Anomalous sea surface structures as an object of statistical topography

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By exploiting ideas of statistical topography, we analyze the stochastic boundary problem of emergence of anomalous high structures on the sea surface. The kinematic boundary condition on the sea surface is assumed to be a closed stochastic quasilinear equation. Applying the stochastic Liouville equation, and presuming the stochastic nature of a given hydrodynamic velocity field within the diffusion approximation, we derive an equation for a spatially single-point, simultaneous joint probability density of the surface elevation field and its gradient. An important feature of the model is that it accounts for stochastic bottom irregularities as one, but not a single, perturbation. Hence, we address the assumption of the infinitely deep ocean to obtain statistic features of the surface elevation field and the squared elevation gradient field. According to the calculations, we show that clustering in the absolute surface elevation gradient field happens with the unit probability. It results in the emergence of rare events such as anomalous high structures and deep gaps on the sea surface almost in every realization of a stochastic velocity field.

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I. INTRODUCTION

Stochastic structures, like clusters, in random media results from the parametric excitation of various physical fields inherent to the media. Applications where clustering plays a significant role abound in physics and include hydrodynamics (passive scalars in turbulent flows), magnetohydrodynamics (passive vectors as a magnetic field with turbulence), and wave propagation of diverse nature (acoustical and radio waves, light, and laser emission) in random media, to name a few. When addressing these problems one usually utilizes the kinematic approximation and bears in mind two crucial properties:

(1) At fixed points in space, field realizations in time are random processes of a peculiar type characterized by the existence of pikes emerging in random time intervals. These intervals are long lasting and consist of weak impulses. This typical realization of a random process results from a log-normal simultaneous probability distribution with a flat tail. It is the flat tail that results in intense but short-term impulses. The basic statistical values are simultaneous probability density, moments, the Lyapunov exponent, and typical realization curve, indicating the general features of the random processes. It is crucial that by taking into account only the named statistical characteristics of a random process, structure formation cannot be accounted for. Some one-dimensional problems, governed by ordinary differential equations, can possess only a dynamical localization [1–3].

(2) Structures in a stochastic field manifest themselves in physical space. It can be treated by means of appropriate statistical analysis originated from statistical topography theory [4–7] for stochastic fields, whereas the simplest problem formulation with the spatial statistical uniformity ensures that all random field statistical features do not depend on the spatial coordinate. Hence, the equation governing the single-point probability density coincides in form with the equation of probability density in every given point; however, these equations have a different meaning. This implies that one should use different statistical approaches to study this equation. We refer the reader to the monographs and papers [6–9] where these problems have been considered in detail.

By now, a great body of literature has been concerned with anomalously high structures which are observable in a vast range of media, including random ones, for instance, rogue waves in the ocean [10–13], capillary waves [14], plasma physics [15], the branching of electron flows in semiconductor devices [16,17], and extreme acoustic [18] and optical [19–21] waves. In this work, we are mainly interested in surface ocean structures. Despite significant efforts (see reviews in Refs. [10,12] and, with an emphasis on statistical properties, Refs. [5,22–24]), there is still no exhaustive explanation for how these structures are born and what constitutes their main statistical properties. Partial explanations, based on the observation that such events correspond to heavy distribution tails differing from the Gaussian one, can be found in Ref. [25].

It is worth noting that one of the approaches to treat rogue waves is to consider them as structures that emerge at random places in random instants of time. The causes and mechanisms of rogue wave appearance attract attention of many researchers (see, for example, Ref. [25–27] and references therein). Some of them analyze dynamical systems on the basis of the Schrödinger equation that can be treated analytically or numerically. Usually, when dealing with random process realizations in time, the emergence of rare but intense impulses is identified as rogue waves. This identification seems to be debatable due to the reasoning laid out above.

Thus, in this work, we adhere to the idea that rogue waves are, in general, generated as a result of clustering and coherency that appear in uniformly random fields. We suggest

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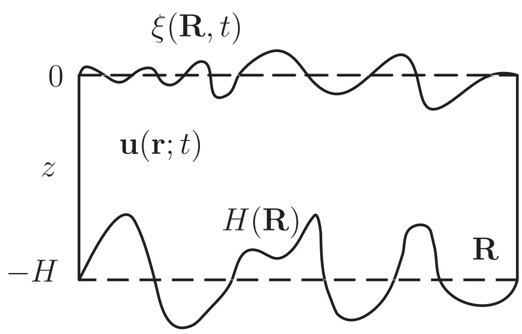

FIG. 1. Sea surface fluctuations.

addressing the rogue wave problem on the sea surface based on the statistical topography ideas [4,6,8,9] which describe stochastic structure formation in random fields [7,13]. The main goal of the paper is to study the generation of rogue waves by making use of developed methods that have been shown to be fruitful in solving stochastic problems with parametric excitation [7,8,28].

The paper is organized as follows. Section II describes the problem; in particular, the equations under study are introduced. Then the random velocity field correlation function is discussed in two cases, the hydrodynamic turbulence and wave turbulence. In Sec. III, the derivation of the probability density equation is presented in the general form and the role of random bottom irregularities is discussed. Section IV provides the statistical analysis of probability density equations. Here we consider two cases of the random velocity field separately to obtain probability density equations for the surface elevation and its gradient. In Sec. V, we investigate the statistical topography characteristics of the random field of the surface elevation gradient. The paper is concluded in Sec. VI. A simpler equation qualitatively analogous to the surface gradient equation is considered in the appendix.

II. PROBLEM STATEMENT

A. Sea surface structures

Let $\mathbf{r} = \{r_i\}$ be the spatial coordinates with $i = 1,2,3$, $z = r_3$, for the vertical coordinate, and R_α, R_β, $(\alpha,\beta = 1,2)$ for the horizontal coordinates orthogonal to z. Then, $\mathbf{r} = \{\mathbf{R},z\}$. The three-dimensional velocity field $\mathbf{u}(\mathbf{r};t)$ is then the sum of vertical and horizontal components such as $u_i(\mathbf{r};t) = \{u_\alpha(\mathbf{R},z;t), w(\mathbf{R},z;t)\}$.

The water boundary elevation on the sea surface is imposed as a kinematic boundary condition $z = \xi(\mathbf{R},t)$ (see Fig. 1) as follows:

$$\frac{d}{dt}\xi(\mathbf{R},t) = w(\mathbf{R},z;t)\Big|_{z=\xi(\mathbf{R},t)}, \tag{1}$$

where $\frac{d}{dt}\xi(\mathbf{R},t)$ is the total derivative of the surface elevation.

Boundary condition (1) can be considered as a closed stochastic quasilinear equation within the kinematic approximation, i.e., with prescribed statistical features of the velocity fields $\mathbf{u}(\mathbf{R},z;t)$ and $w(\mathbf{R},z;t)$:

$$\frac{\partial \xi(\mathbf{R},t)}{\partial t} + u_\alpha[\mathbf{R},\xi(\mathbf{R},t),t]\frac{\partial \xi(\mathbf{R},t)}{\partial R_\alpha} = w[\mathbf{R},\xi(\mathbf{R},t);t], \tag{2}$$

with initial condition $\xi(\mathbf{R},0) = \xi_0(\mathbf{R})$. We accept the Einstein notation over summation. Equation (2) governs wave generation on the sea surface that is induced by the vertical component of the hydrodynamical field. By differentiating Eq. (2) over $\mathbf{R}$, one can obtain an equation for the elevation gradient $p_\beta(\mathbf{R},t) = \frac{\partial \xi(\mathbf{R},t)}{\partial R_\beta}$, which characterizes the surface slope,

$$\begin{aligned}\frac{\partial p_\beta(\mathbf{R},t)}{\partial t} &+ \left[\frac{\partial u_\alpha(\mathbf{R},z;t)}{\partial R_\beta}\Big|_{z=\xi(\mathbf{R},t)} + \frac{\partial u_\alpha(\mathbf{R},\xi(\mathbf{R},t);t)}{\partial z}p_\beta(\mathbf{R},t)\right]p_\alpha(\mathbf{R},t) + u_\alpha(\mathbf{R},\xi(\mathbf{R},t),t)\frac{\partial p_\alpha(\mathbf{R},t)}{\partial R_\beta}\\ &= \frac{\partial w(\mathbf{R},z;t)}{\partial R_\beta}\Big|_{z=\xi(\mathbf{R},t)} + \frac{\partial w(\mathbf{R},\xi(\mathbf{R},t);t)}{\partial z}p_\beta(\mathbf{R},t),\end{aligned} \tag{3}$$

with initial conditions $\mathbf{p}(\mathbf{R},0) = \mathbf{p}_0(\mathbf{R}) = \frac{\partial \xi_0(\mathbf{R})}{\partial \mathbf{R}}$. It is worth noting that there is the second boundary condition related to bottom irregularities (see Fig. 1). According to the kinematic approximation, this boundary condition is a functional, so for variational derivatives $\xi(\mathbf{R},t)$ and $\mathbf{p}(\mathbf{R},t)$ there are the relations

$$\begin{aligned}\frac{\delta \xi(\mathbf{R},t)}{\delta \mathbf{u}(\mathbf{R}',z',t')} &\sim \theta[z' - H(\mathbf{R})]\theta\left(t-t'\right),\\ \frac{\delta \mathbf{p}(\mathbf{R},t)}{\delta \mathbf{u}(\mathbf{R}',z',t')} &\sim \theta[z' - H(\mathbf{R})]\theta(t-t'),\end{aligned} \tag{4}$$

where $\theta\,(z)$ is the Heaviside step function.

B. Liouville equation

Let us introduce a joint indicator surface elevation function and its gradient,

$$\varphi(\mathbf{R},t;\xi,\mathbf{p}) = \delta\left[\xi(\mathbf{R},t) - \xi\right]\delta\left[\mathbf{p}(\mathbf{R},t) - \mathbf{p}\right]. \tag{5}$$

Taking into account dynamical conditions, one can derive a linear Liouville equation [7,13],

$$\begin{aligned}\frac{\partial \varphi(\mathbf{R},t;\xi,\mathbf{p})}{\partial t} &= -\frac{\partial}{\partial \xi}w(\mathbf{R},\xi;t)\varphi(\mathbf{R},t;\xi,\mathbf{p})\\ &\quad - \left[u_\alpha(\mathbf{R},\xi;t)\frac{\partial}{\partial R_\alpha} - \frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial \xi}p_\alpha\right]\varphi(\mathbf{R},t;\xi,\mathbf{p})\\ &\quad - \frac{\partial}{\partial p_\beta}\left[\frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial R_\beta} + \frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial \xi}p_\beta\right]p_\alpha\varphi(\mathbf{R},t;\xi,\mathbf{p})\\ &\quad - \frac{\partial}{\partial p_\beta}\left[\frac{\partial w(\mathbf{R},\xi;t)}{\partial R_\beta} + \frac{\partial w(\mathbf{R},\xi;t)}{\partial \xi}p_\beta\right]\varphi(\mathbf{R},t;\xi,\mathbf{p})\end{aligned} \tag{6}$$

with initial conditions

$$\varphi(\mathbf{R},0;\xi,\mathbf{p}) = \delta\left[\xi - \xi_0(\mathbf{R})\right]\delta[\mathbf{p} - \mathbf{p}_0(\mathbf{R})].$$

Equation (6) governs the joint surface elevation probability density and its spatial gradient for a dynamical system with deterministic parameters and random initial conditions.

C. Statistical characteristics of the random velocity field

1. *Field correlation functions* $\mathbf{u}(\mathbf{r},t)$

We are interested in the case of generating sea structures by use of a random hydrodynamical velocity field $\mathbf{u}(\mathbf{r},t)$ in the form of a random Gaussian field which is statistically uniform and isotropic in space and statistically stationary in time. Its correlation and spectral functions read

$$B_{ij}(\mathbf{r}-\mathbf{r}',t-t') = \langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\rangle = \int d\mathbf{k}E_{ij}(\mathbf{k},t-t')e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}, \quad (7)$$

$$E_{ij}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int d\mathbf{r}B_{ij}(\mathbf{r},t)e^{-i\mathbf{k}\mathbf{r}}.$$

In the general case of a given random velocity field $\mathbf{u}(\mathbf{r},t)$, the spectral function $E_{ij}(\mathbf{k},t)$ has the form

$$E_{ij}(\mathbf{k},t) = E^s(k,t)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) + \frac{k_ik_j}{k^2}E^p(k,t).$$

Let us introduce the function

$$B_{ij}(\mathbf{r}) = \int_0^\infty d\tau B_{ij}(\mathbf{r},\tau), \quad (8)$$

which characterizes all the statistical features when addressing the problem in the diffusion approximation.

Asymptotically as $t \gg \tau_0$, under the diffusion approximation a solution of the initial dynamical system is a Markovian random field. Given the smallness of all statistical effects on the temporal scales close to the temporal correlation radius τ_0 [7], the applicability conditions are posed.

Further, we will use the second spatial derivatives of the correlation function of a random velocity field $\mathbf{u}(\mathbf{r},t)$ at the coordinate's origin. These derivatives satisfy the tensor equality [6,7],

$$-\frac{\partial^2 B_{ij}(0)}{\partial r_k \partial r_l} = \frac{D^s}{d(d+2)}[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}] + \frac{D^p}{d(d+2)}(\delta_{kl}\delta_{ij} + \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}), \quad (9)$$

where parameters are

$$\begin{aligned} D^s &= \int d\mathbf{k}\, k^2E^s(k) = 4\pi\int_0^\infty dk\, k^4E^s(k) \\ &= \frac{1}{d-1}\int_0^\infty d\tau\, \langle \boldsymbol{\omega}(\mathbf{r},t+\tau)\boldsymbol{\omega}(\mathbf{r},t)\rangle, \\ D^p &= \int d\mathbf{k}\, k^2E^p(k) = 4\pi\int_0^\infty dk\, k^4E^{\mathrm{p}}(k) \\ &= \int_0^\infty d\tau\left\langle \frac{\partial\mathbf{u}(\mathbf{r},t+\tau)}{\partial\mathbf{r}}\frac{\partial\mathbf{u}(\mathbf{r},t)}{\partial\mathbf{r}}\right\rangle, \end{aligned} \quad (10)$$

and d is the spatial dimension, $\boldsymbol{\omega}(\mathbf{r},t) = \operatorname{curl}\mathbf{u}(\mathbf{r},t)$ is the velocity field curl, and $\partial\mathbf{u}(\mathbf{r},t)/\partial\mathbf{r}$ is the divergence.

We are concerned with two types of random velocity fields: (i) incompressible hydrodynamic turbulence and (ii) random-wave hydrodynamic fields. We believe that the first type produces *stationary structures*, while the second one produces wavelike *propagating structures*.

2. *Hydrodynamic turbulence*

In the first case, the spectral function looks like

$$E_{ij}(\mathbf{k},t) = E^s(k,t)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) \quad (11)$$

with the three-dimensional velocity field variance

$$\sigma_{\mathbf{u}}^2 = \langle\mathbf{u}^2(\mathbf{r},t)\rangle = 2\int d\mathbf{k}E^s(k,0). \quad (12)$$

Variable $B_{ij}(\mathbf{r})$ in (8) in this case takes the form

$$B_{ij}(\mathbf{r}) = \int_0^\infty d\tau B_{ij}(\mathbf{r},\tau) = \int d\mathbf{k}E^s(k)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}, \quad (13)$$

where $E^s(k) = \int_{\mathbf{0}}^\infty d\tau E^s(k,\tau)$ characterizes temporal correlation radius τ_0-$B_{ii}(\mathbf{0}) = \sigma_{\mathbf{u}}^2\tau_0$. Equation (9) then transforms to

$$-\frac{\partial^2 B_{ij}(0)}{\partial r_k\partial r_l} = \frac{D^s}{d(d+2)}[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}]. \quad (14)$$

3. *Wave turbulence*

The second type leads to the correlation function

$$B_{ij}(\mathbf{r},t) = \int d\mathbf{k}\frac{k_ik_j}{k^2}E^p(k)e^{-\lambda k^2 t}\cos\{\mathbf{k}\mathbf{r} - \omega(\mathbf{k})t\}, \quad (15)$$

where $\omega = \omega(\mathbf{k}) > 0$ determines a variance curve of the wave motion and λ indicates the wave attenuation.

The velocity field variance then is

$$\sigma_{\mathbf{u}}^2 = \langle\mathbf{u}^2(\mathbf{r},t)\rangle = \int d\mathbf{k}E^p(k), \quad (16)$$

and analogous variable (13) is defined as

$$B_{ij}(\mathbf{r}) = \int_0^\infty dt B_{ij}(\mathbf{r},t) = \int d\mathbf{k}\frac{k_ik_j}{k^2}\left[E_1^p(k)\cos\mathbf{k}\mathbf{r} + E_2^p(k)\sin\mathbf{k}\mathbf{r}\right], \quad (17)$$

where

$$E_1^p(k) = \frac{E^p(k)\lambda k^2}{\lambda^2k^4 + \omega^2(k)}, \quad E_2^p(k) = \frac{E^p(k)\omega(k)}{\lambda^2k^4 + \omega^2(k)}.$$

4. *Field correlation functions* $u_\alpha(\mathbf{R},\xi;t)$ *and* $w(\mathbf{R},\xi;t)$

Statistical properties of the components $u_\alpha(\mathbf{R},\xi;t)$ and $w(\mathbf{R},\xi;t)$ of the hydrodynamics field $\mathbf{u}(\mathbf{r},t)$ in Liouville equation (6) determine the required statistical characteristics.

a. Hydrodynamic turbulence. The correlation functions follow from the relations (11)–(14) for the two-dimensional vector $\mathbf{K}$ and one-dimensional projection

$$z = \xi,$$

$$B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi',t-t') = \langle u_\alpha(\mathbf{R},\xi;t)u_\beta(\mathbf{R}',\xi';t')\rangle = \int d\mathbf{k}\, E^s(k,t-t')\left(\delta_{\alpha\beta}-\frac{K_\alpha K_\beta}{k^2}\right)e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')}, \tag{18}$$

where $\mathbf{k}=\{\mathbf{K},k_z\}$, $\mathbf{K}=\{K_a\}$, $\alpha=1,2$, and $k=\sqrt{\mathbf{K}^2+k_z^2}$.

The correlations $B_{\alpha w}$ and B_{ww} are

$$B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi';t-t') = -\int d\mathbf{k}E^s(k,t-t')\frac{K_\alpha k_z}{k^2}e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')},$$
$$B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi';t-t') = \int d\mathbf{k}\, E^s(k,t-t')\left(1-\frac{k_z^2}{k^2}\right)e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')}. \tag{19}$$

It follows from Eqs. (18) that the second derivative of the correlation function $B_{\alpha\beta}(\mathbf{R},\xi)$ at the coordinate origin is

$$\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\gamma \partial R_\delta} = -\int d\mathbf{k}E^s(k)K_\gamma K_\delta\left(\delta_{\alpha\beta}-\frac{K_\alpha K_\beta}{k^2}\right) = -\frac{1}{2}\int d\mathbf{k}E^s(k)K^2\delta_{\alpha\beta}\delta_{\gamma\delta}+\frac{1}{8}\int d\mathbf{k}E^s(k)\frac{K^4}{k^2} \times(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{20}$$

Other correlation functions then satisfy

$$\begin{aligned}\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi} &= \int d\mathbf{k}E^s(k)\frac{K^2k_z^2}{k^2},\\ \frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi} &= -\int d\mathbf{k}E^s(k)\left(2k_z^2-\frac{K^2k_z^2}{k^2}\right),\\ \frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial R_\gamma \partial R_\gamma} &= -\int d\mathbf{k}E^s(k)\left(K^2-\frac{K^2k_z^2}{k^2}\right),\\ \frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial \xi \partial \xi} &= -\int d\mathbf{k}E^s(k)\left(k_z^2-\frac{k_z^4}{k^2}\right).\end{aligned} \tag{21}$$

It is worth recalling that all the statistical characteristics of the velocity field components relate to the statistical characteristics of the field $\mathbf{u}(\mathbf{r},t)$. The relation is clearly seen in the polar coordinates $k_z=k\cos\theta$, $K^2=k^2\sin^2\theta$, after integrating over the angle variables.

b. Wave turbulence. One can derive analogous formulas in the case of stochastic wave motion. Utilizing correlation (17) in the form

$$B_{ij}(\mathbf{R},w) = \int d\mathbf{k}\frac{k_ik_j}{k^2}\left[E_1^p(k)\cos\mathbf{kr}+E_2^p(k)\sin\mathbf{kr}\right],$$

one gets

$$B_{\alpha\beta}(\mathbf{R},w) = \int d\mathbf{k}\frac{K_\alpha K_\beta}{k^2}\left[E_1^p(k)\cos\mathbf{kr}+E_2^p(k)\sin\mathbf{kr}\right],$$
$$B_{w\beta}(\mathbf{R},w) = \int d\mathbf{k}\frac{k_zK_\beta}{k^2}\left[E_1^p(k)\cos\mathbf{kr}+E_2^p(k)\sin\mathbf{kr}\right],$$
$$B_{ww}(\mathbf{R},w) = \int d\mathbf{k}\frac{k_z^2}{k^2}\left[E_1^p(k)\cos\mathbf{kr}+E_2^p(k)\sin\mathbf{kr}\right]. \tag{22}$$

The second derivative of the correlation function at the coordinate's origin is

$$-\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\gamma \partial R_\delta} = \int d\mathbf{k}\frac{K_\alpha K_\beta K_\gamma K_\delta}{k^2}E_1^p(k) = \frac{1}{8}\int d\mathbf{k}E_1^p(k)\frac{K^4}{k^2} \times(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{23}$$

The other correlations satisfy

$$\begin{aligned}\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi} &= \frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi} = \frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial R_\gamma \partial R_\gamma} = -\int d\mathbf{k}E_1^p(k)\frac{K^2k_z^2}{k^2},\\ \frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial \xi \partial \xi} &= -\int d\mathbf{k}E_1^p(k)\frac{k_z^4}{k^2}.\end{aligned} \tag{24}$$

III. DERIVATION OF PROBABILITY DENSITY EQUATION

The cases considered imply that the joint surface elevation probability density and its gradient are indicator function (5), averaged over an ensemble of the random velocity field realizations $\mathbf{u}(\mathbf{r},t)$, i.e.,

$$P(\mathbf{R},t;\xi,\mathbf{p}) = \langle\varphi(\mathbf{R},t;\xi,\mathbf{p})\rangle_{\mathbf{u}}. \tag{25}$$

Now one can obtain an equation relating the joint surface elevation probability density and its gradient. To do that, one averages Liouville equation (6) over an ensemble of the random velocity field realization $\mathbf{u}(\mathbf{r},t)$. To divide velocity correlations $\mathbf{u}(\mathbf{r},t)$, one can make use of the Furutsu-Novikov theorem [6,29,30], accounting for sea bottom irregularities $H(\mathbf{R})$ [see Fig. 1 and Eq. (4)] [7]:

$$\langle u_i(\mathbf{R},\xi,t)R[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{\xi},\tau)]\rangle_{\mathbf{u}} = \int d\mathbf{R}'\int_{-\infty}^{\xi+0}\theta[\xi'-H(\mathbf{R})]d\xi'\int_0^t dt'B_{ij} \times(\mathbf{R}-\mathbf{R}',\xi-\xi',t-t')\left\langle\frac{\delta R\left[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)\right]}{\delta u_j(\mathbf{R}',\xi',t')}\right\rangle_{\mathbf{u}}. \tag{26}$$

The bottom topography has the mean value $\langle H(\mathbf{R})\rangle=-H$, where H is the mean depth of the sea. Taking into account that topographic irregularities are statistically independent on the hydrodynamical velocity field and averaging (26) over an ensemble of realizations, one gets

$$\langle u_i(\mathbf{R},\xi,t)R[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{\xi},\tau)]\rangle_{\mathbf{u}} = \int d\mathbf{R}'\int_{-\infty}^{\xi+0}\langle\theta(\xi'-H(\mathbf{R}))\rangle_H d\xi'\int_0^t dt'B_{ij} \times(\mathbf{R}-\mathbf{R}',\xi-\xi',t-t')\left\langle\frac{\delta R\left[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)\right]}{\delta u_j(\mathbf{R}',\xi',t')}\right\rangle_{\mathbf{u}}, \tag{27}$$

where function $\langle\theta[\xi-H(\mathbf{R})]\rangle_H$ is an integral probability function of the topographic irregularity distribution $H(\mathbf{R})$, i.e., it is the probability $P\{\xi>H(\mathbf{R})\}$. In the case of a statistically uniform random field $H(\mathbf{R})$, this function does not depend on spatial coordinates $\mathbf{R}$, and then we get $\langle\theta[\xi - H(\mathbf{R})]\rangle_H = P\{\xi > H(\mathbf{R})\} = P(H;\xi)$. In the case of an infinitely deep sea model (when $H \to \infty$ in Fig. 1), function $P(H;\xi) \to 1$.

Given the diffusion approach [6,7], Eq. (27) simplifies to the form

$$\langle u_i(\mathbf{R},\xi,t)R[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{\xi},\tau)]\rangle_{\mathbf{u}} = \int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{ij}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta R[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_j(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}},$$

where function $B_{ij}(\mathbf{r})$ is governed by Eq. (8), i.e.,

$$B_{ij}(\mathbf{R},\xi) = \int_0^\infty d\tau B_{ij}(\mathbf{R},\xi,\tau). \tag{28}$$

Particularly,

$$\langle u_\alpha(\mathbf{R},\xi,t)\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{\xi},\tau)]\rangle_{\mathbf{u}} = \int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}$$
$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}},$$

$$\langle w(\mathbf{R},\xi,t)\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{\xi},\tau)]\rangle_{\mathbf{u}} = \int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{w\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}$$
$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0} d\xi' P(H;\xi') B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}},$$

where $\mathbf{u}(\mathbf{r},t)$ $(k = \sqrt{\mathbf{K}^2 + k_z^2})$ are correlation functions of the random hydrodynamical velocity field,

$$B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi') = \langle u_\alpha(\mathbf{R},\xi;t)u_\beta(\mathbf{R}',\xi';t')\rangle = \int_0^\infty d\tau \int d\mathbf{K}dk_z\, E_{\alpha\beta}(k,\tau)e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')}, \tag{29}$$

$$B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi') = \langle u_\alpha(\mathbf{R},\xi;t)w(\mathbf{R}',\xi';t')\rangle = \int_0^\infty d\tau \int d\mathbf{K}dk_z\, E_{\alpha z}(k,\tau)e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')}, \tag{30}$$

$$B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi') = \langle w(\mathbf{R},\xi;t)w(\mathbf{R}',\xi';t')\rangle = \int_0^\infty d\tau \int d\mathbf{K}dk_z\, E_{zz}(k,\tau)e^{i\mathbf{K}(\mathbf{R}-\mathbf{R}')+ik_z(\xi-\xi')}. \tag{31}$$

Now the probability density equation (25) transforms to

$$\begin{aligned}\frac{\partial P(t;\xi,\mathbf{p})}{\partial t} = &-\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')\frac{\partial}{\partial R_\alpha}\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle\\
&-\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi')\frac{\partial}{\partial R_\alpha}\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle\\
&+\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' \frac{\partial B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial\xi}p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle\\
&+\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' \frac{\partial B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial\xi}p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle\\
&-\frac{\partial}{\partial\xi}\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{w\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle\\
&-\frac{\partial}{\partial\xi}\int d\mathbf{R}'\int_{-\infty}^{\xi+0} P(H;\xi')d\xi' B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi')\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle\end{aligned} \tag{32}$$

$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial R_\gamma}\frac{\partial}{\partial p_\gamma}p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial R_\gamma}\frac{\partial}{\partial p_\gamma}p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{\alpha\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$+\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{\alpha w}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\alpha\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$-\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{w\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial R_\beta}\frac{\partial}{\partial p_\beta}\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$-\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial R_\beta}\frac{\partial}{\partial p_\beta}\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$-\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{w\beta}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial \xi}\frac{\partial}{\partial p_\beta}p_\beta\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle$$
$$-\int d\mathbf{R}'\int_{-\infty}^{\xi+0}P(H;\xi')d\xi'\frac{\partial B_{ww}(\mathbf{R}-\mathbf{R}',\xi-\xi')}{\partial \xi}\frac{\partial}{\partial p_\beta}p_\beta\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle. \quad (33)$$

Then, rewriting Liouville equation (6) in an integral form, one obtains the equations featuring the corresponding variational derivatives [7,13]:

$$\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}=\delta(\mathbf{R}-\mathbf{R}')\left[-\delta(\xi-\xi')\frac{\partial}{\partial R_\beta}+\frac{\partial\delta(\xi-\xi')}{\partial\xi}p_\beta\right]P(\mathbf{R},t;\xi,\mathbf{p})$$
$$+\frac{\partial\delta(\mathbf{R}-\mathbf{R}')}{\partial R_\gamma}\delta(\xi-\xi')\frac{\partial}{\partial p_\gamma}p_\beta P(\mathbf{R},t;\xi,\mathbf{p})+\delta(\mathbf{R}-\mathbf{R}')\frac{\partial\delta(\xi-\xi')}{\partial\xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\beta P(\mathbf{R},t;\xi,\mathbf{p}),$$
$$\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}=-\frac{\partial}{\partial\xi}\delta(\mathbf{R}-\mathbf{R}')\delta(\xi-\xi')P(\mathbf{R},t;\xi,\mathbf{p})$$
$$-\frac{\partial\delta(\mathbf{R}-\mathbf{R}')}{\partial R_\gamma}\delta(\xi-\xi')\frac{\partial}{\partial p_\gamma}P(\mathbf{R},t;\xi,\mathbf{p})-\delta(\mathbf{R}-\mathbf{R}')\frac{\partial\delta(\xi-\xi')}{\partial\xi}\frac{\partial}{\partial p_\gamma}p_\gamma P(\mathbf{R},t;\xi,\mathbf{p}). \quad (34)$$

Provided the uniform initial conditions $\xi_0(\mathbf{R})=0$ and $\mathbf{p}_0(\mathbf{R})=0$, all the single-point statistical characteristics do not depend on the spatial coordinate $\mathbf{R}$, i.e.,

$$P(\mathbf{R},t;\xi,\mathbf{p})\equiv P(t;\xi,\mathbf{p}).$$

Formulae (34) can be simplified

$$\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta u_\beta(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}=\delta(\mathbf{R}-\mathbf{R}')\frac{\partial\delta(\xi-\xi')}{\partial\xi}p_\beta P(t;\xi,\mathbf{p})+\frac{\partial\delta(\mathbf{R}-\mathbf{R}')}{\partial R_\gamma}\delta(\xi-\xi')\frac{\partial}{\partial p_\gamma}p_\beta P(t;\xi,\mathbf{p})$$
$$+\delta(\mathbf{R}-\mathbf{R}')\frac{\partial\delta(\xi-\xi')}{\partial\xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\beta P(t;\xi,\mathbf{p}), \quad (35)$$

$$\left\langle\frac{\delta\varphi[\mathbf{u}(\widetilde{\mathbf{R}},\widetilde{z},\tau)]}{\delta w(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{u}}=-\frac{\partial}{\partial\xi}\delta(\mathbf{R}-\mathbf{R}')\delta(\xi-\xi')P(t;\xi,\mathbf{p})-\frac{\partial\delta(\mathbf{R}-\mathbf{R}')}{\partial R_\gamma}\delta(\xi-\xi')\frac{\partial}{\partial p_\gamma}P(t;\xi,\mathbf{p})$$
$$-\delta(\mathbf{R}-\mathbf{R}')\frac{\partial\delta(\xi-\xi')}{\partial\xi}\frac{\partial}{\partial p_\gamma}p_\gamma P(t;\xi,\mathbf{p}). \quad (36)$$

One can further simplify equations for the function $P(t;\xi,\mathbf{p})$ by using the scalar probability density $P(t;\xi,p)$ or $P(t;\xi,I)$, where $p=|\mathbf{p}|$, $I=|\mathbf{p}|^2$.

From now on, we concern ourselves with the case of uniform initial conditions. Spatial irregularities imply diffusion in the $\{\mathbf{r}\}$ space. However, our focus is on spatial structure forming due to diffusion in the phase space $\{\xi,\mathbf{p}\}$. Then, incorporating Eqs. (35)

and (36) into Eq. (33) and integrating it over spatial coordinates $\mathbf{R}'$, ξ', one can derive the required equation

$$\begin{aligned}\frac{\partial P(t;\xi,\mathbf{p})}{\partial t} &= -\frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\alpha \partial R_\beta}\frac{\partial}{\partial p_\gamma}p_\gamma P(t;\xi,\mathbf{p}) + \mathrm{P}(H;\xi)\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma P(t;\xi,\mathbf{p})\\
&\quad -\frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\beta^2 P(t;\xi,\mathbf{p}) + \frac{1}{2}P(H;\xi)\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma P(t;\xi,\mathbf{p})\\
&\quad + B_{ww}(\mathbf{0},0)\frac{\partial^2}{\partial \xi^2}[\mathrm{P}(H;\xi)P(t;\xi,\mathbf{p})] + B_{ww}(\mathbf{0},0)\frac{\partial}{\partial \xi}\left\{\frac{\partial \mathrm{P}(H;\xi)}{\partial \xi}\frac{\partial}{\partial p_\delta}p_\delta P(t;\xi,\mathbf{p})\right\}\\
&\quad - \mathrm{P}(H;\xi)\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\gamma \partial R_\delta}\frac{\partial}{\partial p_\gamma}p_\alpha\frac{\partial}{\partial p_\delta}p_\beta P(t;\xi,\mathbf{p}) + \frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\left[\frac{\partial}{\partial p_\gamma}p_\gamma + \frac{\partial}{\partial p_\gamma}p_\gamma\frac{\partial}{\partial p_\delta}p_\delta\right]P(t;\xi,\mathbf{p})\\
&\quad -\frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma\frac{\partial}{\partial p_\delta}p_\delta p_\beta p_\beta P(t;\xi,\mathbf{p}) + \frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma p_\delta\frac{\partial}{\partial p_\delta}P(t;\xi,\mathbf{p})\\
&\quad + \frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{w\beta}(\mathbf{0},0)}{\partial R_\beta \partial \xi}\left[\frac{\partial}{\partial p_\gamma}p_\gamma + \frac{\partial}{\partial p_\gamma}\frac{\partial}{\partial p_\delta}p_\delta p_\gamma\right]P(t;\xi,\mathbf{p}) - \frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial R_\gamma \partial R_\gamma}\frac{\partial}{\partial p_\delta}\frac{\partial}{\partial p_\delta}P(t;\xi,\mathbf{p})\\
&\quad + \frac{1}{2}\mathrm{P}(H;\xi)\frac{\partial^2 B_{w\beta}(\mathbf{0},0)}{\partial \xi \partial R_\beta}\frac{\partial}{\partial p_\gamma}p_\gamma\frac{\partial}{\partial p_\delta}p_\delta P(t;\xi,\mathbf{p}) - \mathrm{P}(H;\xi)\frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial \xi \partial \xi}\left[\frac{\partial}{\partial p_\gamma}p_\gamma + \frac{\partial}{\partial p_\gamma}p_\gamma\frac{\partial}{\partial p_\delta}p_\delta\right]P(t;\xi,\mathbf{p}).\end{aligned} \tag{37}$$

Since we make use of the kinematic approximation, random bottom irregularities appear in the probability density equation as a monotonic function, $P(H;\xi)$, ranging from 0 to 1. Irregularities appear also in the diffusion coefficients. Therefore, these functions have maximal influence in the case of the infinitely deep sea, i.e., if $P(\infty;\xi) = 1$. We stress that the influence of the bottom irregularities on forming the hydrodynamical velocity field is not in question.

Given the approximation, we study in the next section how a complex stochastic velocity field structure affects the statistical properties of the sea surface elevation and its gradient.

IV. STATISTICAL ANALYSIS OF PROBABILITY DENSITY EQUATIONS

A. Surface elevation probability density

First, we consider the case when the sea surface elevation does not correlate with its gradient. By integrating Eq. (37) over $\mathbf{p}$, one arrives at an unexpected conclusion in

$$\frac{\partial}{\partial t}P(t;\xi) = B_{ww}(\mathbf{0},0)\frac{\partial^2}{\partial \xi^2}P(t;\xi). \tag{38}$$

So the probability density of the surface elevation random field $\xi(\mathbf{R},t)$ is the Gaussian distribution

$$P(t;\xi) = \frac{1}{\sqrt{4\pi B_{ww}(\mathbf{0},0)t}}\exp\left\{-\frac{\xi^2}{4B_{ww}(\mathbf{0},0)t}\right\},$$

not depending on nonlinearity of initial equation (2) with the variance

$$\langle \xi^2(\mathbf{R},t)\rangle = 2B_{ww}(\mathbf{0},0)t.$$

The diffusion coefficient $B_{ww}(\mathbf{0},0)$ in Eq. (38) is naturally linked to the variance of the random hydrodynamical velocity field, $\mathbf{u}(\mathbf{r},t)$.

Then from (19), we have

$$B_{ww}(\mathbf{0},0) = \int d\mathbf{k}\, E^s(k,0)\left(1 - \frac{k_z^2}{k^2}\right).$$

Transiting to $\mathbf{k}$, $k_z = k\cos\theta$, $K^2 = k^2\sin^2\theta$, integrating over the angle variables, and using Eq. (12), one gets the equality for hydrodynamic turbulence,

$$B_{ww}(\mathbf{0},0) = \frac{2}{3}\int d\mathbf{k}\, E^s(k,0) = \frac{1}{3}\sigma_{\mathbf{u}}^2.$$

By analogy, in case of wave turbulence, using Eq. (15) and Eq. (16), one gets the equation

$$B_{ww}(\mathbf{0},0) = \frac{1}{3}\int d\mathbf{k}\, E^p(k,0) = \frac{1}{3}\sigma_{\mathbf{u}}^2.$$

Summarizing, one can draw a conclusion that the complex structure of a hydrodynamic velocity field in the deep sea cannot be a *direct cause* of a stochastic structure on the sea surface.

B. Probability density of the surface elevation gradient

By integrating Eq. (37) over ξ, in the case of the deep sea, one arrives at the equation

$$\begin{aligned}\frac{\partial}{\partial t}P(t;\mathbf{p}) &= -\frac{1}{2}\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\alpha \partial R_\beta}\frac{\partial}{\partial p_\delta}p_\delta P(t;\mathbf{p})\\
&\quad + 2\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma\left(1 + \frac{\partial}{\partial p_\delta}p_\delta\right)P(t;\mathbf{p})\\
&\quad - \frac{1}{2}\frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma\left(1 + \frac{\partial}{\partial p_\delta}p_\delta\right)p_\beta^2 P(t;\mathbf{p})\\
&\quad - \frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\gamma \partial R_\delta}\frac{\partial}{\partial p_\gamma}p_\alpha\frac{\partial}{\partial p_\delta}p_\beta P(t;\mathbf{p})\\
&\quad - \frac{1}{2}\frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial R_\gamma \partial R_\gamma}\frac{\partial}{\partial p_\delta}\frac{\partial}{\partial p_\delta}P(t;\mathbf{p})\\
&\quad - \frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial \xi \partial \xi}\frac{\partial}{\partial p_\gamma}p_\gamma\left(1 + \frac{\partial}{\partial p_\delta}p_\delta\right)P(t;\mathbf{p}),\end{aligned} \tag{39}$$

where $\mathbf{p}(\mathbf{R},t)$ is the probability density of the surface elevation gradient.

To simplify, we go from Eq. (39) to the random field $I(\mathbf{R},t) = \mathbf{p}^2(\mathbf{R},t)$, multiply (39) by $\delta(p_\mu^2 - I)$, and integrate over $\mathbf{p}$. As a result, we obtain the following equation, with $P(t;I)$ being the probability density:

$$\frac{\partial}{\partial t}P(t;I) = \frac{\partial}{\partial I}I\left[-\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\alpha \partial R_\beta} + 4\frac{\partial^2 B_{\alpha w}(\mathbf{0},0)}{\partial R_\alpha \partial \xi}\left(1 - 2\frac{\partial}{\partial I}I\right) + \frac{\partial^2 B_{\alpha\alpha}(\mathbf{0},0)}{\partial \xi \partial \xi}\left(1 + 2\frac{\partial}{\partial I}I\right)I\right]P(t;I)$$
$$+2\frac{\partial^2 B_{\alpha\beta}(\mathbf{0},0)}{\partial R_\gamma \partial R_\delta}\frac{\partial}{\partial I}\left[\int d\mathbf{p}(\delta_{\gamma\delta}p_\alpha p_\beta + \delta_{\alpha\delta}p_\gamma p_\beta)\delta\left(p_\mu^2 - I\right)P(t;I) - 2\frac{\partial}{\partial I}\int d\mathbf{p}\, p_\gamma p_\alpha p_\delta p_\beta \delta\left(p_\mu^2 - I\right)P(t;I)\right]$$
$$-2\frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial R_\gamma \partial R_\gamma}\frac{\partial}{\partial I}I\frac{\partial}{\partial I}P(t;I) - 2\frac{\partial^2 B_{ww}(\mathbf{0},0)}{\partial \xi \partial \xi}\frac{\partial}{\partial I}I\left(1 + 2\frac{\partial}{\partial I}I\right)P(t;I). \tag{40}$$

To carry out further calculations, one needs to know the correlation function structure, which varies depending on the application.

1. Hydrodynamic turbulence

Now we incorporate Eqs. (20) and (21) into Eq. (40) and use spherical coordinates for the vector $\mathbf{k}$. Then integrating over the angle variables, we get the final equation for the probability density, $P(t;I)$,

$$\frac{\partial}{\partial \tau}P(\tau;I) = \frac{\partial}{\partial I}I\left(1 + 4\frac{\partial}{\partial I}I + I\right)P(\tau;I)$$
$$+2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}I^2 P(\tau;I) + 2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}P(\tau;I), \tag{41}$$

where $\tau = \frac{8}{15}D^s t$ is the dimensionless time, while the variable D^s satisfies Eq. (10), i.e.,

$$D^s = \int d\mathbf{k}\, k^2 E^s(k) = 4\pi\int_0^\infty dk\, k^4 E^s(k)$$
$$= \frac{1}{2}\int_0^\infty d\tau\, \langle \boldsymbol{\omega}(\mathbf{r},t+\tau)\boldsymbol{\omega}(\mathbf{r},t)\rangle, \tag{42}$$

where $\boldsymbol{\omega}(\mathbf{r},t) = \operatorname{curl}\mathbf{u}(\mathbf{r},t)$ is the velocity field curl.

Equation (41) can be represented in the form

$$\frac{\partial}{\partial \tau}P(\tau;I) = \frac{\partial}{\partial I}I(1+I)P(\tau;I)$$
$$+2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}(1+I)^2 P(\tau;I).$$

2. Wave turbulence

Incorporating the second derivatives of the velocity field correlation functions (23) and (24) into Eq. (40), we use spherical coordinates for the vector $\mathbf{k}$. Then integrating over the angle variables, we derive the equation for the probability density $P(t;I)$. It is interesting that this equation precisely coincides with the equation for hydrodynamical turbulence, i.e.,

$$\frac{\partial}{\partial \tau}P(\tau;I) = \frac{\partial}{\partial I}I P(\tau;I) + 4\frac{\partial}{\partial I}I\frac{\partial}{\partial I}I P(\tau;I)$$
$$+\frac{\partial}{\partial I}I^2 P(\tau;I) + 2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}I^2 P(\tau;I)$$
$$+2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}P(\tau;I), \tag{43}$$

where $\tau = \frac{2}{15}D^p t$ is the dimensionless time, while the variable D^p satisfies Eq. (10), i.e.,

$$D^p = \int d\mathbf{k}\, k^2 E_1^p(k) = 4\pi\int_0^\infty dk\, k^4 E_1^p(k)$$
$$= \int_0^\infty d\tau\left\langle \frac{\partial \mathbf{u}(\mathbf{r},t+\tau)}{\partial \mathbf{r}}\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}\right\rangle, \tag{44}$$

with $\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}$ being the velocity field divergence.

V. STATISTICAL TOPOGRAPHY OF RANDOM FIELD FOR THE SURFACE ELEVATION GRADIENT

First, we note that the probability density equation for the random field $I(\tau;\mathbf{R})$, described above, provided there is a spatial uniformity, governs the statistical properties of the surface elevation gradient field. At any fixed point in the space $\widetilde{\mathbf{R}}$, function $I(\tau;\widetilde{\mathbf{R}})$ is a random process in time with the simultaneous probability density independent of $\widetilde{\mathbf{R}}$ governed by the obtained equations. In the physical space $\{\mathbf{R}\}$, there may appear a structure of the field $I(\tau;\mathbf{R}) = |\mathbf{p}(\tau;\mathbf{R})|^2$ as a physical object. This structure appears as closed regions with a high gradient concentration, conventionally called clustering. The equations describing this process are fairly complex due to two *concurrent* phenomena. On one hand, the field $I(\tau;\mathbf{R})$ is induced by a random Gaussian field from the initial condition; on the other hand, there is a parametric excitation of the field. Certain statistical parameters, characterizing this excitation within separate realizations of the random field, can be obtained analytically.

A. Moment equation and the Lyapunov characteristic parameter

Consider the n-th moment function the $I(\tau;\mathbf{R})$-$\langle I^n(\tau;\mathbf{R})\rangle$. By multiplying the probability density equation by I^n, and integrating over I, one obtains

$$\frac{\partial}{\partial \tau}\langle I^n(\tau;\mathbf{R})\rangle = n(4n-1)\langle I^n(\tau;\mathbf{R})\rangle + n(2n-1)$$
$$\times\langle I^{n+1}(\tau;\mathbf{R})\rangle + 2n^2\langle I^{n-1}(\tau;\mathbf{R})\rangle. \tag{45}$$

In particular, as $n = 1$, $\langle I(\tau;\mathbf{R})\rangle = \sigma_{\mathbf{p}}^2(\tau;\mathbf{R})$, the surface elevation gradient variance is

$$\frac{\partial}{\partial \tau}\langle I(\tau;\mathbf{R})\rangle = 3\langle I(\tau;\mathbf{R})\rangle + \langle I^2(\tau;\mathbf{R})\rangle + 2.$$

It means that there is no steady-state distribution as $\tau \to \infty$. However, formally there is a steady-state probability distribution,

$$P(\infty,I)=\frac{3}{2}\frac{1}{(1+I)^{5/2}}. \tag{46}$$

It seems that only the very *tail* of this distribution coincides to some extent with a flat *tail* of the probability distribution at large time.

By analogy, one obtains the equation for the Lyapunov characteristic parameter, $\frac{\partial}{\partial\tau}\langle\ln I(\tau)\rangle$,

$$\frac{\partial}{\partial\tau}\langle\ln I(\tau;\mathbf{R})\rangle=-1-\langle I(\tau;\mathbf{R})\rangle,$$

i.e., the *Lyapunov exponent* decreases rapidly in time as $\tau\to\infty$,

$$e^{\langle\ln I(\tau;\mathbf{R})\rangle}\sim\exp\left\{-\tau-\int_0^\tau d\tau'\langle I(\tau';\mathbf{R})\rangle\right\}. \tag{47}$$

The field $I(\tau;\mathbf{R})$ decreases rapidly almost at every point in space as time τ increases, therefore the field may clusterize in small space areas.

B. Integral probability distribution equation

The integral probability distribution function for the probability density $P(\tau;I)$ is defined as

$$\Phi(\tau,I)=\int_0^I dI'P(\tau;I')=\langle\theta\left(I-I(\tau;\mathbf{R})\right)\rangle_{\mathbf{u}}, \tag{48}$$

and it is the probability of an event $P(I(\tau;\mathbf{R})<I)$. Due to parametric excitation, the function rapidly approaches unity as time increases. So the value

$$\begin{aligned}s_{\rm hom}\left(\tau,\overline{I}\right)&=\langle\theta(I(\tau;\mathbf{R})-\overline{I})\rangle_{\mathbf{u}}\\&=1-\langle\theta(\overline{I}-I(\tau;\mathbf{R}))\rangle_{\mathbf{u}}\end{aligned} \tag{49}$$

is the probability of an event $P[I(\tau;\mathbf{R})>\overline{I}]$. Then, in accordance with statistical topography theory [6–9,13,28]), this function in the case of a statistically uniform field delineates geometrically a specific area, where the field $I(\tau;\mathbf{R})$ exceeds any given value $\overline{I}$. As the field decreases almost at every point in space, this probability approaches zero, indicating that the basic statistic characteristics (moments) concentrate inside this small area. As the specific gradient is squared, localized inside the area, and satisfies the equation

$$\langle i(\tau;I>\overline{I})\rangle=\langle I(\tau)\rangle-\int_0^{\overline{I}}dI\,IP(\tau;I), \tag{50}$$

this means that as this small area decreases further, the intense gradient pushes water mass upward, creating *high narrow structures*, as well as pushes them downward, inducing *relatively short narrow gaps*. This corresponds to very rare intense fluctuations of the surface elevation Gaussian field $\xi(\mathbf{R},t)$. These results can be verified by means of numerical simulations addressing the probability density and integral probability distribution.

By incorporating relation $P(\tau;I)=\frac{\partial}{\partial I}\Phi(\tau,I)$ into Eqs. (41) and (43), and integrating over I, one arrives at the following equation with the integral probability distribution:

$$\begin{aligned}\frac{\partial}{\partial\tau}\Phi(\tau,I)&=I\frac{\partial}{\partial I}\Phi(\tau,I)+4I\frac{\partial}{\partial I}I\frac{\partial}{\partial I}\Phi(\tau,I)+I^2\frac{\partial}{\partial I}\Phi(\tau,I)\\&\quad+2I\frac{\partial}{\partial I}I^2\frac{\partial}{\partial I}\Phi(\tau,I)+2I\frac{\partial^2}{\partial I^2}\Phi(\tau,I)\end{aligned} \tag{51}$$

with the initial condition $\Phi(0,I)=1$ and boundary conditions

$$\Phi(\tau,0)=0,\quad\Phi(\tau,\infty)=1.$$

Equation (51) can be rewritten as

$$\frac{\partial}{\partial\tau}\Phi(\tau,I)=I(1+I)\frac{\partial}{\partial I}\Phi(\tau,I)+2I\frac{\partial}{\partial I}(1+I)^2\frac{\partial}{\partial I}\Phi(\tau,I). \tag{52}$$

It is also worth noting that a *typical realization curve* $I^*(\tau)$, which is the main characteristic of the random process evolution in time, is defined as

$$\Phi[\tau,I^*(\tau)]=\tfrac{1}{2}. \tag{53}$$

VI. CONCLUSION

The main result of the paper is that we, first, derived equations relating the joint simultaneous probability density for a perturbed surface field $\xi(\mathbf{R},t)$, and its gradient $\mathbf{p}(\mathbf{R},t)$ under the influence of the vertical component in the hydrodynamic velocity field. Second, we have analyzed statistical features of these equations. In the case of a spatially uniform medium as in the deep sea, we have obtained that the surface perturbations are a Gaussian random field. This means that the hydrodynamic field complex structure cannot directly cause stochastic structure formation on the sea surface. The fields $\xi(\mathbf{R},t)$ and $\mathbf{p}(\mathbf{R},t)$ do not correlate, as the equation for the probability density, and the integral probability functions (41) and (48) for a positive field $I(\mathbf{R},t)=\mathbf{p}^2(\mathbf{R},t)$, are intricate and do not depend on hydrodynamical-velocity-field statistical structure. Then, making use of the equations, we have a recurrence equation for the field $I(\mathbf{R},t)$ moments and an equation for the Lyapunov characteristic exponent. These equations show that all the statistical moments increase in time, as the Lyapunov exponent asymptote rapidly decreases at long times.

On one hand, at every point in the space $\widetilde{\mathbf{R}}$, the random function $I(\tau;\widetilde{\mathbf{R}})$ is a random process in time. Its simultaneous probability density is independent of $\widetilde{\mathbf{R}}$, satisfying Eq. (41), which is the Fokker-Plank equation of a Markov process. Time evolution of this process is described by a *typical realization curve* [6,7], which unfortunately cannot be expressed analytically. An analysis of the behavior of the Lyapunov exponent points out indirectly that a random process realization $I(\tau;\widetilde{\mathbf{R}})$ decreases almost at every point in space with the unit probability. Therefore, the field $I(\tau;\mathbf{R})$ begins to clusterize in some small volume in space, with rare but intense fluctuations (intermittence) about a typical realization curve. These fluctuations make all the moment functions grow in time.

On the other hand, in the physical space $\{\mathbf{R}\}$, there may appear a structure forming inside the field $I(\tau;\mathbf{R})=|\mathbf{p}(\tau;\mathbf{R})|^2$ as closed areas with a high gradient concentration (clustering) which is also governed by the derived equations, (41) and (48). This suggests that as this small area decreases further, the large gradient pushes water mass upward, creating *high narrow structures*, as well as pushes them downward, inducing *relatively short narrow gaps*. This corresponds to very rare intense fluctuations of the surface elevation Gaussian field $\xi(\mathbf{R},t)$ with a typical realization curve coinciding with the mean value to be zero.

Equations (41) and (48) are intricate, possessing two major phenomena. First, the field $I(\tau;\mathbf{R})$ is generated due to a random Gaussian velocity field, and, second, there is a parametric excitation at work in the system. As a concluding remark, it is worth noting that equations similar to (41) and (43) have been obtained in the problem of turbulent dynamos for a hydrodynamic flow with possible analytical solutions [6,7,31,32]. Relevant calculations can be found in the appendix.

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APPENDIX: THE SIMPLEST TURBULENT DYNAMO PROBLEM

The simplest permissible equation in a dimensionless form is

$$\frac{\partial}{\partial\tau}P(\tau;E)=\left(\frac{\partial}{\partial E}E+2\frac{\partial}{\partial E}E\frac{\partial}{\partial E}E+2\frac{\partial}{\partial E}E\frac{\partial}{\partial E}\right)\times P(\tau;E) \quad \text{(A1)}$$

with the initial condition $P(0;E)=\delta(E-\beta)$.

This equation appears in the problem of energy clustering in the magnetic field of a random hydrodynamic field $\mathbf{u}(\mathbf{r},t)$, with nondivergent magnetic field $\mathbf{H}(\mathbf{r},t)$ [6,7,31,32]),

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\mathbf{u}(\mathbf{r},t)\right]\mathbf{H}(\mathbf{r},t)=\left[\mathbf{H}(\mathbf{r},t)\frac{\partial}{\partial\mathbf{r}}\right]\mathbf{u}(\mathbf{r},t), \quad \text{(A2)}$$

with uniform initial condition $\mathbf{H}(\mathbf{r},0)=\mathbf{H}_0$.

Let us consider a model hydrodynamic field $\mathbf{u}(\mathbf{r},t)=\mathbf{v}(t)f(\mathbf{kr})$, where $\mathbf{v}(t)$ is a Gaussian vector random process, the *white noise*, and $f(\mathbf{kr})$ is a periodic function. Choosing the x axis along vector $\mathbf{k}$, one can see that the velocity field becomes dependent only on one spatial coordinate, $f(\mathbf{kr})=f(kx)$. In this case, the x component of the magnetic field is conserved, i.e., $H_x(\mathbf{r},t)=H_{x0}$, where the transversal component of the magnetic field $\mathbf{H}_\perp(x,t)$ satisfies the equation

$$\left[\frac{\partial}{\partial t}+v_x(t)\frac{\partial}{\partial x}f(x)\right]\mathbf{H}_\perp(x,t)=\mathbf{v}_\perp(t)\frac{\partial f(x)}{\partial x}H_{x0}, \quad \text{(A3)}$$

with the initial condition $\mathbf{H}_\perp(x,0)=\mathbf{H}_{\perp 0}$. The probability density of the component is governed by Eq. (A1). The velocity field model, $\mathbf{u}(\mathbf{r},t)=\mathbf{v}(t)\sin 2(\mathbf{kr})$, was first introduced in Ref. [33], and it allowed us to obtain an analytical solution as a continuity equation for the scalar density field $\rho(\mathbf{r},t)$,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t)+\frac{\partial}{\partial\mathbf{r}}\mathbf{u}(\mathbf{r},t)\rho(\mathbf{r},t)=0,\quad \rho(\mathbf{r},0)=\rho_0,$$

and as well as a solution of Eq. (A3) for a vector magnetic field. Thus it helped to follow the emergence and evolution of the clustering process in separate realizations of the velocity field.

Equation (A1) is significantly simpler than the one obtained above, so it allowed us to get an asymptotic energy moment solution at large times,

$$\langle E^n(\tau)\rangle\sim A_n e^{n(2n-1)\tau},$$

which corresponds to the log-normal law with a Gaussian generation. The Lyapunov exponent was also estimated as

$$e^{\langle\ln E(\tau)\rangle}=\beta e^{-\tau}.$$

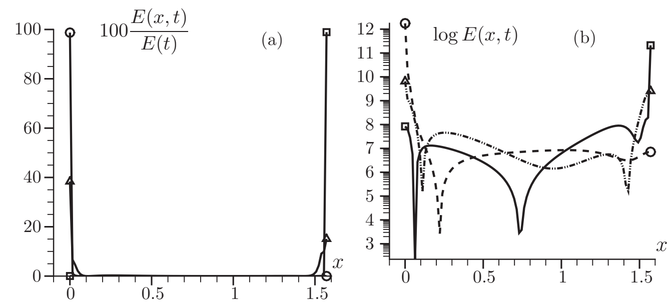

FIG. 2. The dynamics of a cluster disappearing at point 0 and its emergence at point $\pi/2$. The circle marks time $t=10.4$; (triangle) $t=10.8$, (square) $t=11.8$.

This means that the magnetic field decreases at almost every point in space or, in other words, a clustering occurs. This was simulated numerically as $\beta=0$ [6,7,31,32]. Results are also shown in Fig. 2(a). The figure demonstrates a part of the energy, contained within a cluster, divided by the total energy at a time. Figure 2(b) illustrates the dynamics of perturbations in the magnetic energy as they move from one boundary to another.

Equations (A1), (41), and (48) comprise terms corresponding to the Gaussian field generation, which then define energy generation at short times. For simplicity, we demonstrate this for a two-dimensional case.

The probability density of the Gaussian vector field $\mathbf{H}_\perp(\mathbf{R},t)$ in the case of a uniform problem is

$$\frac{\partial}{\partial\tau}P(\tau;\mathbf{H}_\perp)=\frac{1}{2}\frac{\partial^2}{\partial\mathbf{H}_\perp^2}P(\tau;\mathbf{H}_\perp),$$

with the solution $P(\tau;\mathbf{H}_\perp)=\frac{1}{2\pi\tau}\exp(-\frac{\mathbf{H}_\perp^2}{2\tau})$.

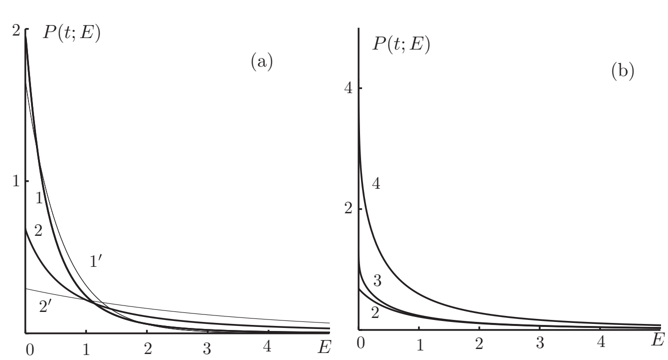

FIG. 3. Probability density (A1) at times (a) $\tau=0.3$ (curve 1) and $\tau=1.7$ (curve 2). Regular lines correspond to the Gaussian process (A4) $\tau=0.3$-1′, $\tau=1.7$-2′. (b) The same as in (a) but at times $\tau=1.7$-2, $\tau=5.0$-3, $\tau=8.3$-4.

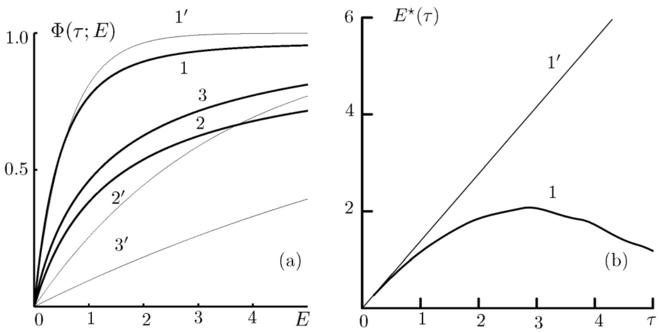

FIG. 4. (a) Integral probability function (A6) at times $\tau = 0.3$ (curve 1), $\tau = 1.7$ (curve 2), and $\tau = 5.0$ (curve 3); regular curves 1′, 2′, 3′ correspond to the Gaussian distribution. (b) Typical realization curves of the magnetic field energy, 1, and the Gaussian process, 1′.

Therefore, the probability density of the transversal energy $E = \mathbf{H}_\perp^2(\mathbf{R},t)$ is

$$P(\tau;E) = \frac{1}{2\tau}\exp\left(-\frac{E}{2\tau}\right), \qquad \text{(A4)}$$

which satisfies

$$\frac{\partial}{\partial\tau}P(\tau;E) = 2\frac{\partial}{\partial E}E\frac{\partial}{\partial E}P(\tau;E).$$

Hence, the integral probability distribution function satisfies

$$\frac{\partial}{\partial\tau}\Phi(\tau;E) = 2E\frac{\partial^2}{\partial E^2}\Phi(\tau;E),$$

with the solution

$$\Phi(\tau;E) = 1 - \exp\left(-\frac{E}{2\tau}\right). \qquad \text{(A5)}$$

The last relation yields the following relation for a typical realization curve:

$$E^*(\tau) = (2\ln 2)\tau.$$

Clustering occurs provided the curve decreases as compared to a linear growth in the Gaussian process. Figure 3 depicts numerical solutions of Eq. (A1).

The figure implies that from the beginning the probability density decreases similarly to the Gaussian distribution at a decreasing speed, thus the Gaussian field generation prevails. At time moments of order of $\tau = 1{,}7$, it drastically changes, and now clustering starts to playing a major role.

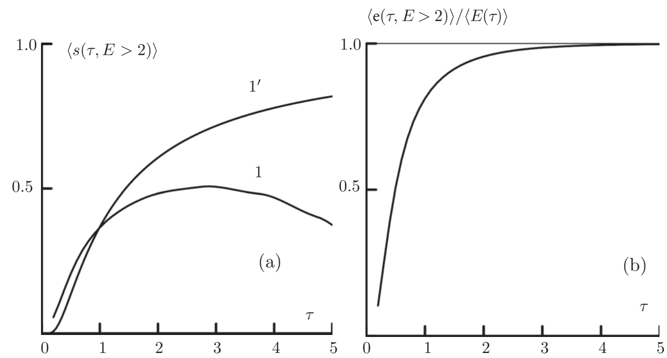

FIG. 5. (a) Specific area for distribution (A1), where the magnetic field energy exceeds 2. (b) Specific energy localized in this area.

Now we consider the integral probability function of the magnetic filed energy. From Eq. (A1), it follows that

$$\frac{\partial}{\partial\tau}\Phi(\tau;E) = \left[E + 2E\frac{\partial}{\partial E}(E+1)\right]\frac{\partial}{\partial E}\Phi(\tau;E). \qquad \text{(A6)}$$

Figure 4(a) shows its numerical solution, with the typical realization curve Fig. 4(b).

The figure indicates that at relatively short times, up to $\tau = 1{,}7$, the speed of growth of the integral function decreases, which is characteristic of the Gaussian process. At large times, the speed of growth increases as inherent to the log-normal distribution. Figure 4(b) shows that as $\tau \geqslant 3{,}0$, the typical realization curve of the process (A1) decreases, which indicates clustering. However, it should be stressed that the clustering begins earlier, but it begins to prevail over the generation at this time.

Figure 5(a) shows the time dependency of a specific area (49) of the regions where the magnetic field energy exceeds the maximal value on the typical realization curve, i.e., $E > 2$. In the case of a Gaussian distribution, this area approaches unity, signifying no clustering. In the case of a magnetic field, the area decreases, indicating the emergence of clustering at time $\tau \sim 1{,}7$. Figure 5(b) deals with dynamics of the specific energy (50) normalized to the total energy (50), which is localized inside these areas.

The results for a magnetic field qualitatively correspond to the surface elevation gradient field discussed in the main body of the paper. The effect of generation will be of the same order. Clustering will be more intense as follows from Lyapunov exponent equation (47), decreasing rapidly as compared to the magnetic field.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Solids* (John Wiley, New York, 1988).
[3] V. I. Klyatskin and A. I. Saichev, Sov. Phys. Usp. **35**, 231 (1992).
[4] M. B. Isichenko, Rev. Mod. Phys. **64**, 961 (1992).
[5] V. I. Klyatskin, *Stochastic Equations through the Eye of the Physicist: Basic Concepts, Exact Results and Asymptotic Approximations* (Elsevier, Amsterdam, 2005).
[6] V. I. Klyatskin, *Lectures on Dynamics of Stochastic Systems* (Elsevier, Amsterdam, 2010).
[7] V. I. Klyatskin, *Stochastic Equations: Theory and Applications in Acoustics, Hydrodynamics, Magnetohydrodynamics, and Radiophysics, Vol. 2*, Understanding Complex System (Springer, International Publishing, Switzerland, 2015).
[8] V. I. Klyatskin, Phys. Usp. **55**, 1152 (2012).
[9] V. I. Klyatskin, Rus. J. Math. Phys. **20**, 295 (2013).

[10] C. Kharif, E. Pelinovsky, and A. Slunyaev, *Rogue Waves in the Ocean* (Springer, Berlin, 2009).

[11] K. Dysthe, H. E. Krogstad, and P. Muller, Annu. Rev. Fluid Mech. **40**, 287 (2008).

[12] E. J. Heller, L. Kaplan, and A. Dahlen, J. Geophys. Res. **113**, C09023 (2008).

[13] V. I. Klyatskin, Theor. Math. Phys. **180**, 850 (2014).

[14] M. Shats, H. Punzmann, and H. Xia, Phys. Rev. Lett. **104**, 104503 (2010).

[15] W. M. Moslem, R. Sabry, S. K. El-Labany, and P. K. Shukla, Phys. Rev. E **84**, 066402 (2011).

[16] M. A. Topinka, B. J. LeRoy, R. M. Westervelt, S. E. J. Shaw, R. Fleischmann, E. J. Heller, K. D. Maranowski, and A. C. Gossard, Nature (London) **410**, 183 (2001).

[17] B. Liu and E. J. Heller, Phys. Rev. Lett. **111**, 236804 (2013).

[18] M. A. Wolfson and S. Tomsovic, J. Acoust. Soc. Am. **109**, 2693 (2001).

[19] C. Bonatto, M. Feyereisen, S. Barland, M. Giudici, C. Masoller, José R. Rios Leite, and J. R. Tredicce, Phys. Rev. Lett. **107**, 053901 (2011).

[20] J. Dudley, C. Finot, G. Millot, J. Garnier, G. Genty, D. Agafontsev, and F. Dias, Eur. Phys. J. Spec. Top. **185**, 125 (2010).

[21] D.R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature **450**, 1054 (2007).

[22] Y. A. Kravtsov, Rep. Prog. Phys. **55**, 39 (1992).

[23] S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Principles of Statistical Radiophysics: Wave Propagation through Random Media* (Springer, Berlin, 1989).

[24] M.-R. Alam, Geophys. Res. Lett. **41**, 8477 (2014).

[25] J. J. Metzger, R. Fleischmann, and T. Geisel, Phys. Rev. Lett. **112**, 203903 (2014).

[26] V. P. Ruban, JETP Lett. **97**, 686 (2013).

[27] V. E. Zakharov, and A. A. Gelash, Phys. Rev. Lett. **111**, 054101 (2013).

[28] V. I. Klyatskin, Theor. Math. Phys. **176**, 1252 (2013).

[29] K. Furutsu, J. Res. N.B.S. **D-67**, 303 (1963).

[30] E. A. Novikov, Sov. Phys. JETP **20**, 1290 (1965).

[31] V. I. Klyatskin, and O. G. Chkhetiani, JETP **109**, 345 (2009).

[32] V. I. Klyatskin, Phys. Usp. **54**, 441 (2011).

[33] V. I. Klyatskin and K. V. Koshel, Phys. Usp. **43**, 717 (2000).