Dynamics of a differential-difference integrable (2 + 1)-dimensional system

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A Kadomtsev-Petviashvili- (KP-) type equation appears in fluid mechanics, plasma physics, and gas dynamics. In this paper, we propose an integrable semidiscrete analog of a coupled (2 + 1)-dimensional system which is related to the KP equation and the Zakharov equation. N-soliton solutions of the discrete equation are presented. Some interesting examples of soliton resonance related to the two-soliton and three-soliton solutions are investigated. Numerical computations using the integrable semidiscrete equation are performed. It is shown that the integrable semidiscrete equation gives very accurate numerical results in the cases of one-soliton evolution and soliton interactions.

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I. INTRODUCTION

Nonlinear evolution equations (NEEs) appear in almost all branches of physics, such as fluid mechanics, plasma physics, optical fibers, and solid state physics. The nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction, and convection are very important in nonlinear wave equations. Since the concept of solitons for the Korteweg–de Vries (KdV) equation was introduced, there has been considerable interest in this kind of special NEE, such as the Burgers equation, the nonlinear Schrödinger (NLS) equation, and the Boussinesq equation.

Compared with one-dimensional NEEs, (2 + 1)-dimensional coupled systems are more attractive in describing nonlinear phenomena in real physical situations. Some (2 + 1)-dimensional NEEs exhibit not only localized coherent structures such as curved-line solitons, half-straight-line solitons, and dromions [1,2], but also inelastic interactions, e.g., resonance [3], reconnection [4], and annihilation [5].

Zakharov formulated the system of equations

$$iE_t + \frac{1}{2}E_{xx} - nE = 0, \tag{1}$$

$$n_{tt} - n_{xx} - 2(|E|^2)_{xx} = 0 \tag{2}$$

for the ionic sound wave under the action of a ponderomotive force due to a high-frequency field and for a Langmuir wave [6]. Here $Ee^{-i\omega_p t}$ is the normalized electric field of the Langmuir oscillation, n is the normalized density perturbation, x is the normalized spatial variable, t is the time variable, and the subscripts denote partial derivatives. For an ion sound wave propagating in only one direction, for example, in the positive x direction, one can suppose that

$$n_t \cong -n_x. \tag{3}$$

Under this assumption Eq. (2) can be simplified as follows:

$$n_t + n_x + (|E|^2)_x = 0. \tag{4}$$

The interactions of solitons of the system (1) and (4) were studied by the inverse scattering technique in [7].

In the present work, we consider the following (2 + 1)-dimensional soliton equation:

$$iu_t + u_{xx} + uv = 0, \tag{5a}$$

$$v_t + v_y + (|u|^2)_x = 0, \tag{5b}$$

where $i = \sqrt{-1}$, u is a complex function of two scaled space coordinates x and y and time t, and v is a real function. Equations (5) are similar to the integrable Zakharov equations (1) and (4) when $x = y$ in Eq. (5b). Maccari [8] obtained Eqs. (5) by an asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling from the Kadomtsev-Petviashvili (KP) equation. He also constructed the Lax pair for the system. The Painlevé property of the system (5) was investigated in [9] and its doubly periodic solutions were given by using the extended Jacobian elliptic function expansion method [10]. Traveling wave solutions of the system were obtained in [11,12]. The interaction dynamics between the two solitons, especially the soliton resonant interactions, was studied in [13]. However, it appears that N-soliton solutions of the system (5) have not been given by use of the Hirota method.

Over the decades, integrable discretizations of soliton equations have received considerable attention [14–17]. Ablowitz and Ladik proposed a method to construct integrable discrete analogs of soliton equations based on Lax pairs [18,19]. Hirota proposed a bilinear method to construct integrable discrete analogs of soliton equations based on bilinear equations [20–22]. Applications of integrable discretizations of soliton equations have been considered in various fields [23–27]. In our recent work, we proposed an integrable semidiscrete analog of coupled integrable dispersionless equations [28,29]. The key step there is the discretization of bilinear differential operators under gauge invariance. Considering the physical background and potential application of the (2 + 1)-dimensional system (5), we aim to study its semidiscrete analog and the dynamics of soliton solutions of the semidiscrete system.

The remainder of this paper is organized as follows. In Sec. II, we derive N-soliton solutions of the system (5) by use of the Hirota method. In Sec. III, we present a semidiscrete analog of the system in the spatial direction. In Sec. IV, numerical computations of the semidiscrete system are performed. Interactions of multisoliton solutions, especially the resonance

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of two solitons, are investigated by means of their asymptotic behaviors in Sec. V. Conclusions are given in Sec. VI. Finally we present the N-soliton solution of the semidiscrete system by the Pfaffian technique in the Appendix.

II. BILINEAR FORM AND SOLITON SOLUTIONS

Through the dependent-variable transformations

$$u = \frac{g}{f}, \quad v = 2(\ln f)_{xx}, \tag{6}$$

where g and f are complex and real functions of x, y, and t, respectively, the bilinear form of system (5) is expressed as

$$\left(iD_t + D_x^2\right) g \cdot f = 0, \tag{7}$$

$$(D_x D_t + D_x D_y) f \cdot f + g g^* = 0, \tag{8}$$

where the bilinear Hirota operators in the first term apply to a pair of functions. Here the bilinear differential operator is defined by [30]

$$D_{x_1}^{n_1} D_{x_2}^{n_2} a \cdot b \equiv \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1'}\right)^{n_1} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2'}\right)^{n_2} \times a(x_1,x_2) \cdot b(x_1',x_2')|_{x_1'=x_1, x_2'=x_2}. \tag{9}$$

In [13], one-soliton and two-soliton solutions of (7) and (8) were found. We get the result that the one-soliton solution can be expressed in the form

$$g = \exp(\eta_1), \tag{10}$$

$$f = 1 + a(1,1^*) \exp(\eta_1 + \eta_1^*), \tag{11}$$

with $\eta_1 = k_1 x + p_1 y + i k_1^2 t$ and $a(1,1^*) = -\frac{1}{2(k_1+k_1^*)(ik_1^2 - ik_1^{*2} + p_1 + p_1^*)}$. Here k_1 and p_1 are complex constants and η^* denotes the complex conjugate of η.

The two-soliton solution is in the following form:

$$g = \exp(\eta_1) + \exp(\eta_2) + a(1,2,1^*) \exp(\eta_1 + \eta_2 + \eta_1^*) + a(1,2,2^*) \exp(\eta_1 + \eta_2 + \eta_2^*), \tag{12}$$

$$f = 1 + a(1,1^*) \exp(\eta_1 + \eta_1^*) + a(1,2^*) \exp(\eta_1 + \eta_2^*) + a(2,1^*) \exp(\eta_2 + \eta_1^*) + a(2,2^*) \exp(\eta_2 + \eta_2^*) + a(1,2,1^*,2^*) \exp(\eta_1 + \eta_2 + \eta_1^* + \eta_2^*), \tag{13}$$

with $\eta_j = k_j x + p_j y + i k_j^2 t$ $(j = 1,2)$. Here the coefficients are defined by the formulas

$$a(j,l^*) = -\frac{1}{2\left(k_j + k_l^*\right)\left(ik_j^2 - ik_l^{*2} + p_j + p_l^*\right)}, \tag{14}$$

$$a(i,j) = 2(k_i - k_j)\left(-ik_i^2 + ik_j^2 - p_i + p_j\right), \tag{15}$$

$$a(i^*,j^*) = 2(k_i^* - k_j^*)\left(ik_i^{*2} - ik_j^{*2} - p_i^* + p_j^*\right), \tag{16}$$

$$a(i,j,k^*) = a(i,j) a(i,k^*) a(j,k^*), \tag{17}$$

$$a(i,j^*,k^*) = a(i,j^*) a(i,k^*) a(j^*,k^*), \tag{18}$$

$$a(i,j,k^*,l^*) = a(i,j) a(i,k^*) a(i,l^*) a(j,k^*) a(j,l^*) a(k^*,l^*), \tag{19}$$

where k_j and p_j are complex constants. In the same way, we can construct the three-soliton solution,

$$\begin{aligned} g = {} & \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) \\ & + a(1,2,1^*) \exp(\eta_1 + \eta_2 + \eta_1^*) + a(1,3,1^*) \exp(\eta_1 + \eta_3 + \eta_1^*) + a(2,3,2^*) \exp(\eta_2 + \eta_3 + \eta_2^*) \\ & + a(1,2,2^*) \exp(\eta_1 + \eta_2 + \eta_2^*) + a(1,3,3^*) \exp(\eta_1 + \eta_3 + \eta_3^*) + a(2,3,3^*) \exp(\eta_2 + \eta_3 + \eta_3^*) \\ & + a(1,2,3^*) \exp(\eta_1 + \eta_2 + \eta_3^*) + a(1,3,2^*) \exp(\eta_1 + \eta_3 + \eta_2^*) + a(2,3,1^*) \exp(\eta_2 + \eta_3 + \eta_1^*) \\ & + a(1,2,3,1^*,2^*) \exp(\eta_1 + \eta_2 + \eta_3 + \eta_1^* + \eta_2^*) + a(1,2,3,1^*,3^*) \exp(\eta_1 + \eta_2 + \eta_3 + \eta_1^* + \eta_3^*) \\ & + a(1,2,3,2^*,3^*) \exp(\eta_1 + \eta_2 + \eta_3 + \eta_2^* + \eta_3^*), \end{aligned} \tag{20}$$

$$\begin{aligned} f = {} & 1 + a(1,1^*) \exp(\eta_1 + \eta_1^*) + a(2,2^*) \exp(\eta_2 + \eta_2^*) + a(3,3^*) \exp(\eta_3 + \eta_3^*) \\ & + a(1,2^*) \exp(\eta_1 + \eta_2^*) + a(2,1^*) \exp(\eta_2 + \eta_1^*) + a(2,3^*) \exp(\eta_2 + \eta_3^*) + a(3,2^*) \exp(\eta_3 + \eta_2^*) \\ & + a(1,3^*) \exp(\eta_1 + \eta_3^*) + a(3,1^*) \exp(\eta_3 + \eta_1^*) \\ & + a(1,2,1^*,2^*) \exp(\eta_1 + \eta_2 + \eta_1^* + \eta_2^*) + a(1,3,1^*,3^*) \exp(\eta_1 + \eta_3 + \eta_1^* + \eta_3^*) \\ & + a(1,2,1^*,3^*) \exp(\eta_1 + \eta_2 + \eta_1^* + \eta_3^*) + a(1,3,1^*,2^*) \exp(\eta_1 + \eta_3 + \eta_1^* + \eta_2^*) \\ & + a(1,2,2^*,3^*) \exp(\eta_1 + \eta_2 + \eta_2^* + \eta_3^*) + a(2,3,1^*,2^*) \exp(\eta_2 + \eta_3 + \eta_1^* + \eta_2^*) \\ & + a(1,3,2^*,3^*) \exp(\eta_1 + \eta_3 + \eta_2^* + \eta_3^*) + (2,3,1^*,3^*) \exp(\eta_2 + \eta_3 + \eta_1^* + \eta_3^*) \\ & + a(2,3,2^*,3^*) \exp(\eta_2 + \eta_3 + \eta_2^* + \eta_3^*) + a(1,2,3,1^*,2^*,3^*) \exp(\eta_1 + \eta_2 + \eta_3 + \eta_1^* + \eta_2^* + \eta_3^*), \end{aligned} \tag{21}$$

where the coefficients are defined as in (14)–(19), and in general

$$a(i_1,i_2,\ldots,i_n,j_1^*,\ldots,j_m^*) = \prod_{1\leqslant k<l\leqslant n} a(i_k,i_l) \prod_{1\leqslant k\leqslant n,1\leqslant l\leqslant m} a(i_k,j_l^*) \prod_{1\leqslant k<l\leqslant m} a(j_k^*,j_l^*). \tag{22}$$

From the above expressions for the one-, two-, and three-soliton solutions, we know that the exact N-soliton solution of Eqs. (5) is in the following form:

$$f = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{N}\mu_j\eta_j + \sum_{j=N+1}^{2N}\mu_j\eta_{j-N}^* + \sum_{1\leqslant i<j}^{2N}\mu_i\mu_j A_{ij}\right], \tag{23}$$

$$g = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{N}\nu_j\eta_j + \sum_{j=N+1}^{2N}\nu_j\eta_{j-N}^* + \sum_{1\leqslant i<j}^{2N}\nu_i\nu_j A_{ij}\right], \tag{24}$$

where

$$\eta_j = k_j x + p_j y + i k_j^2 t, \quad j = 1,2,\ldots,N, \tag{25}$$

$$\eta_j^* = \text{conjugate of } \eta_j, \quad j = 1,2,\ldots,N, \tag{26}$$

$$\exp(A_{i,j}) = a(i,j), \quad i < j = 2,3,\ldots,N, \tag{27}$$

$$\exp(A_{i,N+j}) = a(i,j^*) \quad i,j = 1,2,\ldots,N, \tag{28}$$

$$\exp(A_{N+i,N+j}) = a(i^*,j^*), \quad i < j = 2,3,\ldots,N. \tag{29}$$

Here α_j and γ_j are both real parameters relating respectively to the amplitude and phase of the ith soliton. The sum $\sum_{\mu=0,1}^{(e)}$ indicates summation over all possible combinations of $\mu_i = 0,1$ under the condition

$$\sum_{j=1}^{N}\mu_j = \sum_{j=1}^{N}\mu_{N+j}, \tag{30}$$

and $\sum_{\nu=0,1}^{(o)}$ indicates summation over all possible combinations of $\nu_i = 0,1$ under the condition

$$\sum_{j=1}^{N}\nu_j = \sum_{j=1}^{N}\nu_{N+j} + 1. \tag{31}$$

The form of the N-soliton solution (23) and (24) is the same as that of the combined Schrödinger-KdV equation in [31]. The proof of the N-soliton solution here can be completed by induction and is similar to the one in [31]. The reader can find the details there.

It is known that soliton solutions of many integrable systems (e.g., of Schrödinger type or the KP equation of B-type) can be expressed in Pfaffian form. In the Appendix, we construct the N-soliton solution to the semidiscrete system (32) by using the Pfaffian technique.

III. INTEGRABLE DISCRETE ANALOG OF THE (2+1)-DIMENSIONAL SYSTEM

We consider the discrete system

$$\left[iD_t + \frac{4}{\epsilon^2}\sinh^2\left(\frac{D_n}{2}\right)\right]g_n\cdot f_n = 0, \tag{32a}$$

$$\frac{4}{\epsilon}(D_t + D_y)f_{n+1}\cdot f_n + g_{n+1}g_n^* + g_n g_{n+1}^* = 0, \tag{32b}$$

where the bilinear difference operator $\exp(\delta D_n)$ in the sinh function is defined by

$$\exp(\delta D_n)a\cdot b \equiv a(n+\delta)b(n-\delta), \tag{33}$$

and the parameter ϵ can be regarded as a spatial discrete step. With the variable transformation

$$u_n = \frac{g_n}{f_n}, \quad w_n = \ln\frac{f_{n+1}}{f_n}, \tag{34}$$

the bilinear equations (32) can be cast into the form

$$iu_{n,t}\epsilon^2 + (u_{n+1}+u_{n-1})e^{w_n-w_{n-1}} - 2u_n = 0, \tag{35a}$$

$$4(w_{n,t}+w_{n,y}) + \epsilon(u_n^*u_{n+1} + u_nu_{n+1}^*) = 0. \tag{35b}$$

Setting

$$v_n = \frac{1}{\epsilon^2}\frac{2\left(f_{n+1}f_{n-1} - f_n^2\right)}{f_n^2} = \frac{2}{\epsilon^2}(e^{w_n-w_{n-1}} - 1) \tag{36}$$

and substituting it into (35a) results in

$$iu_t\epsilon^2 + \left(2u + \epsilon^2 u_{xx}\right)\left(1 + \frac{\epsilon^2}{2}v\right) - 2u + O(\epsilon^2) = 0. \tag{37}$$

The coefficient of the term ϵ^2 is

$$\mathrm{i}u_t + u_{xx} + uv = 0. \tag{38}$$

By changing n to $n-1$ in (35b) and subtracting one from the other, we get

$$4(\partial_t+\partial_y)e^{w_n-w_{n-1}} + e^{w_n-w_{n-1}}\epsilon(E-1) \times (u_{n-1}u_n^* + u_{n-1}^*u_n) = 0, \tag{39}$$

or equivalently,

$$4(\partial_t+\partial_y)\left(1+\frac{\epsilon^2}{2}v_n\right) + \epsilon\left(1+\frac{\epsilon^2}{2}v_n\right)(E-1)(u_{n-1}u_n^* + u_{n-1}^*u_n) = 0. \tag{40}$$

Here E is the shift operator $Ea_n = a_{n+1}$. The continuum limit of (40) as $\epsilon \to 0$ is

$$v_t + v_y + (|u|^2)_x = 0. \tag{41}$$

Thus (35) gives a semidiscrete analog of the system (5). From the derivation above, by eliminating w in (35), we obtain the following semidiscrete system for u and v:

$$\mathrm{i}u_{n,t} + \frac{u_{n+1}+u_{n-1}-2u_n}{\epsilon^2} + \frac{(u_{n+1}+u_{n-1})v_n}{2} = 0, \tag{42a}$$

$$v_{n,t} + v_{n,y} + \left(1+\frac{\epsilon^2}{2}v_n\right) \times \frac{u_n(u_{n+1}^* - u_{n-1}^*) + u_n^*(u_{n+1}-u_{n-1})}{2\epsilon} = 0. \tag{42b}$$

Remark 1. By multiplying both sides Eq. (42a) by u_n^*, employing the conjugate, and then subtracting one of the two equations from the other, we have

$$i(|u_n|^2)_t+\frac{1}{\epsilon^2}[(u_{n+1}+u_{n-1})u_n^*-(u_{n+1}^*+u_{n-1}^*)u_n] + \frac{1}{2}[u_n^*v_n(u_{n+1}+u_{n-1})-u_nv_n(u_{n+1}^*+u_{n-1}^*)]=0.$$

By summation over n, we get

$$\frac{d}{dt}\sum_{n=-\infty}^{+\infty}|u_n|^2=0,$$

which proves that the total energy $\sum_{n=-\infty}^{\infty}|u_n|^2$ is conserved. Numerical computation is given in the next section.

Remark 2. One can check that the first bilinear equation of (5), i.e., Eq. (7), is the same as that of the nonlinear Schrödinger equation. It is well known that the Davey-Stewartson equation, a two-dimensional NLS equation that appears as the shallow-water limit of the Benney-Roskes equation, arises from the two-component KP hierarchy [32]. It was pointed out in [33] that the discretization of the NLS equation can be obtained from reduction of the two-component KP hierarchy. Hence we believe that the semidiscrete system (42) must have a relation with the two-component KP hierarchy. Meanwhile, since the $(2+1)$-dimensional system (5) is derived from the KP equation via an asymptotically exact reduction method, the relation between the semidiscrete system (42) and the differential-difference KP equation [34,35]

$$\Delta(u_t+2u_y-2uu_y)=(2+\Delta)u_{yy} \tag{43}$$

deserves further consideration. Here $u=u(y,t,n)$ and Δ denotes the forward difference operator defined by $\Delta f_n = f_{n+1}-f_n$.

The one-soliton solution for (32) has the form

$$f_n=1+b(1,1^*)\exp(\eta_1+\eta_1^*),\qquad g_n=\exp(\eta_1), \tag{44}$$

with

$$\eta_1=k_1n+p_1y+q_1t,\qquad q_1=\frac{i}{\epsilon^2}(e^{k_1}+e^{-k_1}-2), \tag{45}$$

$$b(1,1^*)=-\frac{\epsilon(e^{k_1}+e^{k_1^*})}{4(e^{k_1+k_1^*}-1)(p_1+p_1^*+q_1+q_1^*)}, \tag{46}$$

p_1 and k_1 are complex constants and η_1^* is the complex conjugate of η_1. If we set $x=n\epsilon$ and $k_1=\epsilon\tilde{k}_1$, we get the following asymptotic relation:

$$\eta_1=\tilde{k}_1x+p_1y+q_1t, \tag{47}$$

$$q_1=\frac{i}{\epsilon^2}(e^{\tilde{k}_1\epsilon}+e^{-\tilde{k}_1\epsilon}-2)=i\tilde{k}_1^2+O(\epsilon), \tag{48}$$

$$b(1,1^*)=\frac{-1}{2(\tilde{k}_1+\tilde{k}_1^*)(p_1+p_1^*+q_1+q_1^*)}+O(\epsilon). \tag{49}$$

This shows that the one-soliton solution of the semidiscrete equation yields that of the continuous equation through the continuum limit $\epsilon\to 0$.

The one-soliton solutions for u_n and v_n are expressed as

$$u_n=\frac{g_n}{f_n}=\frac{e^{\eta_1}}{1+b_{11}e^{\eta_1+\eta_1^*}} =\frac{1}{2\sqrt{b_{11}}}e^{i\mathrm{Im}(\eta_1)}\mathrm{sech}\left(\mathrm{Re}(\eta)+\frac{\ln b_{11}}{2}\right), \tag{50}$$

$$v_n=\frac{2}{\epsilon^2}\left(\frac{f_{n+1}f_{n-1}}{f_n^2}-1\right) =\frac{2}{\epsilon^2}\left[\frac{(1+b_{11}e^{2\mathrm{Re}(\eta_1+k_1)})(1+b_{11}e^{2\mathrm{Re}(\eta_1-k_1)})}{(1+b_{11}e^{2\mathrm{Re}(\eta_1)})^2}-1\right], \tag{51}$$

with $b_{11}=b(1,1^*)$. The two-soliton solution of the semidiscrete system has the form

$$\begin{aligned}f_n=&1+b(1,1^*)\exp(\eta_1+\eta_1^*)+b(1,2^*)\exp(\eta_1+\eta_2^*)\\&+b(2,1^*)\exp(\eta_2+\eta_1^*)+b(2,2^*)\exp(\eta_2+\eta_2^*)\\&+b(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*),\end{aligned} \tag{52}$$

$$\begin{aligned}g_n=&\exp(\eta_1)+\exp(\eta_2)+b(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*)\\&+b(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*),\end{aligned} \tag{53}$$

with the coefficients

$$b(i,j^*)=-\frac{\epsilon(e^{k_i}+e^{k_j^*})}{4(e^{k_i+k_j^*}-1)(p_i+p_j^*+q_i+q_j^*)}, \tag{54}$$

$$b(i,j)=-\frac{4(e^{k_i}-e^{k_j})(p_i-p_j+q_i-q_j)}{\epsilon\left(e^{k_i+k_j}+1\right)}, \tag{55}$$

$$b(i^*,j^*)=-\frac{4(e^{k_i^*}-e^{k_j^*})(p_i^*-p_j^*+q_i^*-q_j^*)}{\epsilon\left(e^{k_i^*+k_j^*}+1\right)}, \tag{56}$$

where $\eta_j=k_jn+p_jy+q_jt$, $1\leqslant i,j\leqslant 2$, with complex constants k_1, k_2, p_1, and p_2 and the dispersion relation $q_j=i\frac{e^{k_j}+e^{-k_j}-2}{\epsilon^2}$. Setting $x=n\epsilon$, $k_j=\epsilon\tilde{k}_j$, in the continuum limit $\epsilon\to 0$, we obtain

$$\eta_j=\tilde{k}_jx+p_jy+q_jt,\quad q_j\to i\tilde{k}_j^2, \tag{57}$$

$$b(i,j^*)\to-\frac{1}{2(\tilde{k}_i+\tilde{k}_j^*)(p_i+p_j^*+q_i+q_j^*)}=a(i,j^*), \tag{58}$$

$$b(i,j)\to-2(\tilde{k}_i-\tilde{k}_j)(p_i-p_j+q_i-q_j)=a(i,j), \tag{59}$$

$$b(i^*,j^*)\to-2(\tilde{k}_i^*-\tilde{k}_j^*)(p_i^*-p_j^*+q_i^*-q_j^*)=a(i^*,j^*). \tag{60}$$

Thus we conclude that the two-soliton solutions of the semidiscrete system reduce to those of the continuous system through the continuum limit $\epsilon\to 0$. Substituting (52) and (53) into (34) and (36), we obtain the two-soliton solutions u_n and v_n, respectively.

The exact N-soliton solutions to Eqs. (32) have the forms

$$f_n=\sum_{\mu=0,1}^{(e)}\exp\left[\sum_{j=1}^{N}\mu_j\eta_j+\sum_{j=N+1}^{2N}\mu_j\eta_{j-N}^*+\sum_{1\leqslant i<j}^{2N}\mu_i\mu_jB_{ij}\right], \tag{61}$$

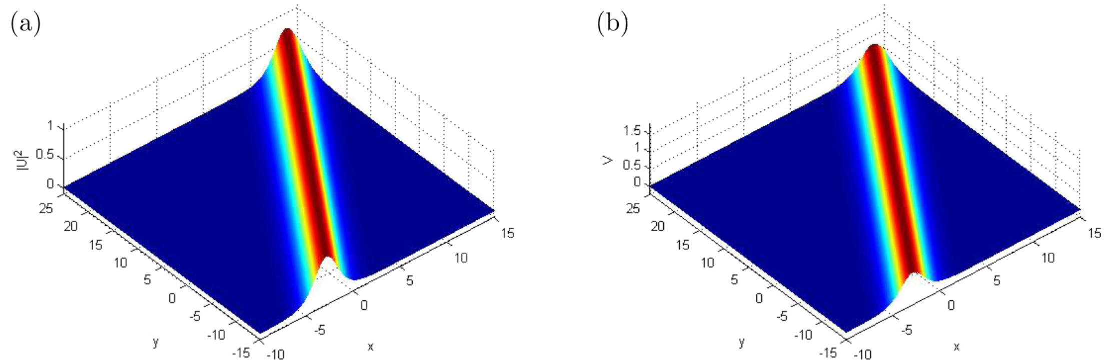

FIG. 1. (Color online) Numerical solution of the one-soliton solution at $t = 4$.

$$g_n = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{N}\nu_j\eta_j + \sum_{j=N+1}^{2N}\nu_j\eta^*_{j-N} + \sum_{1\leqslant i<j}^{2N}\nu_i\nu_j B_{ij}\right], \tag{62}$$

where

$$\eta_j = k_j n + p_j y + q_j t, \quad q_j = i\left(e^{k_j} + e^{-k_j} - 2\right)/\epsilon^2, \tag{63}$$

$$\eta_j^* = \text{complex conjugate of } \eta_j, \quad j = 1,2,\ldots,N, \tag{64}$$

$$\exp(B_{i,j}) = b(i,j), \quad i < j = 2,3,\ldots,N, \tag{65}$$

$$\exp(B_{i,N+j}) = b(i,j^*), \quad i,j = 1,2,\ldots,N, \tag{66}$$

$$\exp(B_{N+i,N+j}) = b(i^*,j^*), \quad i < j = 2,3,\ldots,N. \tag{67}$$

Following the proof of the one- and two-soliton solutions, one can show that the exact N-soliton solutions of the semidiscrete system reduce to those of the continuous system in the continuum limit.

IV. NUMERICAL COMPUTATIONS

In this section, two examples will be used to show that the integrable semidiscretization is a powerful scheme for the numerical solution of the system (5). They include (1) propagation of the one-soliton solution, and (2) interaction of the two-soliton solutions. We employ the Crank-Nicholson scheme for the system (42), the central difference scheme in the y direction, and the Dirichlet condition. We choose the exact one-soliton solution and two-soliton solutions of the system (5) as the initial and boundary values.

Example 1: One-soliton propagation. The parameters taken for the one-soliton solution are $k_1 = 0.6 + 0.3i$, $p_1 = -0.25 - 0.4i$. The number of grid points is taken as 250 in an interval of width 25 in the x domain, which implies a mesh size of $\epsilon = 0.1$. The number of grid points is 400 in an interval of width 40 in the y direction. The time-step size is taken as $\Delta t = 0.05$. Figure 1 displays the numerical solution of the one-soliton problem at $t = 4$. The L_∞ norm is 0.0385 for $|u|^2$ and 0.0422 for v at $t = 4$. It is noted that the numerical error is mainly due to the error of the dispersion relation. In other words, even after a fairly long time, the numerical solution of a soliton preserves its shape very well except for a phase shift.

Example 2: Two-soliton interaction. The parameters taken for the two-soliton solution are $k_1 = 0.6 + 0.3i$, $k_2 = -0.5 + 0.5i$, $p_1 = -0.25 - 0.4i$, $p_2 = 0$. Figure 2 shows the exact two-soliton solution of $|u|^2$ and v. Figure 3 displays the numerical solution for the collision of the two solitons. The profiles show that the collision of two solitons is well simulated.

V. DYNAMIC PROPERTIES

In the following discussion, we fix the discrete step $\epsilon = 1$ in the solutions (61) and (62). For $b(1,2) \neq 0$ in (55), namely, $b(1,2,1^*,2^*) \neq 0$ in (52), the two solitons possess four arms and display regular interaction as shown

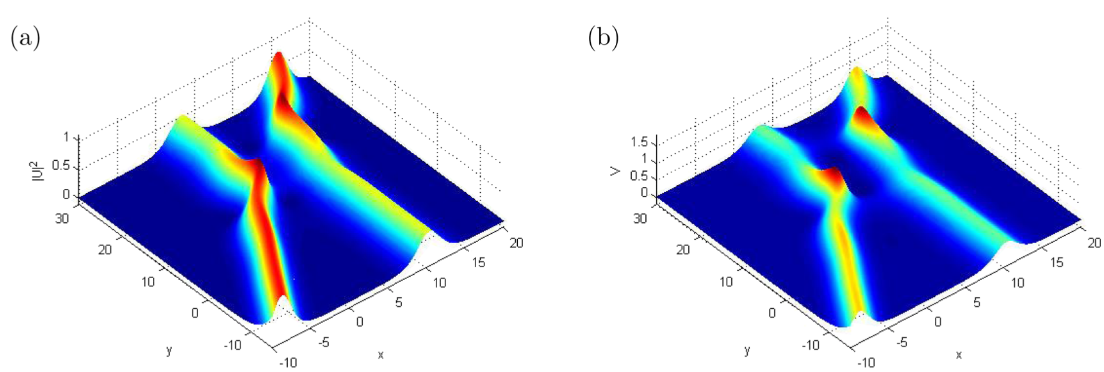

FIG. 2. (Color online) Exact solution for the collision of the two-soliton solution at $t = 4$.

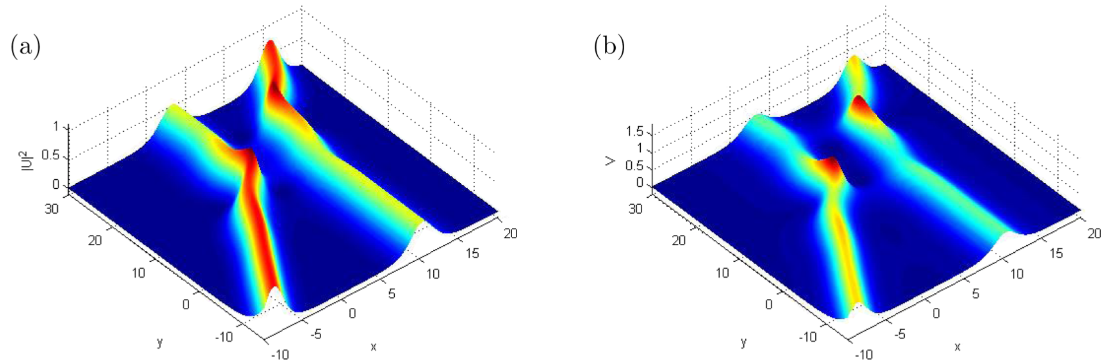

FIG. 3. (Color online) Numerical solution for the collision of two solitons at $t = 4$.

in Fig. 4. One can see that the two obliquely moving solitons pass through each other without affecting each other and keep their original shapes and velocities invariant during the whole propagation. Therefore, the regular interaction between the solitons is completely elastic. The elastic interaction is shown in Fig. 4. The parameters are chosen as $k_1 = 0.6 + 0.3i$, $k_2 = 0.37 - 0.02i$, $p_1 = -0.0185 - 0.192i$, $p_2 = -0.2406903076 - 0.1197442234i$.

When $b(1,2,1^*,2^*) = 0$, that is, $b(1,2) = 0$ in (55), resonant interactions can happen. The resonant interactions in this case are called the "minus resonance" [36,37], namely, after the solitons interact with each other, the amplitudes decrease; sometimes the amplitudes can even reach zero. The resonant situation here is similar to that in the continuous case. In order to analyze the amplitudes of the resonant solitons, we rewrite the two-soliton solutions as follows:

$$u \rightarrow \begin{cases} u_1 = \frac{1}{2\sqrt{b(1,1^*)}} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{\ln b(1,1^*)}{2}\right), & \eta_{2R} \rightarrow -\infty,\ \eta_{1R} \sim 0, \\ u_2 = \frac{1}{2\sqrt{b(2,2^*)}} e^{i\eta_{2I}} \operatorname{sech}\left(\eta_{2R} + \frac{\ln b(2,2^*)}{2}\right), & \eta_{1R} \rightarrow -\infty,\ \eta_{2R} \sim 0, \\ u_3 = 0, & \eta_{1R} \rightarrow +\infty,\ \eta_{1R} - \eta_{2R} \sim 0, \end{cases} \tag{68}$$

and in the second case when $|b(1,2)| \ll 1$,

$$v \rightarrow \begin{cases} v_1 = \frac{1}{4}(e^{2k_{1R}} + e^{-2k_{1R}} - 2)\operatorname{sech}^2\left(\eta_1 + \frac{\ln b(1,1^*)}{2}\right),\ \eta_{2R} \rightarrow -\infty,\ \eta_{1R} \sim 0, \\ v_2 = \frac{1}{4}(e^{2k_{2R}} + e^{-2k_{2R}} - 2)\operatorname{sech}^2\left(\eta_2 + \frac{b(2,2^*)}{2}\right),\ \eta_{1R} \rightarrow -\infty,\ \eta_{2R} \sim 0, \\ v_3 = \frac{B_1 + \sqrt{b(1,2^*)b(2,1^*)b(1,1^*)b(2,2^*)}\operatorname{Re}\left[\left(e^{k_1^* - k_2^*} + e^{k_2^* - k_1^*} - 2\right)\cosh\left(\eta_1 - \eta_2 + \frac{1}{2}\ln\frac{b(1,2^*)b(1,1^*)}{b(2,1^*)b(2,2^*)}\right)\right]}{\left(2\sqrt{b(1,1^*)b(2,2^*)}\cosh\left(\eta_{1R} - \eta_{2R} + \frac{1}{2}\ln\frac{b(1,1^*)}{b(2,2^*)}\right) + 2\sqrt{b(1,2^*)b(2,1^*)}\cosh\left[\eta_{1I} - \eta_{2I} + \frac{1}{2}\ln\frac{b(1,2^*)}{b(2,1^*)}\right)\right]^2}, \\ \quad \eta_{1R} \rightarrow +\infty,\ \eta_{1R} - \eta_{2R} \sim 0, \end{cases} \tag{69}$$

where

$$B_1 = b(1,1^*)b(2,2^*)\left(e^{2k_{1R} - 2k_{2R}} + e^{2k_{2R} - 2k_{1R}} - 2\right) + b(1,2^*)b(2,1^*)\left(e^{2k_{1I} - 2k_{2I}} + e^{2k_{2I} - 2k_{1I}} - 2\right).$$

According to (68) and (69), the interaction between two solitons is investigated in Fig. 5. The parameters are chosen as $k_1 = 0.5 + 0.20i$, $k_2 = 0.37 - 0.02i$, $p_1 = -0.0185 - 0.192i$, $p_2 = -0.2406903076 - 0.1197442234i$. We find that the two solitons possess three branches extending to infinity; this is called the triple-wave structure [37]. Therefore, one can see that for the potential $|u_n|^2$, the amplitude of the third branch is zero. The third branch has a high and steep wave hump for the potential v_n. These phenomena can also be found in the continuous case.

When $b(1,2,1^*,2^*) \rightarrow 0$, as in the continuous case, another type of resonance is shown in Fig. 6. The parameters are chosen as $k_1 = 0.50000001 + 0.20000001i$, $k_2 = 0.37 - 0.02i$, $p_1 = -0.0185 - 0.192i$, $p_2 = -0.2406903076 - 0.1197442234i$ and now $b(1,2) = 5.222108512 \times 10^{-9} + 4.762014844 \times 10^{-9}i$. In Fig. 6(a), the two solitons generate a small-amplitude soliton (in fact the amplitude is close to zero here) in the vicinity of the crossing point, which is different from those in Fig. 5(a). It looks as if the two solitons separate from each other into two parts. But in Fig. 6(b), the line solitons interact to create a particularly high and steep wave hump in the vicinity of the crossing point, which is also different from those in Fig. 5(b). The resonant interaction under the situation $b(1,2,1^*,2^*) \rightarrow 0$ is similar to that in the continuous case.

Three-soliton solutions can be obtained from (61) and (62) by setting $N = 3$. The elastic interaction among three solitons is shown in Fig. 7 with parameters chosen as $k_1 = 0.6 + 0.3i$, $k_2 = 0.37 - 0.02i$, $k_3 = 0.2 + 0.1i$, $p_1 = -0.0185 - 0.192i$, $p_2 = -0.2406903076 - 0.1197442234i$, $p_3 = -0.1 - 0.25i$. The resonant interaction among three solitons is much more complicated

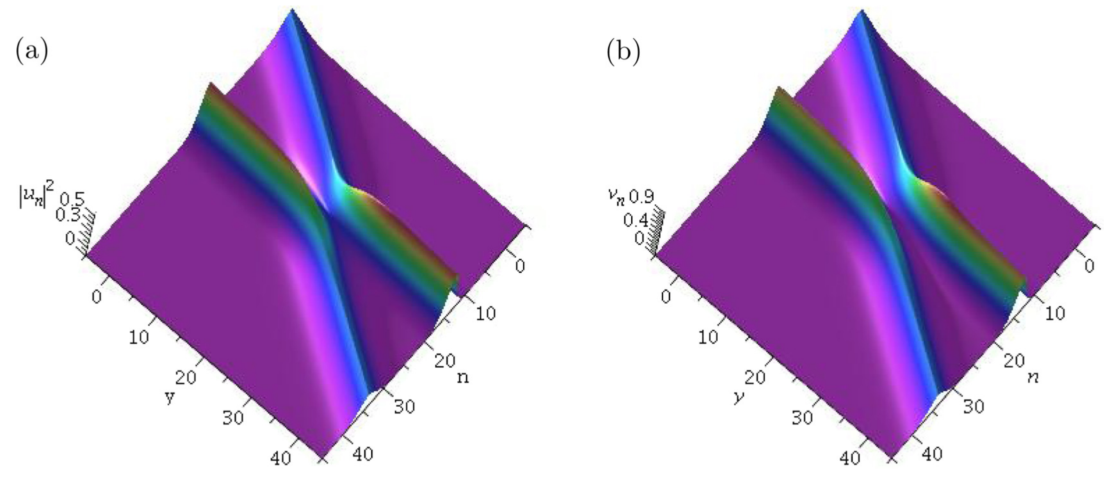

FIG. 4. (Color online) The elastic interaction of two solitons at $t = 20$.

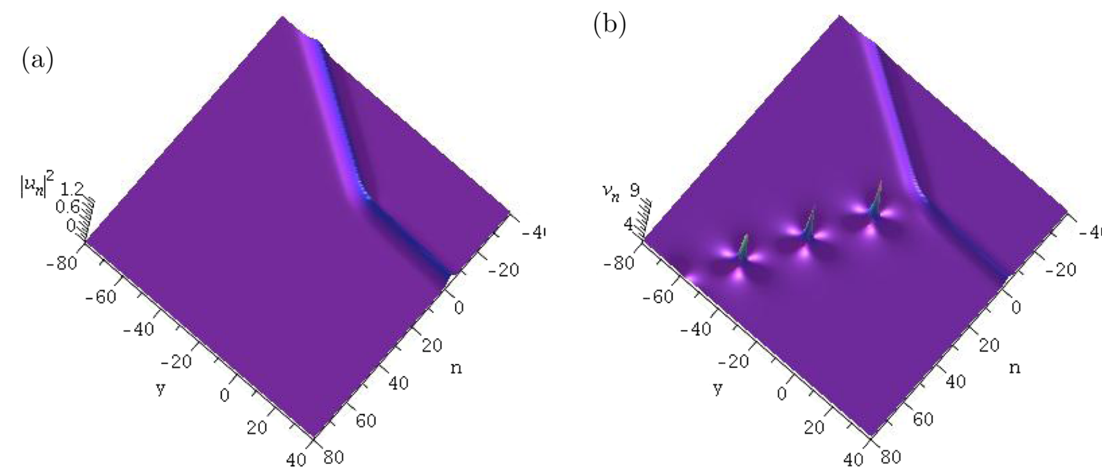

FIG. 5. (Color online) Resonant interactions between two solitons at $t = -10$ when $b(1,2) = 0$.

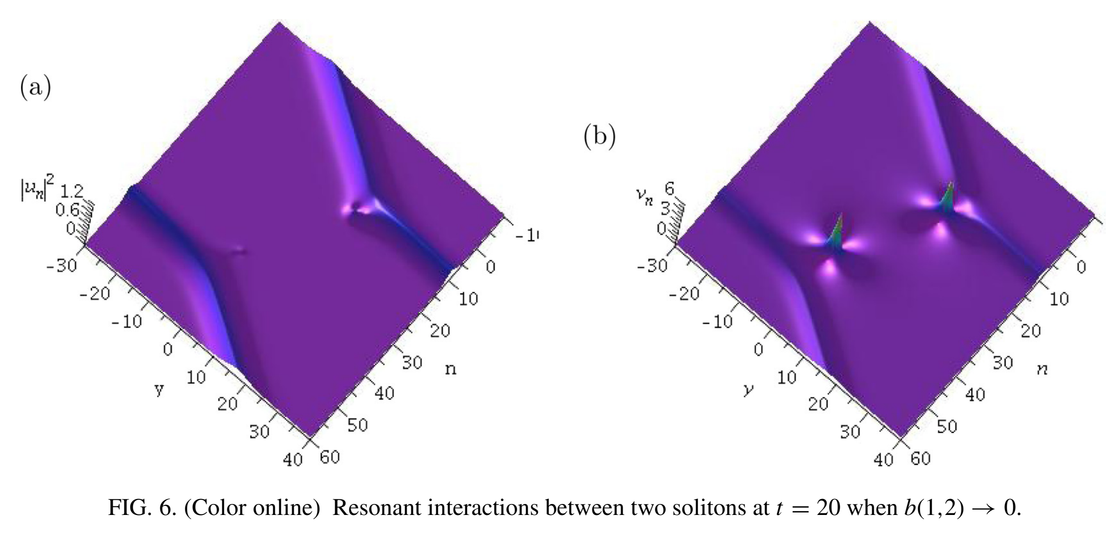

FIG. 6. (Color online) Resonant interactions between two solitons at $t = 20$ when $b(1,2) \to 0$.

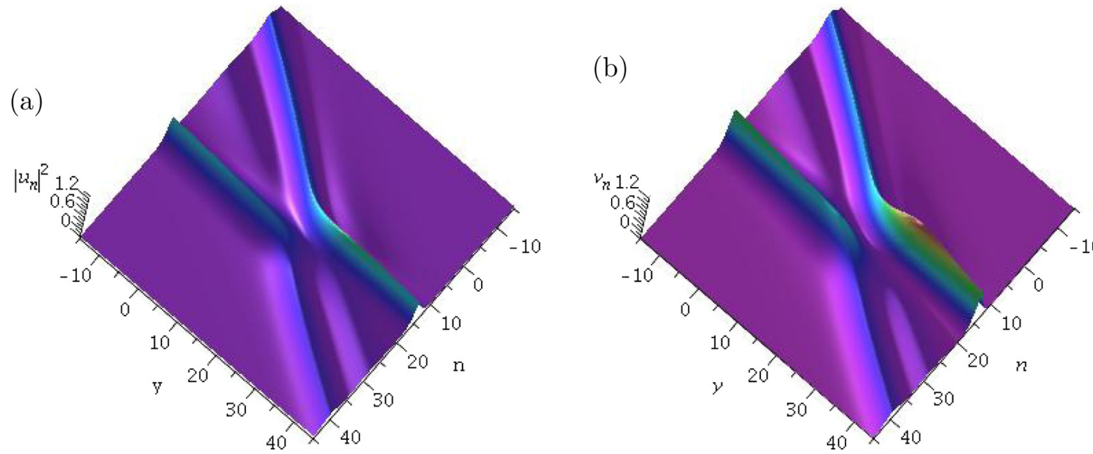

FIG. 7. (Color online) Elastic interactions between three solitons at $t = 20$.

than that of two solitons. Here only one case is depicted in Fig. 8 with the parameters $k_1 = 0.5 + 0.2i$, $k_2 = 0.37 + 0.02i$, $k_3 = 0.24 + 0.257i$, $p_1 = 0.0185 + 0.092i$, $p_2 = -0.173\,412\,314\,8 + 0.164\,255\,776\,6i$, $p_3 = -0.6 - 0.3i$.

VI. CONCLUSION

To summarize, we presented here a semidiscrete integrable version for a $(2+1)$-dimensional system and derived their N-soliton solutions by using the Pfaffian technique. Based on the asymptotic behavior of the two-soliton solutions (52) and (53) and graphical analysis, we analyzed the dynamics of the interactions. It is shown that the regular interaction is completely elastic (i.e., Fig. 4), and two types of resonance occur between the two solitons, both of which are noncompletely elastic (i.e., Figs. 5 and 6). A triple structure (Fig. 5) in the process of the interactions and a high wave hump in the vicinity of the crossing point (i.e., Fig. 6), are observed. Based on the results obtained, it is natural to further consider the integrability of the differential-difference system, such as the Bäcklund transformation, the Lax pair, and infinite conservation laws.

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APPENDIX

A Pfaffian is the square root of a skew-symmetric determinant of order $2n$, and consequently the properties of Pfaffians are closely related to those of determinants [30]. Let $A = \det(a_{j,k})$ $(1 \leqslant j,\, k \leqslant 2n)$, where $a_{j,k} = -a_{k,j}$. The Pfaffian expression of A is

$$A = [\mathrm{Pf}(1,2,3,\ldots,2n)]^2.$$

For example, if $n = 1$, we have

$$\begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = a_{12}^2 = [\mathrm{Pf}(1,2)]^2. \tag{A1}$$

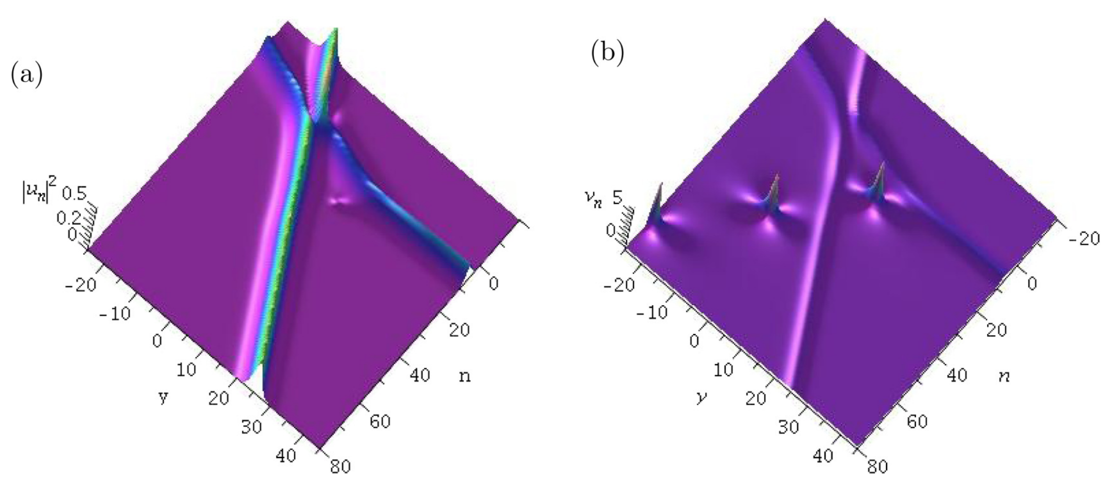

FIG. 8. (Color online) Resonant interactions between three solitons at $t = 20$.

If $n = 2$, we get

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{21} & 0 & a_{23} & a_{24} \\ -a_{31} & -a_{32} & 0 & a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{vmatrix}$$

$$= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 = [\mathrm{Pf}(1,2,3,4)]^2. \quad \text{(A2)}$$

We rewrite the original Eqs. (32) as follows:

$$iD_t g_n \cdot f_n + g_{n+1} \cdot f_{n-1} + g_{n-1} f_{n+1} - 2g_n f_n = 0, \quad \text{(A3a)}$$

$$4(D_t + D_y) f_{n+1} \cdot f_n + g_{n+1} g_n^* + g_n g_{n+1}^* = 0, \quad \text{(A3b)}$$

and define the Pfaffian elements:

$$\mathrm{Pf}(a_i,a_j) = i\frac{e^{k_i} - e^{k_j}}{e^{k_i+k_j} + 1} e^{\eta_i+\eta_j}, \quad \mathrm{Pf}(a_i^*,a_j^*) = -i\frac{e^{k_i^*} - e^{k_j^*}}{e^{k_i^*+k_j^*} + 1} e^{\eta_i^*+\eta_j^*}, \quad \text{(A4a)}$$

$$\mathrm{Pf}(a_i,a_j^*) = -i\frac{e^{k_i} + e^{k_j^*}}{e^{k_i+k_j^*} - 1} e^{\eta_i+\eta_j^*}, \quad \text{(A4b)}$$

$$\mathrm{Pf}(a_i,b_j^*) = (a_i^*,b_j) = 0,\ \mathrm{Pf}(a_i,b_j) = \mathrm{Pf}(a_i^*,b_j^*) = \delta_{ij}, \quad \text{(A4c)}$$

$$\mathrm{Pf}(b_i,b_j) = 0,\ \mathrm{Pf}(b_i^*,b_j^*) = 0, \quad \mathrm{Pf}(b_i,b_j^*) = \frac{\mathrm{i}}{4(p_i + p_j^* + q_i + q_j^*)}, \quad \text{(A4d)}$$

$$\mathrm{Pf}(d_0,\beta) = \mathrm{Pf}(a_j,\beta) = \mathrm{Pf}(a_i^*,\beta) = \mathrm{Pf}(b_j^*,\beta) = 0, \quad \mathrm{Pf}(b_i,\beta) = 1, \quad \text{(A4e)}$$

$$\mathrm{Pf}(d_0,a_j) = e^{\eta_j},\ \mathrm{Pf}(d_0,a_j^*) = e^{\eta_j^*},\ \mathrm{Pf}(d_0,b_i) = \mathrm{Pf}(d_0,b_i^*) = 0, \quad \text{(A4f)}$$

where δ_{ij} is the Kronecker delta function and

$$\eta_i = k_i n + p_i y + q_i t, \qquad q_i = i\left(e^{k_i} + e^{-k_i} - 2\right), \quad i = 1,2,\ldots,N.$$

Theorem 1. The N-soliton solution to Eqs. (32) can be expressed in the Pfaffian form

$$f_n = \mathrm{Pf}(a_1,a_2,\ldots,a_N,a_1^*,a_2^*,\ldots,a_N^*,b_1,b_2,\ldots,b_N,b_1^*, b_2^*,\ldots,b_N^*) = \mathrm{Pf}(\cdot), \quad \text{(A5)}$$

$$g_n = \mathrm{Pf}(d_0,\beta,a_1,a_2,\ldots,a_N,a_1^*,a_2^*,\ldots,a_N^*,b_1, b_2,\ldots,b_N,b_1^*,b_2^*,\ldots,b_N^*) = \mathrm{Pf}(d_0,\beta,\cdot), \quad \text{(A6)}$$

where we use the notation $(\cdot)$ for the sake of simplicity.

We introduce the Pfaffian elements c_p and c_m as

$$\mathrm{Pf}(d_0,c_p) = 0,\ \mathrm{Pf}(c_p,a_i) = \left(-ie^{k_i} - 1\right)e^{\eta_i}, \quad \mathrm{Pf}(c_p,a_i^*) = \left(ie^{k_i^*} - 1\right)e^{\eta_i^*}, \quad \text{(A7)}$$

$$\mathrm{Pf}(d_0,c_m) = 0,\ \mathrm{Pf}(c_m,a_i) = \left(ie^{-k_i} - 1\right)e^{\eta_i}, \quad \mathrm{Pf}\left(c_m,a_i^*\right) = (-ie^{-k_i^*} - 1)e^{\eta_i^*}. \quad \text{(A8)}$$

In what follows, we denote $\mathrm{Pf}(\cdot)$ by $(\cdot)$ for the sake of simplicity. From the properties of Pfaffians, we get the following differential and difference formulas for f_n and g_n,

$$f_{n+1} = \left(d_0,c_p,\cdot\right) + (\cdot), \quad \text{(A9a)}$$

$$g_{n+1} = i\,(d_0,\beta,\cdot) + i(c_p,\beta,\cdot), \quad \text{(A9b)}$$

$$f_{n-1} = (d_0,c_m,\cdot) + (\cdot), \quad \text{(A9c)}$$

$$g_{n-1} = -i(d_0,\beta,\cdot) - i(c_m,\beta,\cdot), \quad \text{(A9d)}$$

$$f_{n,t} = -\left(c_m,c_p,\cdot\right) + i\,(\cdot) + (d_0,c_m,\cdot) - \left(d_0,c_p,\cdot\right), \quad \text{(A9e)}$$

$$g_{n,t} = -(d_0,c_m,c_p,\beta,\cdot) - i(d_0,\beta,\cdot) - (c_p,\beta,\cdot) + (c_m,\beta,\cdot). \quad \text{(A9f)}$$

The substitution of (A9) into (A3a) implies

$$-i\left(d_0,c_m,c_p,\beta,\cdot\right)(\cdot) + i\,(d_0,c_m,\cdot)\,(c_p,\beta,\cdot) - i\left(d_0,c_p,\cdot\right)(c_m,\beta,\cdot) + i\,(d_0,\beta,\cdot)\left(c_m,c_p,\cdot\right) = 0,$$

which vanishes due to the Pfaffian identity [30]

$$\begin{aligned}&(a_1,a_2,a_3,a_4,1,2,\ldots,2m)(1,2,\ldots,2m)\\ &\quad -(a_1,a_2,1,2,\ldots,2m)(a_3,a_4,1,2,\ldots,2m)\\ &\quad +(a_1,a_3,1,2,\ldots,2m)(a_2,a_4,1,2,\ldots,2m)\\ &\quad -(a_1,a_4,1,2,\ldots,2m)(,a_2,a_3 1,2,\ldots,2m).\\ &= 0. \end{aligned} \quad \text{(A10)}$$

Thus we proved that (A5) and (A6) satisfy Eq. (A3a). Furthermore, in order to confirm that (A5) and (A6) satisfy (A3b), we introduce a new auxiliary element β^* and define new Pfaffian entries as follows:

$$(d_0,\beta^*) = (a_j,\beta^*) = (a_i^*,\beta^*) = (b_j,\beta^*) = 0,\ (b_i^*,\beta^*) = 1. \quad \text{(A11)}$$

It is easy to verify that

$$f_{n,t} + f_{n,y} = -\frac{i}{4}(\beta,\beta^*,\cdot), \quad \text{(A12a)}$$

$$f_{n+1,t} + f_{n+1,y} = -\frac{i}{4}\left[\left(\beta,\beta^*,d_0,c_p,\cdot\right) + (\beta,\beta^*,\cdot)\right], \quad \text{(A12b)}$$

$$g_n^* = (d_0,\beta^*,\cdot), \quad \text{(A12c)}$$

$$g_{n+1}^* = -i(d_0,\beta^*,\cdot) - i(c_p,\beta^*,\cdot). \quad \text{(A12d)}$$

(see the Appendix in [38] for reference). By substituting (A12) and (A9) into (A3b), Eq. (A3b) is reduced to the

Pfaffian identity

$$\begin{aligned}
&i[(d_0,c_p,\cdot)+(\cdot)](\beta,\beta^*,\cdot)-i[(\beta,\beta^*,d_0,c_p,\cdot)+(\beta,\beta^*,\cdot)](\cdot)\\
&\quad+[i\,(d_0,\beta,\cdot)+i(c_p,\beta,\cdot)](d_0,\beta^*,\cdot)-[i(d_0,\beta^*,\cdot)+i(c_p,\beta^*,\cdot)]\,(d_0,\beta,\cdot)\\
&=-i\,(\beta,\beta^*,d_0,c_p,\cdot)\,(\cdot)+i\,(d_0,c_p,\cdot)\,(\beta,\beta^*,\cdot)+i(c_p,\beta,\cdot)(d_0,\beta^*,\cdot)-i(c_p,\beta^*,\cdot)\,(d_0,\beta,\cdot)\\
&=0.
\end{aligned} \tag{A13}$$

Thus the bilinear Eq. (A3b) is established and so the theorem is proven. ■

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