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Recursive percolation

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We introduce a simple lattice model in which percolation is constructed on top of critical percolation clusters, and find compelling numerical evidence that it can be repeated recursively any number n of generations. In two dimensions, we determine the percolation thresholds up to $n = 5$. The corresponding critical clusters become more and more compact as n increases, and define universal scaling functions of the standard two-dimensional form and critical exponents that are distinct for any n. This family of exponents differs from previously known universality classes, and cannot be accommodated by existing analytical methods. We confirm that recursive percolation is well defined also in three dimensions.

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The use of percolation theory pervades many parts of science, ranging from material science to geology, epidemiology, and sociology [1,2]. At the percolation threshold it leads to random, scale invariant geometries that have become paradigmatic in theoretical physics and probability theory. Lattice models for percolation [3] have propelled powerful theoretical constructions, leading to a host of exact results, particularly in two dimensions [4–7].

A typical model is bond percolation, in which each link of the lattice is taken to be open with probability p. An important assumption in this model is that the medium is independent of its preceding history. However, in numerous situations this hypothesis is not fulfilled. Examples include the percolation of a liquid in a porous medium such as granular rocks [8], or epidemic spread [9,10], where a renewed percolation (spread) event may depend on the history of sedimentation (immunization). In both cases, the first percolation process imposes a particular type of quenched disorder on the following process. The purpose of this Rapid Communication is to formulate a simple model of such *recursive percolation* and study its properties numerically.

Given a configuration of percolation clusters at criticality, $p^0 = p_c^0$, with superscript $n = 0$ for the original percolation, we define a new ($n = 1$) percolation process on top of them such that occupied bonds are placed with probability p^1 on all the pairs of neighboring sites in the same cluster. One might then expect that any finite probability $p^1 < 1$ would destroy the critical singularity and lead to a subcritical phase where it becomes exponentially difficult to form a large cluster. Contrary to this expectation, we show that there exists a nontrivial critical threshold, $1 > p_c^1 > p_c^0$, separating a subcritical and a critical phase. This means in particular that the construction can be repeated recursively: On top of these critical clusters, one may again study a percolation process and search for its threshold. The same scenario takes place, so that the construction may be repeated any number of times. Surprisingly, the nth generation of percolation clusters thus generated enjoys, at their threshold $p = p_c^n$, distinct critical exponents for any n. The $d = 2$ simulations show that the exponents are universal, i.e., independent of lattice and percolation process (bond or site). Moreover, they tend to finite limits when $n \to \infty$—in the case of a "worn out" medium.

This family of recursive critical exponents for $n \geqslant 1$ does not appear in previously known universality classes [4]. The $d = 2$ exponents cannot be accounted for by existing analytical constructions, including the Coulomb gas (CG) approach to conformal field theory (CFT) [4,5], and the more recent Schramm-Loewner evolution (SLE) [6,7]. These field-theoretical methods have provided a plenitude of information about critical behavior, predicting exact values [11,12] of critical exponents for most two-dimensional lattice models. We find in particular that recursive percolation for $n \geqslant 1$ violates the domain Markov property in the context of SLE theory.

Percolation threshold. We study recursive percolation on periodic $L \times L$ square lattices. The starting point is standard bond percolation [1,2], with the known threshold $p_c^0 = \frac{1}{2}$. From a given set of percolation clusters $\mathcal{C}_0$, henceforth called *standard clusters* for clarity, we define a set of *dense clusters* $\overline{\mathcal{C}}_0$ by filling in all bonds between neighboring sites in the same cluster. Here and elsewhere quantities with (without) an overline refer to the dense (standard) case.

Suppose that the thresholds $p_c^1, \ldots, p_c^{n-1}$ are already known. A configuration of clusters $\mathcal{C}_n$ at generation $n \geqslant 1$, with a given occupation probability p^n, is then defined as follows: For each $i = 1, \ldots, n$ in turn, produce $\mathcal{C}_i$ by performing bond percolation on $\overline{\mathcal{C}}_{i-1}$ with probability $p^i = p_c^i$ if $i < n$, and $p^i = p^n$ if $i = n$.

We have performed extensive simulations for $L = 2^\ell$, with $\ell = 4,5,\ldots,12$, using a variant of the Leath-Alexandrowicz algorithm [13]. The existence of a nontrivial threshold p_c^n is revealed by the crossing properties of the probability R_2^n that one cluster in $\mathcal{C}_n$ wraps both periodic lattice directions (see Fig. 1). The finite-size scaling clearly shows that p_c^n acts as an unstable fixed point for nth generation clusters: The slope of R_2^n near p_c^n increases as $L^{y_t^n}$, where the renormalization exponent $y_t^n = 1/\nu^n$ is generally referred to as the thermal exponent. Away from p_c^n, renormalization flows run into the

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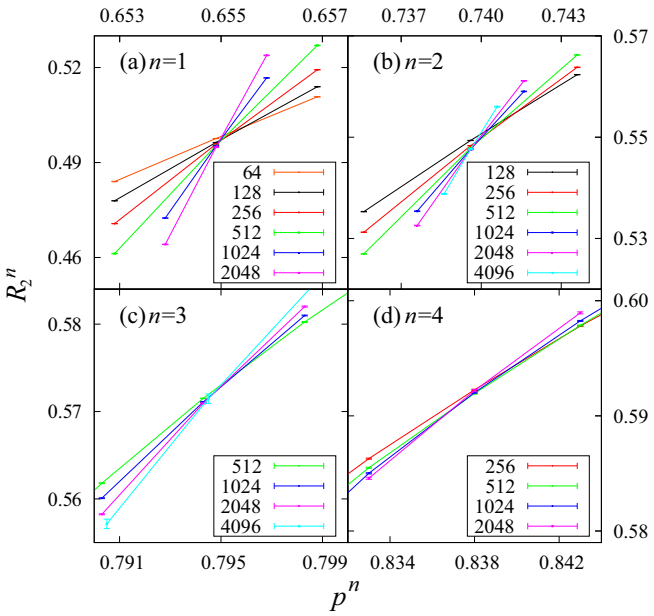

FIG. 1. (Color online) Wrapping probability R_2^n vs p^n for $n = 1,2,3,4$. The superscript n is for the nth generation. Percolation threshold is located by the approximately common crossing for different sizes L, as denoted by different colors.

trivial fixed points $p^n \approx 1$ and $p^n = 0$, respectively; see the Supplemental Material (SM) [14] for more details.

From the scaling of R_2^n near p_c^n (see SM for the fitting formula), we have determined, for $n \leqslant 5$, the thresholds p_c^n, the thermal exponent y_t^n, and the critical value of R_2^n, given in Table I. Notice that the values of p_c^n are close to the simple fraction $(n+1)/(n+2)$, especially for larger n. This suggests that $p_c^n \to 1$ for $n \to \infty$, implying that recursive percolation can be defined for any number of generations.

Observables and scaling. For a given configuration, the occupied bonds are classified into bridges and nonbridges: A bond is a bridge if its deletion leads to the disconnection of a cluster. A pseudobridge is a nonbridge whose removal would change the topology of the associated cluster—i.e., how it wraps the periodic boundary conditions. An efficient algorithm has been introduced in Ref. [15] for the classification of occupied bonds into these three classes; see SM for details. We also measured the size C_1^n of the largest cluster, the number B_R^n of pseudobridges, the length H_1^n of the largest loop [16] surrounding percolation clusters, and the size C_{b1}^n of the largest backbone clusters that are constructed by those nonbridges. At criticality, the finite-size scaling of these observables is governed by a set of critical exponents,

$$C_1^n \propto L^{d_F^n}, \quad H_1^n \propto L^{d_H^n}, \quad B_R^n \propto L^{d_R^n}, \quad C_{b1}^n \propto L^{d_B^n}, \qquad (1)$$

TABLE I. Threshold p_c^n, exponent y_t^n, and wrapping probability R_2^n for nth generation percolation.

n	1	2	3	4	5
y_t^n	0.433(1)	0.273(4)	0.182(4)	0.116(10)	0.09(2)
R_2^n	0.495(1)	0.547(1)	0.571(2)	0.586(3)	0.595(4)
p_c^n	0.654902(10)	0.73954(4)	0.7945(1)	0.8342(8)	0.861(4)

TABLE II. Values of critical exponents. For $n = 0$, the backbone is estimated as $d_B^0 = 1.6431(6)$ from a transfer matrix computation [17] and $d_B^0 = 1.643\,36(10)$ by Monte Carlo simulations [18,19]. Note that d_R^n coincides with y_t^n in Table I. The equality $y_t^0 = d_R^0$ holds true in any spatial dimension [20,21].

n	0	1	2	3	4
d_F^n	1.8958(1)	1.8573(1)	1.8424(1)	1.8357(2)	1.8323(2)
d_B^n	1.6433(3)	1.7596(1)	1.7942(1)	1.8078(2)	1.8148(2)
d_H^n	1.75	1.6083(1)	1.5358(1)	1.4967(1)	1.4723(2)
$\overline{d}_H^n$	1.3333	1.3739(1)	1.3929(1)	1.4026(2)	1.4075(2)
d_R^n	0.751(1)	0.433(1)	0.272(2)	0.182(2)	0.121(3)
$\overline{d}_R^n$	−0.77(3)	−0.429(1)	−0.275(2)	−0.194(6)	−0.15(1)

where d_F^n is the cluster's fractal dimension, d_H^n is the hull dimension, d_R^n is the red-bond exponent, and d_B^n is the backbone dimension. These critical exponents characterize more precisely the critical clusters $\mathcal{C}_n$. Analogous measurements were taken for the dense clusters $\overline{\mathcal{C}}_n$, and our definitions imply that $d_F^n = \overline{d}_F^n$. The "dense" hulls correspond to the accessible perimeters in Ref. [22].

The scaling behavior in Eq. (1) is well confirmed by our numerical data, and the results are shown in Table II [23]. The SM presents the results for additional observables, including the shortest path and the bond density. For the standard case ($n = 0$), it can be derived [24] that the dense bond density is 3/4.

Scaling functions. The recursive percolation can be constructed in an alternative way: Start from a seed site, grow a percolation cluster, construct a $n = 1$ cluster right on top of it from the same seed site, and repeat the process recursively. This is illustrated in Fig. 2. We employ this procedure on periodic $L \times L$ square lattices, and record the probability distribution $P(s,L)$ that the grown cluster is of size s. Figure 3 shows $P(s,L)$ at criticality versus s in a log-log

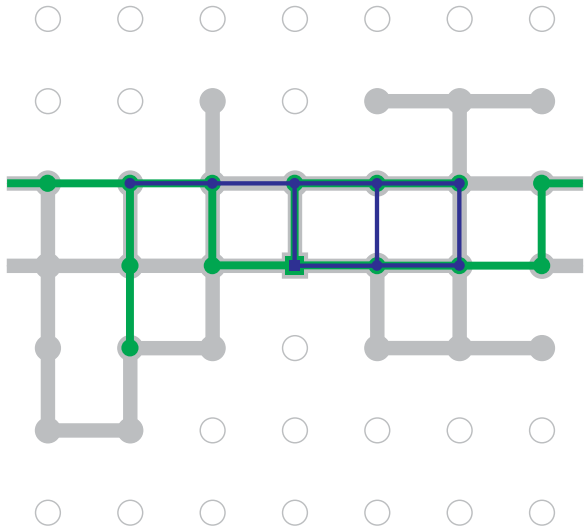

FIG. 2. (Color online) Illustration of recursive single-cluster growing processes. The $n = 0,1,2$ bonds and clusters are marked in gray, green, and black, respectively.

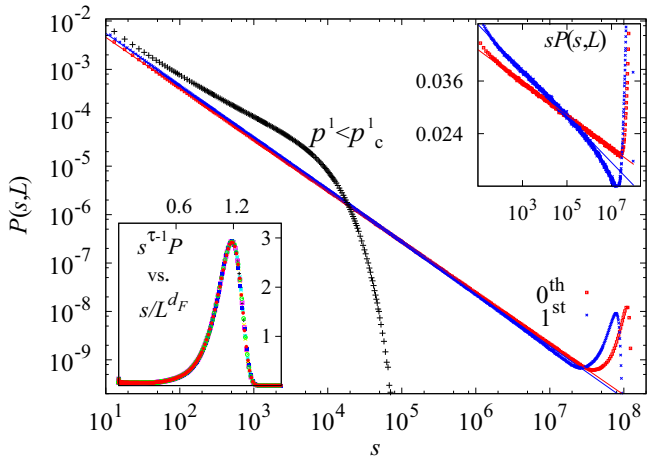

FIG. 3. (Color online) Probability distribution $P(s,L)$ in the single-cluster growing procedure. The size $L = 16\,384$ in the main plot. The red ($n = 0$) and blue ($n = 1$) data are for $p^0 = p_c^0 = 1/2$ and $p^1 = p_c^1 = 0.654\,902$, respectively. The black curve with $p^1 = 0.6 < p_c^1$ displays subcritical behavior. The red/blue straight lines have slopes $1 - \tau \equiv -d/d_F^n$, with $d_F^0 = 91/48$ and $d_F^1 = 1.8573$. The right-top inset shows the product $sP(s,L)$, and the left-bottom corner displays $s^{\tau-1}P(s,L)$ for $n = 1$ vs s/d_F^1, with $L = 256, 512, \ldots, 16\,384$.

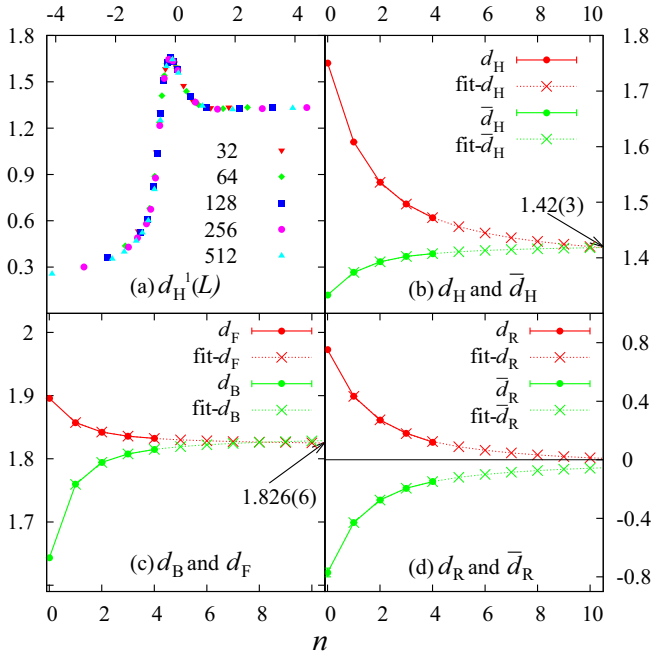

FIG. 4. (Color online) Fractal dimensions. (a) Effective dimensions $d_H^1(L)$ vs the variable $u = (p^1 - p_c^1)L^{y_t^1}$; $d_H^1(L)$ approaches the exact value 4/3 for $u \gg 0$. (b)–(d) The critical exponent d_X^n vs generation n, with X = H,F,R, respectively. The "×" points are from the fitting results. For each of the exponents, the n dependence can be fitted to a common $n \to \infty$ limit, as shown.

scale for $L = 16\,384$. The algebraically decaying behavior of $P(s,L)$ is well displayed in a wide range of size s.

The standard scaling theory yields

$$P(s,L) \sim s^{1-\tau} f(s/L^{d_F}) \quad (\tau = 1 + d/d_F), \tag{2}$$

where f is a universal function and the hyperscaling relation $\tau = 1 + d/d_F$ involves spatial dimension d. A nontrivial question arises: Does Eq. (2), particularly the hyperscaling relation, hold true for $n \geqslant 1$, for which the underlying geometries are already fractal? We apply in Fig. 3 the critical exponents $d_F^0 = 91/48$ [11] and $d_F^1 = 1.8573$. The latter is taken from Table II, obtained from the other construction of recursive percolation. Surprisingly, the two insets of Fig. 3 strongly support that the $n = 1$ recursive percolation enjoys the scaling form in Eq. (2) with original dimensionality $d = 2$.

We show in Fig. 4(a) the effective hull dimension d_H^1 against the variable $u = (p^1 - p_c^1)L^{y_t^1}$, defined as $d_H^1(L) \equiv \log_2 [H_1^1(2L)/H_1^1(L)]$. With the choice $y_t^1 = 0.433(1)$ the data for all sizes L collapse perfectly to reveal the universal scaling function. For $u = 0$ one has the $\mathcal{C}_n$ universality class, here with $d_H^1 = 1.6083(1)$, while for $u > 0$ there is a flow to the $\overline{\mathcal{C}}_{n-1}$ universality class, as expected, now with $\overline{d}_H^0 = 4/3$. The flow for $u < 0$ is to the trivial fixed point with $d_H = 0$.

Critical exponents. In two dimensions, the field-theoretical methods [11,12] predict the exact results for percolation and q-state Fortuin-Kasteleyn (FK) clusters (a correlated percolation model)

$$\begin{aligned} d_F^0 &= 2 - (6-g)(g-2)/(8g) = 91/48, \\ d_H^0 &= 1 + 2/g = 7/4, \\ d_R^0 &= (4-g)(4+3g)/(8g) = 3/4, \end{aligned} \tag{3}$$

where $g = 8/3$ for percolation, in which case the above results are rigorous [6,7]. The CG duality transformation $g \to 16/g$ relates $d_H^0 \to \overline{d}_H^0$, and leads to the duality relation $(d_H^0 - 1)(\overline{d}_H^0 - 1) = 1/4$ [12].

By comparing Eq. (3) to the numerical results in Table II, we obtain that, for $n \geqslant 1$, (1) d_F^n cannot be described by the exact d_F^0 formula in Eq. (3) that has a minimum $d_{F,\min} = (2 + \sqrt{3})/2 \approx 1.866$ at $g = 2\sqrt{3}$, and (2) the recursive clusters violate the domain Markov property, since d_H^n and $\overline{d}_H^n$ data do not satisfy the duality relation.

Limiting clusters. The n dependence of critical exponents is illustrated in Figs. 4(b)–4(d). It is shown that as n increases, the exponents for $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$ approach each other. They can be convincingly fitted to ratios of low-degree polynomials, with a common limit for the standard and dense exponents. The common limiting values for $n \to \infty$ are estimated as

$$\begin{aligned} d_H^\infty &= \overline{d}_H^\infty \simeq 1.42(3), \\ d_B^\infty &= \overline{d}_B^\infty = d_F^\infty = \overline{d}_F^\infty \simeq 1.826(6), \\ d_R^\infty &= \overline{d}_R^\infty \simeq 0.00(6). \end{aligned} \tag{4}$$

The numerical values of Table II also appear to satisfy $d_R^n = -\overline{d}_R^n$ for each n; we have no explanation for this. In particular, the common $n \to \infty$ limit $d_R^\infty = \overline{d}_R^\infty = 0$ most probably holds true. These results provide substantial evidence that the difference between standard and dense clusters—the property that permitted us to define recursive percolation in the first place—disappears when $n \to \infty$. The variation of critical

exponents (Table II) and the convergence $p_c^n \to 1$ (Table I), all monotonic, strongly support the conjecture that recursive percolation defines a distinct universality class for any finite n.

The above results can be used to characterize the limiting clusters $\mathcal{C}_\infty = \overline{\mathcal{C}}_\infty$ in various ways. The fact that $d_R^\infty = 0$ means that the number of red bonds in the limiting clusters does not grow with L. This is compatible with the observation [Fig. 4(c)] that the difference between the clusters and their backbones vanishes in the limit. In other words, the limiting clusters are dense objects, with only few leaves or dangling ends. Moreover, they are devoid of deep fjords, since their hulls and external perimeter scale in the same way [Fig. 4(b)].

Another set of clusters having similar characteristics are the FK clusters of the $q = 4$ state Potts model, whose hulls behave as the level lines of a free Gaussian field with central charge $c = 1$. These Potts clusters can be described by the CG construction with the self-dual choice of the coupling, $g = 4$ [5]. They have $d_R = 0$, $d_F = \frac{15}{8}$, and $d_H = \frac{3}{2}$, coming from Eq. (3).

Despite this resemblance, the $\mathcal{C}_\infty$ clusters are most definitely different from the $q = 4$ FK clusters: The fractal dimensions d_F^∞ and d_H^∞ disagree with Eq. (3).

Universality. Changing the lattice from square to triangular, or the process from bond to site percolation [25], obviously modifies the thresholds p_c^n. However, the critical exponents d_F^n, d_R^n, and d_H^n are found to be unchanged; the critical wrapping probability R_2^n also remains the same for different processes on the same lattice. This demonstrates the universality of recursive percolation. Simulations for $d = 3$ (see SM) produce different exponents, but the thresholds $p_c^n < 1$ remain nontrivial, so we conjecture that recursive percolation is well defined in any d below the upper critical dimension $d_{uc} = 6$ [1,2]. For $d > d_{uc}$, the construction of a critical percolation cluster is basically a branching process with small corrections [26], and we expect $p_c^n = 1$ for $n \geqslant 1$ and trivial exponents (see SM).

Discussion. We introduced a simple lattice model, *recursive percolation*, which represents an infinite family of different universality classes. A crucial element of its definition is that the nth recursive process occurs on the set of dense percolation clusters $\overline{C}_{n-1}$ such that occupied bonds can be placed between those neighboring sites connected via nonlocal paths. Indeed, using instead the standard clusters C_{n-1} would have been tantamount to a trivial modification of p in the $n = 0$ process, leading to $p_c^n = 1$ for all $n \geqslant 1$. In other words, the strengthening along the perimeter $C_{n-1} \to \overline{C}_{n-1}$ makes the clusters subcritical, while $p_c^{n-1} \to p_c^n$ takes them back to criticality, but in a different, n-dependent universality class. This modeling of the wearing out of the medium upon multiple percolation events differs from that of Ref. [8], which has a trivial percolation threshold $p_c^n = 1$ for $n \geqslant 1$. In Ref. [8], the standard nontrapping invasion percolation algorithm is used to generate a percolating site cluster on top of a previous standard site cluster, and thus effectively removes the occupied sites. In a very recent paper [27], a different fragmentation process is introduced and a set of varying dynamic exponents is found.

We also stress that although the underlying medium is fractal for $n \geqslant 1$, recursive percolation belongs to the realm of the original d-dimensional Euclidean space, as witnessed most clearly by the hyperscaling relation in Eq. (2). Moreover, we find that the $d = 2$ critical exponents for $n \geqslant 1$ are beyond the description of field-theoretical methods that are applicable for most two-dimensional lattice models. Several important questions arise: Is recursive percolation conformally invariant at criticality, what is the universality criterion, and how can the exact values of critical exponents be obtained? Do the $n \geqslant 1$ clusters enjoy multifractal properties?

It is worth mentioning that earlier studies of statistical models on top of fractal structures [28] focused either on the case where the underlying structure is a self-similar set, such as the Sierpiński gasket [29], or where the model is random walks of the self-avoiding (SAW) [30–35] or loop-erased [36] types on top of percolation backbones. While the former case is easy, the latter inherits the difficulties of the underlying d-dimensional lattice. Interestingly, SAW on backbones defines a different universality class exactly at $p = p_c$ [32] with multifractal properties [34,35]. In renormalization group language this means that p_c is an unstable fixed point from which the system may flow to either the usual SAW fixed point at $p = 1$, or to a trivial fixed point at $p = 0$. However, recursive percolation differs from these existing works in various ways: It can be defined recursively any number of times, and the critical exponents are incompatible with existing analytical methods.

We conclude by suggesting that the recursive construction presented here, via the study of percolation on percolation clusters, may carry over more generally to the q-state Potts model. For instance, it is well known that q-state FK clusters arise by considering percolation with $p_c = \sqrt{q}/(1 + \sqrt{q})$ on top of q-state Potts spin clusters [37], which are widely applied in cluster-type Monte Carlo methods [38]. Both types of clusters are well defined for arbitrary real $0 \leqslant q \leqslant 4$ [39,40]. It is thus tempting to speculate that on top of q-state FK clusters one may define new q_1-state FK clusters, and that the latter will be critical for a suitable nontrivial choice of the temperature variable, with distinct critical exponents. Future work will show whether this construction is possible and can be repeated recursively.

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