Robust criticality of an Ising model on rewired directed networks

Adam Lipowski,[1] Krzysztof Gontarek,[1] and Dorota Lipowska[2]
[1]*Faculty of Physics, Adam Mickiewicz University, Poznań, Poland*
[2]*Faculty of Modern Languages and Literature, Adam Mickiewicz University, Poznań, Poland*
(Received 15 March 2015; published 2 June 2015)

We show that preferential rewiring, which is supposed to mimic the behavior of financial agents, changes a directed-network Ising ferromagnet with a single critical point into a model with robust critical behavior. For the nonrewired random graph version, due to a constant number of out-links for each site, we write a simple mean-field-like equation describing the behavior of magnetization; we argue that it is exact and support the claim with extensive Monte Carlo simulations. For the rewired version, this equation is obeyed only at low temperatures. At higher temperatures, rewiring leads to strong heterogeneities, which apparently invalidates mean-field arguments and induces large fluctuations and divergent susceptibility. Such behavior is traced back to the formation of a relatively small core of agents that influence the entire system.

DOI: 10.1103/PhysRevE.91.062801 PACS number(s): 89.65.Gh, 89.75.Fb

I. INTRODUCTION

Understanding the complex behavior of financial markets is one of the main objectives of econophysics. Fat-tail non-Gaussian fluctuations, volatility clustering, or rapid decay of autocorrelations of returns characterize most of the financial markets, suggesting that these stylized facts [1] have some more fundamental explanation. Searching for such an explanation, one can resort to an approach particularly suited for physicists, namely, agent modeling [2]. In the spirit of statistical mechanics, one considers a collection of agents involved in interactions resembling the functioning of financial markets. Since buying and selling are the most important activities of such agents, a number of models of financial markets bear some similarity to the two-state percolation [3] or Ising-like models [4].

An important agent interaction is mimicking some other agent's behavior, which suggests a similarity to ferromagnetic systems. However, ferromagnets typically exhibit rather small fluctuations, which is much different from the behavior of financial markets. Ferromagnets exhibit large fluctuations only at the critical point separating ferromagnetic and paramagnetic phases. To place the system at the critical point requires, however, a fine-tuning of control parameters. On the other hand, financial markets seem to be more robust with strong fluctuations appearing without any tuning of parameters. Some models were proposed where mimicking other agents' behavior is compensated by the tendency to be in the minority [5] or where agents with more complex strategies were used [6]. They do reproduce some of the stylized facts, but their considerable complexity hinders a deeper understanding.

Apparently, the analogy with simple ferromagnetic systems is not sufficient to model financial markets and one should search for more suitable extensions. In our opinion, an important ingredient of models of financial markets should be the possibility to choose and sometimes also change neighbors that a given agent would like to mimic. The objective of the present paper is to implement such a rewiring mechanism and to show that it drastically affects the behavior of the model. When the neighbors to be mimicked are selected at random and kept fixed, the model behaves as an ordinary ferromagnet with ferromagnetic and paramagnetic phases separated at a critical point. However, when agents might switch the neighbors and preferentially select those they consider as more influential, the system generically exhibits divergent fluctuations. Such behavior indicates that preferential rewiring induces a robust criticality, which is a required feature of stock-market models [7]. We also examine the mechanism leading to the robust criticality.

II. MODEL WITHOUT REWIRING

In our model we consider N agents represented by spinlike variables $s_i = \pm 1$, $i = 1,2,\dots,N$. At each time step t, each agents decides whether to buy ($s_i = 1$) or sell ($s_i = -1$) an asset. To make the decision, an agent tries to mimic the behavior of its neighbors and the model evolves according to the heat-bath dynamics

$$s_i(t+1) = \begin{cases} 1 & \text{with probability} \quad p = \frac{1}{1+\exp[-2h_i(t)/T]} \\ -1 & \text{with probability} \quad 1-p, \end{cases} \tag{1}$$

where

$$h_i(t) = \sum_j s_j(t) \tag{2}$$

is the local field acting on a given agent i and the summation in is over its neighbors. The control parameter T is the analog of the temperature in the magnetic Ising model and determines the level of fluctuations in the decision process.

The neighborhood of a given agent is set randomly, namely, each agent has a fixed number of z randomly selected neighbors, which it interacts with via the local field. The neighboring relation is not necessarily symmetric: If agent j enters the expression for the local field of agent i, it does not imply that agent i enters the expression for the local field of agent j. In other words, agents are nodes of a directed random network and each node has z out-links (arrows point at the nodes that contribute to the local field). The number of in-links of a given agent, which specifies how many agents it influences, is not fixed and it can vary among agents (of course, the average over all agents equals z). Equal numbers of out-links and unequal numbers of in-links constitute an important feature of our model, which we will refer to as the out-homogeneity.

Taking into account the spin variables, the above rules define actually an Ising ferromagnet on a directed random graph. Models of this kind were already analyzed and shown to exhibit an ordinary ferromagnetic-paramagnetic phase transition belonging to the mean-field universality class [8].

In the following, we present a more detailed analysis of our model for $z = 4$. Due to the out-homogeneity, one can write a relatively simple equation, which governs the evolution of magnetization. Let $P_i(t)$ denote the probability that agent i at time t takes the value $s_i = 1$. Assuming that $P_i(t)$ is spatially homogeneous and does not depend on i, from the heat-bath rules we obtain that

$$P(t+1) = \sum_{k=0}^{4} \binom{4}{k} P^k(t)[1 - P(t)]^{4-k} \times \frac{1}{1 + \exp[-4(k-2)/T]}. \quad (3)$$

Of course, Eq. (3) can be easily rewritten in terms of magnetization $[m(t) = 2P(t) - 1]$, which is common in Ising-model studies. In the steady-state limit ($t \to \infty$) Eq. (3) becomes a fourth-order polynomial equation, which can be easily solved numerically (and with some more effort even analytically). Moreover, the critical temperature T_c can be found using the standard procedure of expanding the $t \to \infty$ limit of Eq. (3) in the vicinity of the critical point. Elementary calculations reveal that T_c obeys

$$2 = \tanh(4/T_c) + 2\tanh(2/T_c). \quad (4)$$

The solution of Eq. (4) can be written as

$$T_c = \frac{4}{\ln[(1+x)/(1-x)]}, \quad (5)$$

where

$$x = \frac{1}{3}\left[1 - 5\sqrt[3]{\frac{2}{11 + 3\sqrt{69}}} + \sqrt[3]{\frac{1}{2}(11 + 3\sqrt{69})}\right]. \quad (6)$$

We thus obtain $T_c \approx 3.089\,82$, approximately.

The factorized form of the probabilities suggests that Eq. (3) is nothing more than the mean-field equation for our model and thus it is only approximate. This would certainly be the case for undirected graphs, where neighbors j and k of agent i are strongly correlated (since i contributes to the local fields of both j and k). For undirected random graphs, some insight into the behavior of the Ising model can be obtained by using a replica method [9] or some recurrence relations based on the similarity of random graphs to Cayley trees [10]. On the other hand, in directed networks, even though j and k are neighbors of i, they are not more correlated than any other two randomly selected nodes (Fig. 1). Since the graph is sparse, we expect that in the limit $N \to \infty$, such correlations are negligible and consequently the factorization in Eq. (3) should be legitimate.

Monte Carlo simulations of our model confirm the above analysis (Fig. 2). Calculating the magnetization for $N = 10^4$ and $z = 4$, we find it to be in very good agreement with $m = 2P(t = \infty) - 1$ obtained from the numerical solution

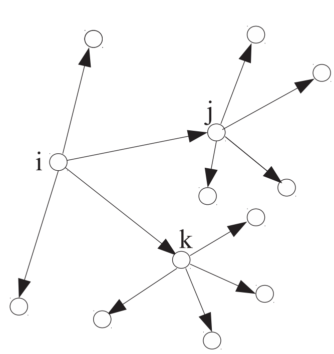

FIG. 1. In a directed random graph, neighbors j and k of node i are not more correlated than any other two randomly selected nodes.

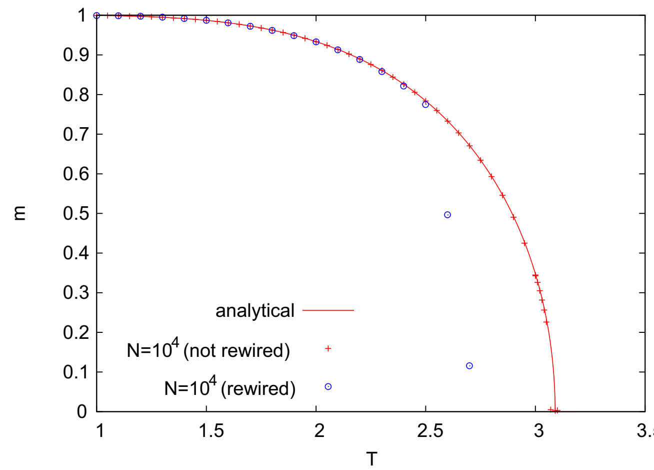

FIG. 2. (Color online) Temperature dependence of the magnetization m for the rewired and nonrewired models ($z = 4$). The data are obtained from Monte Carlo simulations and are compared with the numerical solution of Eq. (3). Simulation and equilibration times are equal to 10^4 Monte Carlo steps.

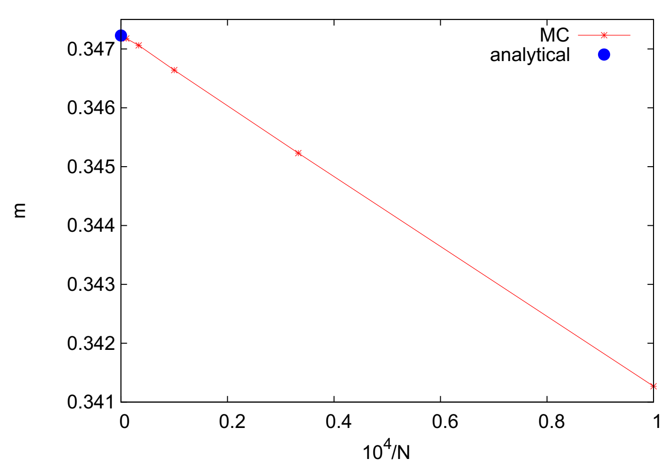

FIG. 3. (Color online) Magnetization m as a function of the inverse of size calculated for the nonrewired model of size $N = 10^4$, 3×10^4, 10^5, 3×10^5, 10^6, and 3×10^6, with $z = 4$ and $T = 3$. Simulation and equilibration times are equal to 10^7 and 10^4 Monte Carlo steps, respectively. In the limit $N \to \infty$, perfect agreement with the solution of Eq. (3) can be seen.

of the steady-state limit of Eq. (3). For $T = 3$, we made much more extensive calculations (Fig. 3). The linear extrapolation $N \to \infty$ based on simulations for $N \leqslant 3 \times 10^6$ gives $m = 0.347\,23(2)$, which is in perfect agreement with $m = 0.347\,225\ldots$ obtained from the numerical solution of Eq. (3). In our opinion, such agreement strongly supports the claim that Eq. (3) is exact (at least in the limit $t \to \infty$).

III. MODEL WITH REWIRING

The model analyzed in the previous section behaves similarly to some other Ising-like models with ferromagnetic and paramagnetic phases separated at the critical point. Our primary motivation is to modify such ordinary ferromagnets so that they would resemble the behavior of financial markets, at least to some extent. We are particularly interested in supplanting a fine-tuned critical point with a more generic critical behavior, which would exist in some, possibly large, temperature range. So far our agents make the decision to buy or sell based on the observation of their z neighbors and the assignment of these neighbors is fixed during the entire evolution of the model. In the present section we modify this rule and allow one to change the neighbors. The rewiring we use is preferential: Each agent has its status equal to the number of in-links that are (currently) attached to it. The selection of a new neighbor takes place with probability proportional to its status [11]. A single step of the dynamics of our model is thus defined as follows: Update spin variables S_i $(i = 1,2,\ldots,N)$ according to the heat-bath algorithm (1) and rewire each agent selecting preferentially anew its z out-links.

Since we keep the dynamics of spin variables basically unchanged, one might expect that Eq. (3) still describes the behavior of our model. Monte Carlo simulations show that to some extent this is indeed the case (Fig. 2) and very good agreement with Eq. (3) can be seen over much of the temperature range. However, close to the critical point $T = T_c$, the rewired model shows much lower and perhaps zero magnetization. It would be desirable to understand the reasons why Eq. (3) is no longer obeyed at higher temperatures.

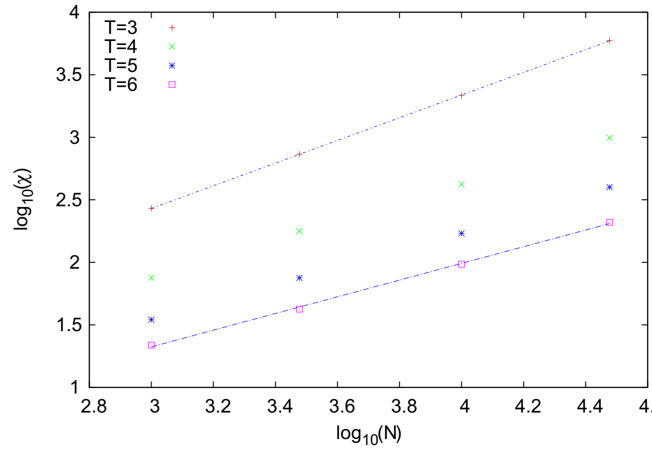

FIG. 5. (Color online) Size dependence of susceptibility χ for the rewired model ($z = 4$). The power-law fit $\chi \sim N^{\alpha}$ shows that α varies from 0.67 for $T = 6$ up to 0.91 for $T = 3$.

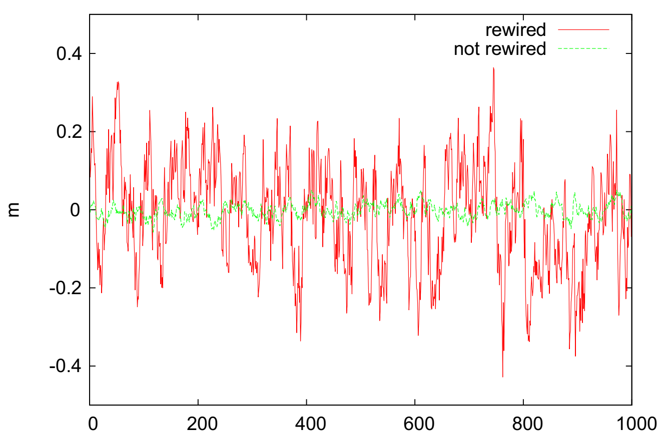

FIG. 4. (Color online) Time dependence of magnetization m for the rewired and nonrewired models ($z = 4$ and $N = 10^4$). Simulations were made for $T = 4$, which for the nonrewired model is deep in the paramagnetic phase.

Possible explanations include the appearance of correlations (which we argue are negligible in the nonrewired case) or a breakdown of homogeneity [which is also one of the assumptions leading to Eq. (3)]. An effort to understand the origin of this behavior will be made in the next section.

What is even more interesting is that the magnetization in the rewired version shows large fluctuations also at temperatures much higher than T_c (Fig. 4). To measure these fluctuations more quantitatively, we calculated the susceptibility χ that up to the temperature factor is equal to the variance of magnetization $\chi = \frac{1}{N}[\langle(\sum_{i=1}^{N} S_i)^2\rangle - \langle\sum_{i=1}^{N} S_i\rangle^2]$. Numerical values indicate that as a function of system size N the susceptibility diverges as $\chi \sim N^{\alpha}$, where $\alpha \sim 0.67$–0.91 depends slightly on temperature (Fig. 5). Such behavior is observed in a large temperature range ($3 \leqslant T \leqslant 6$) for the system size $10^3 \leqslant N \leqslant 3 \times 10^4$. The divergence of susceptibility indicates that the model exhibits a robust critical behavior. Together with data from Fig. 2, this suggests that the model with rewiring has two phases: low-temperature, which is ferromagnetic, and high-temperature, which is critical. It is difficult for us to locate precisely the transition point between these two phases. For longer simulations, it seems to shift slightly toward lower temperatures. Moreover, one cannot exclude that at sufficiently large temperature the critical phase will be replaced with the paramagnetic one (having much smaller fluctuations).

The critical behavior in our model is also robust with respect to the frequency of rewiring. We made simulations with rewiring taking place, e.g., with probability 0.1 (i.e., with probability 0.9, the out-links of a given agent at a given step were left unchanged). Such modification slows down the dynamics but retains the power-law divergence of the susceptibility.

IV. DYNAMICS OF REWIRING

To get some understanding of our model, we look at the structure of the network that emerges during the rewiring. We notice that rewiring is not affected by spin variables

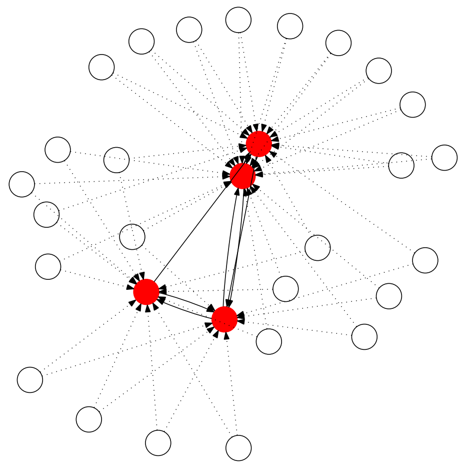

FIG. 6. (Color online) Network structure after simulations of $t = 10^3$ steps for $N = 30$ agents and $z = 2$ [12]. Nodes with no in-links (open circles) represent agents that do not influence the decisions of any other agent (links that go out from them are drawn with dotted lines). Nodes with some in-links (closed red circles) represent the only agents that influence other agents (their out-links are drawn with solid lines).

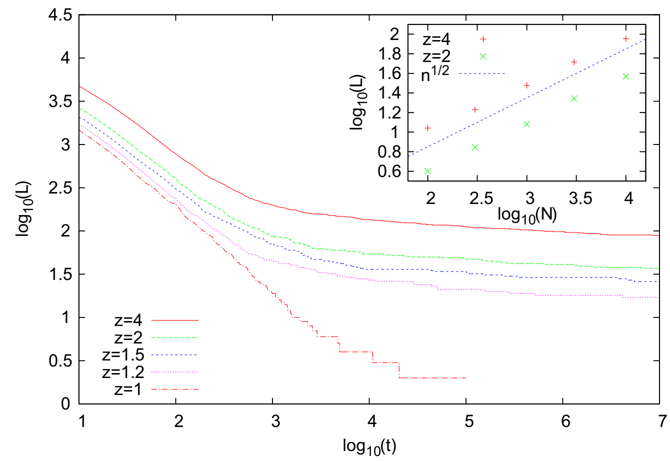

FIG. 8. (Color online) Time dependence of the size of the core (log-log scale); calculations were made for $N = 10^4$. The inset shows that for $z = 2$ and 4 the steady-state size of the core increases approximately as $N^{1/2}$. To generate the network of fractional z ($1 < z < 2$), with probability $2 - z$ we created one out-link and with probability $z - 1$ we created two such links for each agent. Thus fractional z has only an average sense.

and thus might be considered as an independent process (but not vice versa—the spin dynamics depends of course on the structure of the network). Some insight is already obtained from simulations of a small system (Fig. 6). One can notice that most agents have no in-links and thus they do not influence any other agent. There is only a small core of agents that are responsible for the decision formation of the other agents. Such structure appears also for larger systems (Fig. 7). One can notice a substantial heterogeneity of the resulting core as for the number of agents that a given agent is influencing.

A more detailed analysis shows, however, that the core size L slowly diminishes in time (Fig. 8). This is not surprising since once an agent loses all of its in-links, it cannot get them back because the probability to be selected in a rewiring process is proportional to the current number of in-links (which is 0 for such an agent). Although the process of diminishing of L is irreversible, it is extremely slow for $z > 1$. Only at $z = 1$, this process is considerably faster and in a large time interval consistent with t^{-1} decay. Such a slow decay for $z > 1$ suggests that at long (but not infinitely long) time, the core is almost in a steady state and has a certain size. Numerical calculations show that for $z = 2$ and 4 it increases with the system size approximately as $N^{1/2}$ (inset in Fig. 8).

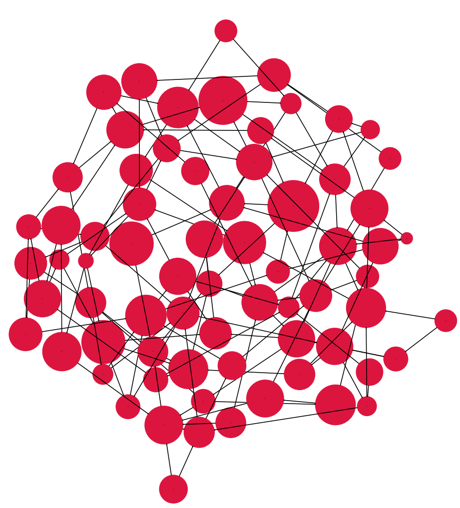

FIG. 7. (Color online) Network structure after simulations of $t = 10^4$ steps for $N = 10^3$ agents and $z = 2$ [12]. Agents with no in-links are omitted. The size of a circle is proportional to the number of in-links.

Some insight into the stability of the core can be obtained from the analysis of the time τ needed for the system to condensate, i.e., for a given z to reach the core size $z + 1$ (which is the smallest core size that the system can reach). Numerical calculations show that for $z > 1$, τ exhibits a fast, possibly exponential, increase with the system size (Fig. 9). Again, a slower increase ($\sim N$) is obtained only at $z = 1$.

Our results in Figs. 8 and 9 show that rewiring for $z = 1$ leads to a rather fast condensation, while for $z > 1$ the dynamics is basically trapped in a core of size $\sim N^{1/2}$. Even though the condensed state could be reached in principle, for large N and $z > 1$ it virtually never happens. The situation is reminiscent of some models with the so-called absorbing states: For some values of control parameters, the absorbing state of the dynamics is basically unavailable and the model remains in the active phase (the lifetime of which in that regime is also exponentially divergent with the system size) [13].

A network structure with a nearly stable core suggests an explanation of the generic divergence of susceptibility that we reported in the previous section. Indeed, since agents are influenced only by agents from the core, on average there are $N^{1/2}$ agents that are influenced by a single agent belonging to the core. Considering core agents as independent (and influencing $N^{1/2}$ other agents), we easily obtain that $\chi \sim N^{1/2}$. Numerical results (Fig. 5) suggest a faster increase

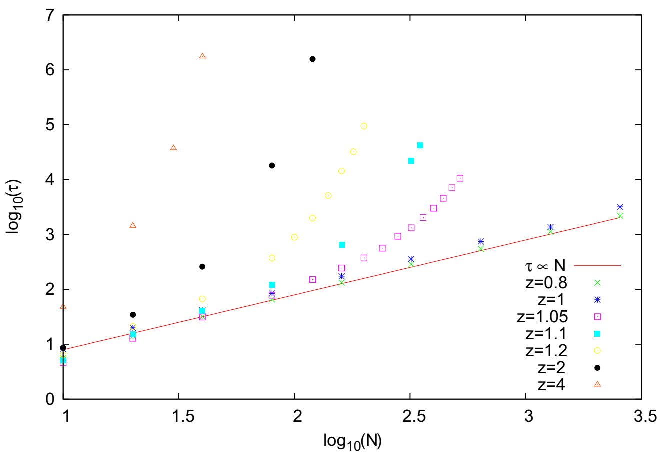

FIG. 9. (Color online) System size dependence of the time τ to condensate, i.e., to reach the core size $z+1$. Notice that even for z slightly larger than 1, τ shows a rapid increase.

at low temperatures (for $T=3$, we obtained $\chi \sim N^{0.91}$) that most likely result from some correlations between core agents. Another factor affecting our simple estimations of the divergence of χ might be some heterogeneity of the core (Fig. 7).

V. CONCLUSION

In the present paper we have shown that preferential rewiring changes an Ising ferromagnet, which has a single critical point, into a model with robust critical behavior. The rewiring mechanism that we used is supposed to mimic the behavior of financial agents who try to follow their neighbors but at the same time have also some freedom to choose the ones to follow. We assume that the preference in the rewiring process is proportional to the number of in-links of a given agent. It is thus not a more or less objective measure of its performance but solely how the agent is perceived by the population of other agents. Similar recipes turned out to be successful in, e.g., some page rank algorithms used by search engines [14] or various recommendation systems [15].

Our model is of some interest from the statistical-mechanics point of view. Due to out-homogeneity, we could write a simple mean-field-like equation [Eq. (3)] that can be used to obtain the magnetization of the model. We argued, however, that for the present model in the nonrewired version this equation should be exact and numerical simulations provide a very strong support for the claim. Very good agreement with this equation was obtained also for the rewired case, but only in the low-temperature regime. We made some attempts to explain why rewiring invalidates this equation at higher temperatures and at the same time leads to the divergence of susceptibility and criticality. In our opinion this is related to the formation of a relatively small subset of agents that retain some in-links and thus drive the entire system.

The change of the dynamical regime in the rewired version at $z=1$ is also of some interest. It is tempting to associate the change with some percolation transition that for random graphs is known to take place at $z=1$ [16]. However, rewiring redistributes links in a highly nonrandom fashion and a possible relation to random graphs is by no means obvious.

We hope that our model might be useful also in the econophysics context. Relatively simple rules that generate a robust criticality might serve as a starting point for further modifications and analysis. For example, one might consider a model where an agent that no longer has any in-links still retains some (small) status ϵ and can be thus selected during the rewiring process. We expect that for small (possibly N-dependent) ϵ such a model would be similar to our ($\epsilon=0$) model, but certainly numerical simulations would be needed to support such a claim. One of the important stylized facts that apparently is missing in our model is volatility clustering. One might hope that some extensions where agents, for example, try to be in the minority (like in the so-called minority games [17]) or use more sophisticated strategies (such as fundamentalist, trend follower, or noise trader) will provide a more realistic description of financial markets and at the same time will retain simplicity of the model.

ACKNOWLEDGMENTS

The research for this work was supported by Narodowe Centrum Nauki (NCN) Grants No. 2013/09/B/ST6/02277 (A.L.) and No. 2011/01/B/HS2/01293 (D.L.) and Ministry of Science and Higher Education Grant No. N N202 488039 (K.G.).

[1] A. Chakraborti, I. M. Toke, M. Patriarca, and F. Abergel, Quant. Financ. **11**, 991 (2011).
[2] B. LeBaron, Quant. Financ. **1**, 254 (2001); E. Samanidou, E. Zschischang, D. Stauffer, and T. Lux, Rep. Prog. Phys. **70**, 409 (2007).
[3] R. Cont and J. P. Bouchaud, Macroecon. Dyn. **4**, 170 (2000).
[4] T. Kaizoji, Physica A **287**, 493 (2000); W.-X. Zhou and D. Sornette, Eur. Phys. J. B **55**, 175 (2007).
[5] S. Bornholdt, Int. J. Mod. Phys. C **12**, 667 (2001); T. Kaizoji, S. Bornholdt, and Y. Fujiwara, Physica A **316**, 441 (2002).
[6] T. Lux and M. Marchesi, Int. J. Theor. Appl. Financ. **3**, 675 (2000).
[7] Some other rewiring mechanisms are known to lead to interesting complex networks: J. Ohkubo and T. Horiguchi, J. Phys. Soc. Jpn. **74**, 1334 (2005); J. Ohkubo, M. Yasuda, and K. Tanaka, Phys. Rev. E **72**, 065104 (2005).
[8] F. W. S. Lima and M. A. Sumour, Physica A **391**, 948 (2012).
[9] M. Leone, A. Vazquez, A. Vespignani, and R. Zecchina, Eur. Phys. J. B **28**, 191 (2002).
[10] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E **66**, 016104 (2002).
[11] Such a selection mechanism is known as a roulette-wheel selection. Recently, an efficient O(1) algorithm of this process was proposed: A. Lipowski and D. Lipowska, Physica A **391**, 2193 (2012).

[12] Network graphs were drawn using GRAPHVIZ: J. Ellson, E. R. Gansner, E. Koutsofios, S. C. North, and G. Woodhull, in *Graph Drawing Software*, edited by M. Junger and P. Mutzel (Springer, Berlin, 2003), pp. 127–148.
[13] H. Hinrichsen, Adv. Phys. **49**, 815 (2000); G. Ódor, Rev. Mod. Phys. **76**, 663 (2004).
[14] S. Brin and L. Page, Comput. Networks ISDN **30**, 107 (1998).
[15] G. Linden, B. Smith, and J. York, IEEE Internet Comput. **7**, 76 (2003).
[16] P. Erdös and A. Rényi, Publ. Math. Debrecen **6**, 290 (1959); M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, Oxford, 2010).
[17] W. B. Arthur, Am. Econ. Rev. **84**, 406 (1994); D. Challet and Y. C. Zhang, Physica A **246**, 407 (1997); D. Challet, M. Marsili, and Y. C. Zhang, *ibid.* **294**, 514 (2001).