) \sim 1/f^2$, whereas additive processes have

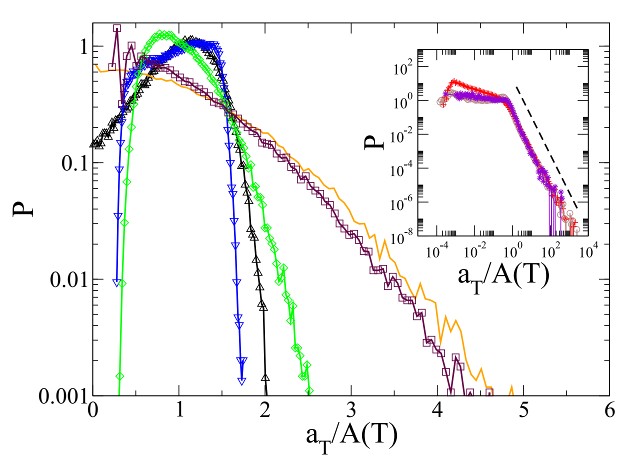

FIG. 3. (Color online) Probability density of the relative amplitude $a_t/A(T)$ of spectra corresponding to the distinct models. Symbols and colors are the same as in the previous figure. Here the main figure is on semilog axis, while the inset is log-log. The dashed line in the inset, which serves as a guide for the eye, shows a $1/x^2$ behavior.

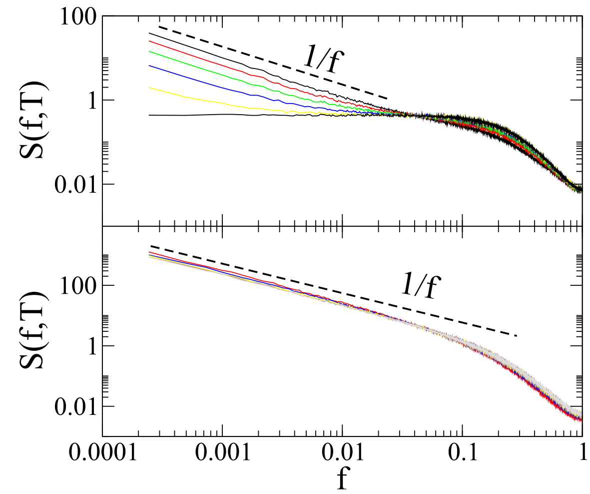

FIG. 4. (Color online) Spectra of C1 (top) and RP1 (bottom) processes with distinct degree of symmetry. In the C1 case the spectrum goes from $1/f$ to flat when increasing symmetry from $b = 0$ to 0.5 with a step of 0.1. In the RP1 case symmetry is irrelevant.

weaker correlation with $S_Z(f,T) \sim 1/f^\beta$, $\beta < 1$. Concerning probability densities of internal fluctuations and amplitudes, multiplicative processes show power laws, whereas additive processes show exponential tails. As an example, in Fig. 3 we plot the probability density of the relative amplitude $a_T/A(T)$, which is a magnitude independent of T, for several representative models. Note that here the models of exponential pulses and intermittent maps exhibit exponential tails despite their multiplicative origin.

V. SYMMETRY

It is worth remarking on the effect of symmetry on the spectral properties of series. Changes of symmetry (parameter b) in the renewal model are irrelevant; only a change in the value of the averaged mean value, $\langle \overline{Y}_T \rangle$, is detected. On the contrary, multiplicative processes are more sensitive to these changes. For instance, the model with correlated pulses keeps its scaling properties, $\langle \overline{Y}_T \rangle \sim \langle \overline{\delta}_T^2 \rangle^{1/2} \sim T^\alpha$, while it is asymmetric ($b \neq 1/2$) but becomes uncorrelated, with a flat spectrum and $\langle \overline{Y}_T \rangle = 0$, when it is symmetric ($b = 1/2$). These properties can be seen in Fig. 4, where the parameter b in the generation of series of pulses R1 and E1 is varied. Note that in the asymmetric case the series shows almost the same properties. Hence, close to a stationary $1/f$ process, symmetry is an indication of the additivity of the model.

VI. CONCLUSIONS

There exist a wide variety of models derived from multiplicative processes that show $1/f^\beta$ integrable spectra with $\beta > 1$. Like the model presented in [2] they are good candidates to explain the low-frequency cutoff paradox of $1/f$ processes. As they are complementary we investigate techniques to distinguish them in an experimental situation. With the help of simulations we show that pure multiplicative processes obtained directly from a stochastic equation, like

M1 and M2, exhibit power laws in both probability densities, $P(Y,T)$ and $P(a_t/A(T))$ (Figs. 3 and 4). Hence they are easily distinguished from other series coming from additive processes such as R1 and R2 that exhibit exponential tails. However, other series related to multiplicative processes, such as E1, E2, and IM, exhibit exponential tails in some of the probability densities and therefore are more difficult to distinguish.

[1] M. A. Caloyannides, J. Appl. Phys. **45**, 307 (1974).
[2] M. Niemann, H. Kantz, and E. Barkai, Phys. Rev. Lett. **110**, 140603 (2013).
[3] M. A. Rodríguez, Phys. Rev. E **90**, 042122 (2014).
[4] J. J. Ramasco, J. M. López, and M. A. Rodríguez, Phys. Rev. Lett. **84**, 2199 (2000).
[5] B. B. Mandelbrot, IEEE Trans. Inf. Theor. **13**, 289 (1967).
[6] S. B. Lowen and M. C. Teich, Phys. Rev. E **47**, 992 (1993).
[7] B. Kaulakys, J. Ruseckas, V. Gontis, and M. Alaburda, Physica A **365**, 217 (2006).
[8] J. Ruseckas and B. Kaulakys, J. Stat. Mech. (2014) P06005.
[9] B. Kaulakys and T. Meskauskas, Phys. Rev. E **58**, 7013 (1998).
[10] Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980).
[11] S. Tasaki and P. Gaspard, J. Stat. Phys. **109**, 803 (2002).