Efficiency at maximum power of a quantum heat engine based on two coupled oscillators

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We propose and theoretically investigate a system of two coupled harmonic oscillators as a heat engine. We show how these two coupled oscillators within undamped regime can be controlled to realize an Otto cycle that consists of two adiabatic and two isochoric processes. During the two isochores the harmonic system is embedded in two heat reservoirs at constant temperatures T_h and $T_c(< T_h)$, respectively, and it is tuned slowly along a protocol to realize an adiabatic process. To illustrate the performance in finite time of the quantum heat engine, we adopt the semigroup approach to model the thermal relaxation dynamics along the two isochoric processes, and we find the upper bound of efficiency at maximum power (EMP) η^* to be a function of the Carnot efficiency $\eta_C(=1 - T_c/T_h)$: $\eta^* \leqslant \eta_+ \equiv \eta_C^2/[\eta_C - (1 - \eta_C)\ln(1 - \eta_C)]$, identical to those previously derived from ideal (noninteracting) microscopic, mesoscopic, and macroscopic systems.

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I. INTRODUCTION

The optimization on the performance of heat engines that proceed in finite time is always carried out within the context of finite-time thermodynamics [1–24], which began with the so-called endoreversible model proposed by Curzon and Ahlborn [1]. Based on a Carnot-like cycle operating between a hot and a cold reservoir at constant temperatures T_h and $T_c(<T_h)$, Curzon and Ahlborn (CA) found the efficiency at maximum power (EMP) to be of the form $\eta_{CA} = 1 - \sqrt{T_c/T_h} = 1 - \sqrt{1 - \eta_C}$, where $\eta_C = 1 - T_c/T_h$ is Carnot efficiency. The issue of the EMP has triggered intensive studies ranging from classical to quantum regime, with special emphasis on the bounds and possible universality of the EMP [10,20,22,25–36]. It is found that, at small relative temperature differences, the EMP η^* can be universally expressed in terms of the Carnot efficiency η_C, $\eta^* = \eta_C/2 + \eta_C^2/8 + O(\eta_C^3)$. This universality [20,27,30,32–36] holds well both for ideal and interacting systems which are ranging from microscopic to macroscopic scale, and it is also independent of sources of irreversibility: heat transfer between a finite temperature difference and internal dissipation (e.g., decoherence and relaxation [11,20]). In a recent article, Wu *et al.* [27] proposed a quantum Otto engine, in which an ideal spin or harmonic system (obeying Fermi-Dirac or Bose-Einstein statistics) as the working substance couples to two heat reservoirs at constant temperatures T_c and T_h, and found the upper bound of EMP, η_+, to be

$$\eta^* \leqslant \eta_+ \equiv \frac{\eta_C^2}{[\eta_C - (1 - \eta_C)\ln(1 - \eta_C)]} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{7\eta_C^3}{96} + O\left(\eta_C^4\right), \tag{1}$$

which can be excellently approximated by $\eta_{CA} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{6\eta_C^3}{96} + O(\eta_C^4)$. This result, also found in microscopic [20], mesoscopic [35], and macroscopic [32] systems, is particularly interesting, since it is independent of properties of these systems (In contrast, the performance characteristics of the classical Otto cycle depends sensitively on the working substance.) Of greater physical relevance are interacting systems instead of ideal systems, and some interacting systems have gained much attention in the topic of performance in finite time of quantum heat engines [19,21,23]. One generic model for investigating various interacting physical systems, including quantum dots [37], trapped atoms [38], and cavity optomechanics [39,40], is a system of coupled harmonic oscillators, which has also an attractive feature of being currently available technology realizing [41,42].

In this paper, we propose a quantum heat engine, in which a system of two coupled harmonic oscillators absorbs (releases) heat from (to) its environment while in contact with two reservoirs at different temperatures. We show how to realize an ideal adiabatic process, with the system being isolated from a heat reservoir, and an isochoric process, where the harmonic oscillators exchange heat with a heat bath. From a semigroup approach [43] we analyze the performance in finite time of the Otto cycle, obtaining the explicit expressions for power output and efficiency. We find that, as in the ideal quantum systems without inclusion of interaction, the EMP for our engine model is also bounded from above by η_+ in Eq. (1).

II. A MODEL OF QUANTUM OTTO ENGINE CYCLE

The system of two coupled harmonic oscillators, with frequencies ω_a and ω_b and with masses m_a and m_b, is sketched in Fig. 1. When these two oscillators are undamped, the system Hamiltonian is the sum of free and interacting parts [44],

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + \frac{1}{2}m_a\omega_a^2x_a^2 + \frac{1}{2}m_b\omega_b^2x_b^2 + \lambda x_a x_b\sqrt{m_a\omega_a m_b\omega_b}, \tag{2}$$

where we have introduced the parameter λ to describe the coupling strength between the two harmonic oscillators a and b. It follows, using the standard transformation: $\hat{c}_a = (m_a\omega_a\hat{x}_a + i\,\hat{p}_a)/\sqrt{2m_a\hbar\omega_a}$, $\hat{c}_b = (m_b\omega_b\hat{x}_b + i\,\hat{p}_b)/\sqrt{2m_b\hbar\omega_b}$, that the

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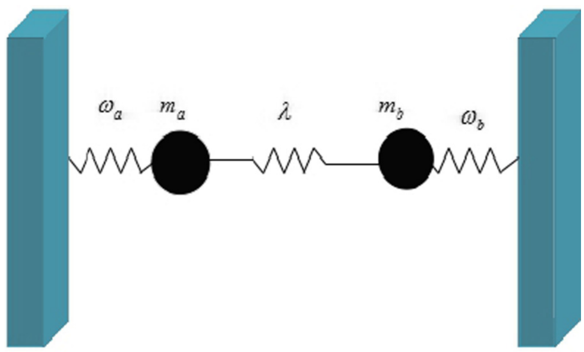

FIG. 1. (Color online) Sketch of the two coupled harmonic oscillators.

Hamiltonian of the system is given by $\hat{H} = \hbar\omega_a(\hat{c}_a^\dagger\hat{c}_a + \frac{1}{2}) + \hbar\omega_b(\hat{c}_b^\dagger\hat{c}_b + \frac{1}{2}) + 2\hbar\lambda(\hat{c}_a + \hat{c}_a^\dagger)(\hat{c}_b + \hat{c}_b^\dagger)$, where $\hat{c}_a(\hat{c}_a^\dagger)$ and $\hat{c}_b(\hat{c}_b^\dagger)$ are the annihilation (creation) operators in terms of bare modes. Here the first term describes two linearly harmonic oscillators, and the second one denotes the coupling of one oscillator to the other. For the reminder of the paper, we will set $\hbar = 1$ for simplicity. Using the normal-mode amplitudes, which are related to the individual coordinates via transformation $x_A = (x_a + x_b)/\sqrt{2}$ and $x_B = (x_a - x_b)/\sqrt{2}$, the Hamiltonian can be expressed in the diagonal form:

$$\hat{H} = \omega_A\big(\hat{A}^\dagger\hat{A} + \tfrac{1}{2}\big) + \omega_B\big(\hat{B}^\dagger B + \tfrac{1}{2}\big), \tag{3}$$

where the new operators $\hat{A}$ and $\hat{B}$ are the boson annihilation operators for the normal-mode excitations of the system, with frequencies

$$\omega_A = \frac{1}{\sqrt{2}}\sqrt{\omega_a^2 + \omega_b^2 - \sqrt{\left(\omega_a^2 - \omega_b^2\right)^2 + 4\lambda^2\omega_a\omega_b}}, \tag{4}$$

$$\omega_B = \frac{1}{\sqrt{2}}\sqrt{\omega_a^2 + \omega_b^2 + \sqrt{\left(\omega_a^2 - \omega_b^2\right)^2 + 4\lambda^2\omega_a\omega_b}}. \tag{5}$$

Analogous to the dressed atom picture [45], these two eigenfrequencies ω_A and ω_B are associated with dressed states for the two coupled harmonic oscillators with frequencies ω_a and ω_b.

A. Adiabatic transition for the coupled harmonic system

Let us assume for simplicity that the parameters of the system composed of two-coupled harmonic oscillators can be fine tuned in order for the relation $\omega_b = r_\omega\omega_a$ with $\omega_a = \text{const}$ to be satisfied. We plot in Fig. 2 the two eigenvalues ω_A and ω_B, as a function of the ratio ω_b/ω_a for fixed coupling strength $\lambda = 0.3\omega_a$ (solid lines), comparing the frequencies of bare modes ω_a and ω_b (dashed lines). When coupling is nonvanishing, the two curves corresponding to the two eigenfrequencies no longer intersect. As shown in Fig. 2, there is a characteristic anticrossing region that can be described by a frequency splitting of $\delta = \omega_B - \omega_A$. At this avoided crossing, the two eigenfrequencies become $\omega_{A,B} = \omega_a\sqrt{1 \pm \frac{\lambda}{\omega_a}}$. In the weak coupling limit, the frequency splitting δ is proportional

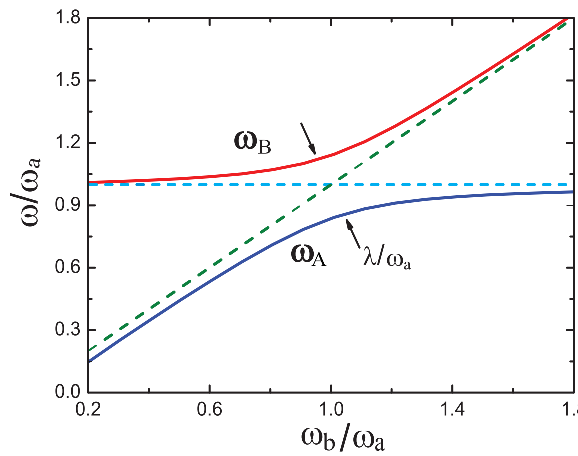

FIG. 2. (Color online) Eigenvalues of the coupled harmonic system, as a function of the ratio of the bare modes ω_b/ω_a for coupling strength $\lambda = 0.3\omega_a$ (solid lines) and $\lambda = 0$ (dashed lines). The frequency splitting $\delta = \omega_B - \omega_A$ is proportional to the dimensionless coupling strength $g = \lambda/\omega_a$.

to the dimensionless coupling strength g: $\delta \propto g$, where we define $g \equiv \lambda/\omega_a$.

If we initially have $\omega_b \ll \omega_a$, we can fine tune slowly the coupled system such that it oscillates at lower eigenfrequency ω_A in Fig. 2 until $\omega_b \gg \omega_a$, indicating the possibility of transferring the energy from one oscillator to the other. The same procedure proceeds if the system initially starts out in the upper eigenfrequency ω_B. That is, when adiabatically tune the system through the coupling region at a very low speed, the system will oscillate at the same eigenfrequency ω_α ($\alpha = A,B$), and its energy will thus remain in the branch in which it was initiated. This scenario that we describe here, to which our analysis will be restricted, is referred to an adiabatic transition [41,46].

B. Dynamics during the interaction interval of the system with a heat bath

If a system is in contact with external fields or heat baths, quantum dynamics is generated and can be used to describe the properties of the system evolution. To find the equation of motion that describes the evolution of a physical quantity along the thermal branch, we resort to the semigroup approach [15,16,43] in which Markovity [15,16] will be imposed on the evolution. Following this approach, an operator $\hat{O}$ (associated with an observable) in the Heisenberg picture is described by the quantum master equation [15,16,18,27]:

$$\frac{d\hat{O}}{dt} = i[\hat{H},\hat{O}] + \frac{\partial\hat{O}}{\partial t} + \mathcal{L}_D(\hat{O}), \tag{6}$$

where $\mathcal{L}_D(\hat{O}) = \sum_\upsilon k_\upsilon(\hat{V}_\upsilon^\dagger[\hat{O},\hat{V}_\upsilon] + [\hat{V}_\upsilon^\dagger,\hat{O}_\upsilon]\hat{V}_\upsilon)$ denotes the Liouville dissipative generator, and $\hat{H}$ is the system Hamiltonian operator. Here $\hat{V}_\upsilon^\dagger$ and $\hat{V}_\upsilon$, Hermitian conjugates, are operators in the Hilbert space for the system, and k_υ are phenomenological positive coefficients. A quantum version of the first law of thermodynamics can be recovered from substitution of $\hat{O} \equiv \hat{H}$ into Eq. (6): $\frac{dE}{dt} = \frac{\bar{d}W}{dt} + \frac{\bar{d}Q}{dt} = \langle\frac{\partial\hat{H}}{\partial t}\rangle + \langle\mathcal{L}_D(\hat{H})\rangle$, with identifying $P = \frac{\bar{d}W}{dt} = \langle \frac{\partial \hat{H}}{\partial t} \rangle$ and $\frac{\bar{d}Q}{dt} = \langle \mathcal{L}_D(\hat{H}) \rangle$ as power and heat flux, respectively.

For a system consisting of a harmonic oscillator, the operators $\hat{V}^\dagger$ and $\hat{V}$ in Eq. (6) can be chosen as the Bosonic creation operator $\hat{\alpha}^\dagger$ and annihilation operator $\hat{\alpha}$ with $\alpha = A,B$. Under the assumption [16] that transitions occur only between adjacent energy levels, we find that the dissipation $\mathcal{L}_D(\hat{O})$ can be expressed as $\mathcal{L}_D(\hat{O}) = k_\uparrow(\hat{\alpha}[\hat{O},\hat{\alpha}^\dagger] + [\hat{\alpha},\hat{O}]\hat{\alpha}^\dagger) + k_\downarrow(\hat{\alpha}^\dagger[\hat{O},\hat{\alpha}] + [\hat{\alpha}^\dagger,\hat{O}]\hat{\alpha})$, where $k_\uparrow$ and $k_\downarrow$ denote the transition rates from the upper to the lower level (for the two adjacent levels) and vice versa. As a result, the motion of the system Hamiltonian becomes

$$\frac{d\langle \hat{H} \rangle}{dt} = -\gamma(\langle \hat{H} \rangle - \langle \hat{H} \rangle^{eq}), \tag{7}$$

where we define $\gamma = k_\downarrow - k_\uparrow$. We assume that the transition rates obey the detailed balance [47]: $\tilde{n}k_\downarrow = (\tilde{n}+1)k_\uparrow$, with $\tilde{n} = 1/(e^{\beta\omega(t)} - 1)$ being the population. From a physical point of view, when a harmonic system couples to a heat reservoir, a single oscillator will jump for one energy state to a lower (upper) one, emitting (absorbing) a photon during the interaction interval. The heat is then exchanged between the system and the reservoir because of the emission and absorption of thermal photons. The relation of the detailed balance, associated with the dynamical information, gives the way the system couples to the heat reservoir, and it must be satisfied such that the system evolves in a specific protocol to achieve thermal equilibrium [27]. Here $\langle H \rangle^{eq}$ as the asymptotic value of $\langle H \rangle$ can be expressed as a function of transition rates $k_\uparrow$ and $k_\downarrow$, $\langle H \rangle^{eq} = \omega n^{eq} = \omega(\frac{k_\uparrow}{k_\downarrow - k_\uparrow} + \frac{1}{2})$, and it can be achieved at thermal equilibrium with $n = (\tilde{n} + 1/2) = \frac{1}{2}\coth(\beta\omega/2)$, where $\beta = 1/(k_B T)$ with the Boltzmann constant k_B and temperature T. In the following we refer to n as the mean population for convenience.

C. Power and efficiency for the quantum Otto cycle

When nonadiabatic transitions between the upper and lower eigenfrequency branches (cf. Fig. 2) are avoided, each band as a protocol can be associated with a different quantum Otto cycle. This Otto cycle along the branch α, with $\alpha = A$ and B, is sketched in the plane of the eigenfrequency ω_α and thermal mean population $n_\alpha = \langle \alpha^\dagger \alpha \rangle$, as shown in Fig. 3. In addition to proceeding two adiabatic processes, the harmonic system undergoes two isochoric processes, with constant frequencies $\omega_{\alpha,h}$ and $\omega_{\alpha,c}$ ($<\omega_{\alpha,h}$), where the system is coupled to a hot and a cold heat reservoir whose inverse temperatures are β_h and β_c ($>\beta_h$), respectively. Switching to a quantitative analysis, the four consecutive steps of the cycle have to be described in detail from a dynamical view.

(1) *Adiabatic compression.* This process in which the system is isolated from a heat reservoir is realized by modulating the frequencies, such that the eigenfrequencies from the initial value $\omega_{\alpha,c}$ to its final value $\omega_{\alpha,h}$ over a time τ_{ch}. Here and hereafter we use $q_{\alpha,i}$, with $i = 1,\ldots,4$, and $\alpha = A,B$, to denote the values of physical quantity q at the special nodes i of the cycle α. As we discussed in detail in Sec. II A, slowly tuning the coupled harmonic system leads to

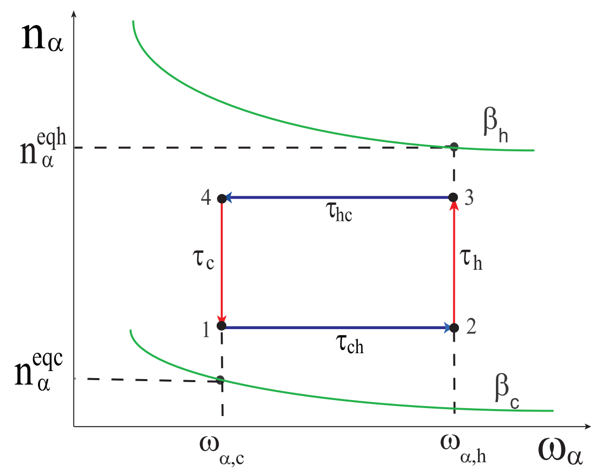

FIG. 3. (Color online) Schematic diagram of a quantum Otto cycle along α branch in the $(\omega_\alpha, n_\alpha)$ plane, with $\alpha = A,B$. The nodes of the cycle i (with $i = 1,2,3,4$) connecting two adiabats $1 \to 2$ and $3 \to 4$ and two isochores $2 \to 3$ and $4 \to 1$ are indicated by circles. n_α^{eqh} and n_α^{eqc} are thermal mean populations of the system approaching thermal equilibrium with the hot and cold reservoirs at inverse temperatures β_h and β_c, respectively.

adiabatical evolution while decoupling from the heat baths. We use $W_{\alpha,ch}$ to represent the total work input along the compression.

(2) *Hot isochore.* At this point system is predominantly coupled to the hot reserver at temperature β_h, and it is allowed to thermalize over a time τ_h, with constant frequency $\omega_\alpha = \omega_{\alpha,h}$. We note from Eqs. (4) and (5) that, for the system with fixed coupling strength λ, such an isochoric process can be realized by keeping the frequencies of bare modes (ω_a and ω_b) constant. Assuming that the irreversible entropy production is caused exclusively by the temperature differences between the system and the heat reservoir [11,20], we find from Eq. (7) that the instantaneous heat flow is given by

$$\frac{\bar{d}Q_{\alpha,h}}{dt} = \omega_{\alpha,h}\frac{dn_\alpha(t)}{dt} = \gamma_{\alpha,h}\big[n_\alpha^{eqh} - n_\alpha(t)\big]\omega_{\alpha,h}, \tag{8}$$

where $\gamma_{\alpha,h}$ denotes the heat conductivity between the system and the hot reservoir, and n_α^{eqh}, the mean population at thermal equilibrium with the hot bath, can be achieved when and only when the duration time is infinitely long. We note the relation $n_\alpha(0) = n(\omega_{\alpha,h},\beta_c)$ and $n_\alpha(\infty) = n_\alpha^{eq}(\omega_{\alpha,h},\beta_h)$ and use Eq. (8), and as a result we have $n_\alpha(t) = n_\alpha^{eqh} + [n_{\alpha,2} - n_\alpha^{eqh})e^{-\gamma_{\alpha,h}t}$ or

$$n_{\alpha,3} = n_\alpha^{eqh} + \big[n_{\alpha,2} - n_\alpha^{eqh}\big]e^{-\gamma_{\alpha,h}\tau_h}. \tag{9}$$

The heat injection from the hot bath during the isochoric process with no work done is calculated according to

$$Q_{\alpha,h} = E_{\alpha,3} - E_{\alpha,2} = (n_{\alpha,3} - n_{\alpha,2})\omega_{\alpha,h}. \tag{10}$$

(3) *Adiabatic expansion.* The eigenfrequency ω changes from $\omega_{\alpha,h}$ to its initial value $\omega_{\alpha,c}$ after time τ_{hc}, while the mean population n_α keeps unchanged. The work done on the system during this process is denoted by $W_{\alpha,hc}$.

(4) *Cold isochore.* The system, with its constant frequency $\omega_{\alpha,c}$, is now coupled to a cold reservoir at inverse temperature

β_c ($>\beta_h$) in a time interval τ_c. In the approach similar to that for the hot isochore, the heat current is obtained,

$$\frac{đQ_{\alpha,c}}{dt} = \omega_{\alpha,c}\frac{dn_{\alpha,c}(t)}{dt} = \gamma_{\alpha,c}\big[n_\alpha^{eqc} - n_\alpha(t)\big]\omega_{\alpha,c}, \tag{11}$$

yielding the following relation:

$$n_{\alpha,1} = n_\alpha^{eqc} + \big[n_{\alpha,4} - n_\alpha^{eqc}\big]e^{-\gamma_{\alpha,c}\tau_c}. \tag{12}$$

Here $\gamma_{\alpha,c}$ is the heat conductivity between the working substance and the cold reservoir and $n_\alpha(t)$ should be restricted by the boundary constraints: $n_\alpha(0) = n_\alpha(\omega_{\alpha,c},\beta_h)$ and $n_\alpha(\infty) = n_\alpha^{eqc}(\omega_{\alpha,c},\beta_c)$. As in the hot isochore, the irreversibility we consider is only due to the finite temperature difference between the system and the cold reservoir, and the heat absorbed by the system from the cold reservoir can be directly calculated as

$$Q_{\alpha,c} = E_{\alpha,1} - E_{\alpha,4} = (n_{\alpha,1} - n_{\alpha,4})\omega_{\alpha,c}. \tag{13}$$

After a quantum Otto cycle consisting of the above consequence of consecutive steps, the total energy of the system which is a state function remains unchanged, namely, $\Delta E_\alpha = Q_{\alpha,h} + Q_{\alpha,c} + W_{\alpha,hc} + W_{\alpha,ch} = 0$. Considering the mean thermal populations $n_{\alpha,1} = n_{\alpha,2}$ and $n_{\alpha,3} = n_{\alpha,4}$, the total work done by the system for a single cycle of the normal mode A or B can be obtained,

$$W_{\alpha,\text{cycle}} = -(W_{\alpha,hc} + W_{\alpha,ch}) = (n_{\alpha,3} - n_{\alpha,2})(\omega_{\alpha,h} - \omega_{\alpha,c}), \tag{14}$$

and the efficiency, the ratio of the total work to the input heat, is written as

$$\eta_\alpha = \frac{W_{\alpha,\text{cycle}}}{Q_{\alpha,h}} = 1 - \frac{\omega_{\alpha,c}}{\omega_{\alpha,h}}. \tag{15}$$

Using Eqs. (9) and (12), we derive the relation as $n_{\alpha,3} - n_{\alpha,2} = (n_\alpha^{eqh} - n_\alpha^{eqc})\frac{(e^{\gamma_{\alpha,c}\tau_c}-1)(e^{\gamma_{\alpha,h}\tau_h}-1)}{e^{\gamma_{\alpha,c}\tau_c+\gamma_{\alpha,h}\tau_h}-1}$, where $n_c^{eq} = \frac{1}{2}\coth(\beta_c\omega_{\alpha,c}/2)$ and $n_\alpha^{eqh} = \frac{1}{2}\coth(\beta_h\omega_{\alpha,h}/2)$ are minimum and maximum values of $n_{\alpha,2}$ and $n_{\alpha,3}$, respectively. The mean thermal populations n_c^{eq} and n_α^{eqh} will be achieved by equilibrating the system with heat baths during the two quasistatic isochores. Substituting this relation into Eq. (14) and using the cycle time $t_{\text{cycle}} = \tau_{adi} + \tau_c + \tau_h$ with $\tau_{adi} \equiv \tau_{hc} + \tau_{ch}$, we obtain the power output for a normal-mode cycle as

$$P_\alpha = (\omega_{\alpha,h} - \omega_{\alpha,c})(n_\alpha^{\text{eqh}} - n_\alpha^{eqc}) \times \frac{(e^{\gamma_{\alpha,c}\tau_c} - 1)(e^{\gamma_{\alpha,h}\tau_h} - 1)}{(e^{\gamma_{\alpha,c}\tau_c+\gamma_{\alpha,h}\tau_h} - 1)(\tau_{adi} + \tau_h + \tau_c)}. \tag{16}$$

Since $\omega_{\alpha,h} - \omega_{\alpha,c} > 0$ and $n_\alpha^{\text{eqh}} - n_\alpha^{eqc} > 0$, the power output for any thermal cycle along either branch A ($n_B = \langle B^\dagger B\rangle = 0$) or branch B ($n_A = \langle A^\dagger A\rangle = 0$) is always positive. As a consequence, heat is absorbed during both cycles A and B from the hot reservoir, some of which is converted into work, while the rest of which is released to the cold reservoir, indicating that the coupled harmonic system operates as a heat engine.

III. EFFICIENCY AT MAXIMUM POWER

Before turning to an optimal analysis, we note that the power output is a product of two functions, $g(\beta_c,\omega_{\alpha,c},\beta_h,\omega_{\alpha,h}) \equiv (\omega_{\alpha,h} - \omega_{\alpha,c})(n_\alpha^{\text{eqh}} - n_\alpha^{eqc})$, which merely depends on the external parameters β and ω, and $f(\tau_c,\tau_h,\tau_{adi}) \equiv \frac{(e^{\gamma_{\alpha,c}\tau_c}-1)(e^{\gamma_{\alpha,h}\tau_h}-1)}{(e^{\gamma_{\alpha,c}\tau_c+\gamma_{\alpha,h}\tau_h}-1)(\tau_{adi}+\tau_h+\tau_c)}$, a function of the time durations for the isochoric and adiabatic processes. Provided that the external constraints of the heat engine are given, optimizing the objective function P_α is equivalent to optimizing the time-dependent function $f(\tau_c,\tau_h,\tau_{adi})$ [27].

For all practical purposes we focus on optimizing an engine cycle by varying the external constrains of the heat engine, ω_a and ω_b, under the assumption that the time required for completing the two adiabats, τ_{adi}, is independent of the frequencies. In such a case, optimizing the power output P_α becomes equivalent to optimizing the function $g(\beta_c,\omega_{\alpha,c},\beta_h,\omega_{\alpha,h})$, where g can be expressed as a function of the frequencies of the bare modes, $g = g(\omega_{a,c},\omega_{a,h},\omega_{b,c},\omega_{b,h};\beta_c,\beta_h)$. For two given heat reservoirs with constant temperatures β_c and β_h, practically we vary the two frequencies ω_a and ω_b of oscillators a and b to obtain the maximum power output. Mathematically, when we use the extremal condition, $\frac{\partial P_\alpha}{\partial w_{a,c}} = \frac{\partial P_\alpha}{\partial w_{\alpha,c}}\frac{\partial w_{\alpha,c}}{\partial w_{a,c}} = 0$ and $\frac{\partial P_\alpha}{\partial w_{b,c}} = \frac{\partial P_\alpha}{\partial w_{\alpha,c}}\frac{\partial w_{\alpha,c}}{\partial w_{b,c}} = 0$, where $\frac{\partial w_{\alpha,c}}{\partial w_{a,c}} \neq 0$ and $\frac{\partial w_{\alpha,c}}{\partial w_{b,c}} \neq 0$ [as readily proved from Eqs. (4) and (5)], we then have $\partial P_\alpha/\partial\omega_{\alpha,c} = 0 (=\partial P_\alpha/\partial\omega_{a,c} = \partial P_\alpha/\partial\omega_{b,c})$, leading to

$$\frac{\beta_c x_{\alpha,c}(\omega_{\alpha,h} - \omega_{\alpha,c})}{1 - x_{\alpha,c}} = \frac{x_{\alpha,h} - x_{\alpha,c}}{1 - x_{\alpha,h}}, \tag{17}$$

where we have used $x_{\alpha,c} \equiv e^{-\beta_c\omega_{\alpha,c}}$ and $x_{\alpha,h} \equiv e^{-\beta_h\omega_{\alpha,h}}$. Similarly, we set $\partial P_\alpha/\partial\omega_{\alpha,h} = 0$ to determine the optimal upper bounds of $\omega_{a,h}$ and $\omega_{b,h}$, and we arrive at

$$\frac{\beta_h x_{\alpha,h}(\omega_{\alpha,h} - \omega_{\alpha,c})}{1 - x_{\alpha,h}} = \frac{x_{\alpha,h} - x_{\alpha,c}}{1 - x_{\alpha,c}}. \tag{18}$$

This set of of two nonlinear equations [(17) and (18)], derived previously in heat engines based on various kinds of noninteracting quantum systems [20,27], can be used to determine the optimal values $\omega_{\alpha,c}$ and $\omega_{\alpha,h}$ at maximum power. Based on the same approach as the one in Ref. [27], it is not very difficult to verify using Eqs. (17) and (18) that the EMP η_α^* takes the form

$$\eta_\alpha^* = \eta_+ = \frac{\eta_C^2}{[\eta_C - (1 - \eta_C)\ln(1 - \eta_C)]} \simeq 1 - \sqrt{1 - \eta_C}, \tag{19}$$

thereby leading to an excellent approximation:

$$\frac{\omega_{\alpha,c}}{\omega_{\alpha,h}} \simeq \sqrt{\frac{\beta_h}{\beta_c}}. \tag{20}$$

In the high-temperature limit when $\beta\omega \ll 1$ as well as $n \simeq 1/(\beta\omega)$, we find from Eq. (8) and (11) that the heat transport is identified as Newton's heat-transfer law adopted in Ref. [1], implying that the heat current described by Eqs. (8) or (11) would be close to the Newton's heat transfer law at finite temperatures. Furthermore, the irreversibility for our model is assumed to be exclusively caused by temperature differences between the working subsystem and the heat reservoir, as done in the CA (endoreversible) model [1]. These may help to explain why the difference between η_+ and η_{CA} is extremely small [see Eq. (1)].

Whether the Otto cycle proceeds along either branch A or branch B, the expression of EMP [Eq. (20)] is universal and independent of the coupling strength. In our model, the coupling harmonic system is adiabatically oscillating at a given normal-mode branch α while proceeding an adiabatic process. When slowly tuning the system through the coupling region during an adiabatic process, it is convenient to consider the system parameters within the bare mode picture rather than in dressed normal modes. For the optimal cycle, using Eqs. (4), (5), and (20), it follows that there exists a relation $\frac{w_{a,c}}{w_{a,h}} = \sqrt{\frac{1+r_{\omega,h}^2-\sqrt{4g^2r_{\omega,h}+(r_{\omega,h}-1^2)^2}}{1+r_{\omega,c}^2-\sqrt{4g^2r_{\omega,c}+(r_{\omega,c}-1^2)^2}}}\sqrt{\frac{\beta_h}{\beta_c}}$ for branch A or $\frac{w_{a,c}}{w_{a,h}} = \sqrt{\frac{1+r_{\omega,h}^2+\sqrt{4g^2r_{\omega,h}+(r_{\omega,h}-1^2)^2}}{1+r_{\omega,c}^2+\sqrt{4g^2r_{\omega,c}+(r_{\omega,c}-1^2)^2}}}\sqrt{\frac{\beta_h}{\beta_c}}$ for branch B, where the definitions of $r_{\omega,c} \equiv \omega_{b,c}/\omega_{a,c}$ and $r_{\omega,h} \equiv \omega_{b,h}/\omega_{a,h}$ have been used, thereby indicating that $\frac{w_{a,c}}{w_{a,h}} = \sqrt{\frac{\beta_h}{\beta_c}}$ for any branch α when $r_{\omega,c} = r_{\omega,h}$.

Before ending this section, we would like to point out that our analysis above was restricted to the ideal adiabatic phase where the system varies at a very low speed and thus satisfies the adiabatic condition. At a high tuning speed, in which the change of the frequency difference between the two bare modes in time reads $\nu = \frac{\partial(\omega_b-\omega_a)}{\partial t}$, the diabatic transition arises with its probability [41,46]: $p_{dia} = \exp\left(\frac{-\pi\delta^2}{2\nu}\right)$, which is situated between $0 \leqslant p_{dia} \leqslant 1$ and is the probability of switching between two normal-mode branches when the system passing through the anticrossing region. Accordingly, the probability of an adiabatic transition is determined by $p_{adia} = 1 - p_{dia}$. If the Otto cycle consists of diabatic transitions, we use $\eta^*_{\alpha,dia}$ to represent its EMP. Since negative work output is induced by the nonadiabatic dissipation [20,27,48], the EMP $\eta^*_{\alpha,dia}$ must be smaller than its counterpart $\eta^*_\alpha = \eta_+$ [cf. Eq. (19)] for the cycle with no diabatic transitions, namely, $\eta^*_{\alpha,dia} < \eta_+$. Without loss of generality, the EMP can be given by $\eta^*_\alpha = \eta_+ p_{adia} + \eta^*_{\alpha,dia} p_{dia} = \eta_+(1 - p_{dia}) + \eta^*_{\alpha,dia} p_{dia}$, yielding $\eta^*_\alpha \leqslant \eta_+$. That is, the expression of η_+ in Eq. (1) is certainly the upper bound of the EMP for our engine model.

IV. CONCLUSIONS

To summarize, we have proposed a quantum Otto engine working with a system of two coupled harmonic oscillators with frequencies ω_a and $\omega_b(>\omega_a)$. An adiabatic process is realized by varying the frequency ω_b while keeping the frequency ω_a constant, provided that the nonadiabatic transition between two normal-mode branches is avoided. Based on the semigroup approach, we modeled the thermal relaxation dynamics and determined the heat transferred between the system and the heat bath along an isochoric process. We considered the EMP for the Otto cycle which undergoes along any normal-mode branch by optimizing power with respect to the frequencies of bare modes. Our result shows that the EMP is bounded from above by the universal expression of η_+ as given in Eq. (1), whose validity has thus been extended from noninteracting systems to interacting systems.

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