Landau-like theory for universality of critical exponents in quasistationary states of isolated mean-field systems

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An external force dynamically drives an isolated mean-field Hamiltonian system to a long-lasting quasistationary state, whose lifetime increases with population of the system. For second order phase transitions in quasistationary states, two nonclassical critical exponents have been reported individually by using a linear and a nonlinear response theories in a toy model. We provide a simple way to compute the critical exponents all at once, which is an analog of the Landau theory. The present theory extends the universality class of the nonclassical exponents to spatially periodic one-dimensional systems and shows that the exponents satisfy a classical scaling relation inevitably by using a key scaling of momentum.

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I. INTRODUCTION

Universality of critical exponents is one of central issues in studying phase transitions. For continuous phase transitions in mean-field systems, the Landau theory is a powerful tool to understand the universality and a scaling relation [1]. The idea of the Landau theory is to construct a pseudo free energy $\mathcal{F}(T,M,h)$ in the form of polynomial by using the Landau expansion,

$$\mathcal{F}(T,M,h) = \frac{a(T-T_{\rm c})}{2}M^2 + \frac{b}{4}M^4 + \cdots - hM, \tag{1}$$

where T is temperature, $T_{\rm c}$ its critical value, M the magnetization, h the external field, and a and b positive constants. To search the minimal points we consider the condition

$$\frac{\partial \mathcal{F}}{\partial M} = a(T-T_{\rm c})M + bM^3 + \cdots - h = 0. \tag{2}$$

Let h be sufficiently small and $M = m + \delta m$, where m and δm represent, respectively, the spontaneous part and the response to the small external field h. Then, Eq. (2) is divided into the spontaneous part

$$a(T-T_{\rm c})m + bm^3 + \cdots = 0 \tag{3}$$

and the response part

$$[a(T-T_{\rm c}) + 3m^2]\delta m + 3bm(\delta m)^2 + b(\delta m)^3 + \cdots - h = 0. \tag{4}$$

Picking up the first two leading terms in each considering situation, one can compute the critical exponents $\beta = 1/2, \gamma_\pm = 1$ and $\delta = 3$, which are defined as

$$m \propto (T_{\rm c}-T)^\beta, \quad \left.\frac{d(\delta m)}{dh}\right|_{h\to 0} \propto |T-T_{\rm c}|^{-\gamma_\pm}, \quad m \propto h^{1/\delta}, \tag{5}$$

where γ_+ and γ_- are defined in the paramagnetic (Para) high-temperature side and the ferromagnetic (Ferro) low-temperature side, respectively, and δ at the critical point. These exponents satisfy the scaling relation $\gamma_\pm = \beta(\delta - 1)$ [1].

Before reaching thermal equilibria discussed by the Landau theory, isolated mean-field Hamiltonian systems are dynamically trapped in long-lasting quasistationary states (QSSs), which are vast comparing with thermal equilibria [2–5]. The lifetime of a QSS diverges as population of the system [6,7], and it is therefore possible that observable states are solely QSSs in large population systems. Elliptical galaxies and the great red spot of Jupiter are given as examples of QSSs [5,7]. The long lifetime naturally induces a question: Are the critical exponents in the literature of dynamics the same with of statistical mechanics? Recently this question was answered negatively. Dynamics of the mean-field systems is described by the Vlasov equation [8], and a linear [9,10] and a nonlinear [11] response theories are proposed based on the Vlasov dynamics. The former gives $\gamma_+ = 2\beta$ but $\gamma_- = \beta/2$ [12], and the latter $\delta = 3/2$ [11]. These exponents satisfy the Widom scaling relation $\gamma_- = \beta(\delta - 1)$ irrespective of the value of β.

However, due to lack of a Landau like theory for QSSs, which a clue to show the universality of critical exponents, it has not been clarified how wide the universality class is and accordingly whether the scaling relation holds inevitably or accidentally. There are two obstacles to discuss universality of the critical exponents in the literature of dynamics.

One is that the exponents are obtained only in the Hamiltonian mean-field (HMF) model [13,14] and partially in the α-HMF model [15]. Such systems have particles moving on the unit circle, and interaction has the first Fourier mode only. It is not obvious that systems having higher Fourier modes, for instance, the generalized HMF model [16], also have the same nonclassical critical exponents. Indeed, a non-Hamiltonian model of phase oscillators, whose continuous version has similar features with the Vlasov equation, gives $\beta = 1/2$ for a single sinusoidal coupling, but $\beta = 1$ in a general coupling [17]. The other is that computation of the exponent $\delta = 3/2$ is independent of γ_-. The nonlinear response formula gives a self-consistent equation for the magnetization, and δ is obtained by expanding the equation in the Para side, while γ_- is defined in the Ferro side.

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As the first step to construct the Landau like theory in QSSs, we provide a simple way to obtain an expanded equation of the nonlinear response formula [11,18], which is valid in both the Para and the Ferro sides, and even at the critical point, for spatially periodic one-dimensional (1D) systems with generic interactions. From the equation, we compute the critical exponents and show that holding the scaling relation is inevitable.

This article is organized as follows. The model and setting are introduced in Sec. II. In Sec. III we first expand the nonlinear response formula [11] around the reference state. By use of this expansion, in Sec. III A we derive the critical exponents $\gamma_\pm$ and δ for the HMF model and show that the scaling relation $\gamma_- = \beta(\delta - 1)$ is inevitable. In Sec. III B this result is generalized to the Para-Ferro transition in more general models introduced in Sec. II. We present a summary and discussion in Sec. IV.

II. MODEL AND SETTING

We consider a spatially periodic 1D model described by the N-body Hamiltonian

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N}\sum_{i,j=1}^{N} V(q_i - q_j) + \Theta(t)\sum_{i=1}^{N} H_{\rm ext}(q_i), \tag{6}$$

where $q_i \in (-\pi,\pi]$ is the position of ith particle, and $p_i \in \mathbb{R}$ the conjugate momentum. We assume that the interaction $V(q)$ is even and is expanded into the Fourier series

$$V(q) = -\sum_{k=1}^{K} V_k \cos kq, \quad V_k \neq 0, \tag{7}$$

where K is finite. $H_{\rm ext}$ represents contribution from the external force, and $\Theta(t)$ is the Heaviside step. That is, the external force kicks in at $t = 0$. We remark that the following theory can be extended to the external force which goes to be constant asymptotically instead of the step function. We also assume that the external part $H_{\rm ext}(q)$ is expanded into the Fourier series

$$H_{\rm ext}(q) = -\sum_{k=1}^{K} h_k \cos kq, \tag{8}$$

where h_k is the conjugate force of $\cos kq$ and is assumed to be small constant. This model (6) includes the HMF model [14] by setting $K = 1$ and $V_1 = 1$, and the generalized HMF model [16] by $K = 2$, $V_1 = \Delta$ and $V_2 = 1 - \Delta$.

The corresponding single body effective Hamiltonian is

$$\begin{aligned} \mathcal{H}[f](q,p,t) &= p^2/2 + \mathcal{V}[f](q,t) + \Theta(t)\mathcal{H}_{\rm ext}(q), \\ \mathcal{V}[f](q,t) &= -\sum_{k=1}^{K} V_k (M_{kx}\cos kq + M_{ky}\sin kq), \\ \mathcal{H}_{\rm ext}(q) &= -\sum_{k=1}^{K} h_k \cos kq, \end{aligned} \tag{9}$$

where the order parameters are defined as

$$(M_{kx}, M_{ky}) = \int_\mu (\cos kq, \sin kq) f(q,p,t)\, dq\, dp, \tag{10}$$

with $\mu = (-\pi,\pi] \times \mathbb{R}$. The single body distribution function f is governed by the Vlasov equation

$$\partial_t f + \{\mathcal{H}[f], f\} = 0, \quad f(q,p,0) = f_{\rm I}(q,p), \tag{11}$$

where the Poisson bracket $\{a,b\}$ is given by

$$\{a,b\} = \frac{\partial a}{\partial p}\frac{\partial b}{\partial q} - \frac{\partial a}{\partial q}\frac{\partial b}{\partial p}. \tag{12}$$

We assume $M_{ky} = 0 (k = 1, \ldots, K)$ and M_{kx} is simply denoted by M_k, which is divided into $M_k = m_k + \delta m_k$ where m_k and δm_k are the spontaneous part and the response to the external field, respectively. In the following, we focus on the phase transition between the Para phase ($m_1 = \cdots = m_K = 0$) and the Ferro phase ($m_1, \ldots, m_K \neq 0$ in general).

We start from a stable stationary state $f_{\rm I}$ at $t < 0$ and exert the external force at $t = 0$. We assume that the external force drives the state to another stable stationary state $f_{\rm A}$ asymptotically. The two stationary states, $f_{\rm I}$ and $f_{\rm A}$, give the Hamiltonians $\mathcal{H}_{\rm I} = \mathcal{H}[f_{\rm I}]$ and $\mathcal{H}_{\rm A} = \mathcal{H}[f_{\rm A}]$, respectively, which differ from each other in general. Due to the 1D nature and integrability of $\mathcal{H}_{\rm I/A}$, angle-action variables $(\theta_{\rm I/A}, J_{\rm I/A})$ are available and $\mathcal{H}_{\rm I/A}$ depends on $J_{\rm I/A}$ only. We denote the angle averages of an observable Y as

$$\langle Y\rangle_{\rm I} = \frac{1}{2\pi}\int_0^{2\pi} Y(q,p)\, d\theta_{\rm I}, \quad \langle Y\rangle_{\rm A} = \frac{1}{2\pi}\int_0^{2\pi} Y(q,p)\, d\theta_{\rm A}, \tag{13}$$

where, for instance, the subscript I of $\langle\cdot\rangle_{\rm I}$ represents to take the average over each connected iso-$J_{\rm I}$ curve.

III. EXPANSION OF NONLINEAR RESPONSE FORMULA

The nonlinear response theory provides the asymptotic state $f_{\rm A}$, which is roughly represented as $f_{\rm A} = \langle f_{\rm I}\rangle_{\rm A}$ (see Ref. [11] for details and also Ref. [19]). Jeans theorem [7,20] states that $f(q,p)$ is stationary if and only if it depends on (q,p) solely through the first integrals. Thus, we may have functions $F_{\rm I}$ and $F_{\rm A}$ satisfying

$$f_{\rm I}(q,p) = F_{\rm I}(\mathcal{H}_{\rm I}(q,p)), \quad f_{\rm A}(q,p) = F_{\rm A}(\mathcal{H}_{\rm A}(q,p)). \tag{14}$$

Our job is to expand $f_{\rm A}$ around the reference state $f_{\rm I}$ for computing the small response.

We assume that $F_{\rm I}$ is given and smooth, but the form and smoothness of $F_{\rm A}$ are not obvious due to existence of the average $\langle\cdot\rangle_{\rm A}$ [11]. We, therefore, expand $f_{\rm A}$ by extracting $f_{\rm I}$ from the averaged form $\langle F_{\rm I}(\mathcal{H}_{\rm I})\rangle_{\rm A}$. The idea is to use the fact that the bracket $\langle\cdot\rangle_{\rm A}$ can be removed for any function $\psi(\mathcal{H}_{\rm A})$ as $\langle\psi(\mathcal{H}_{\rm A})\rangle_{\rm A} = \psi(\mathcal{H}_{\rm A})$, since the bracket represents the average over an iso-$J_{\rm A}$ curve while $\mathcal{H}_{\rm A}$ is constant along the curve. Keeping this fact in mind and denoting the order of external force as $O(\mathcal{H}_{\rm ext}) = O(h)$, we expand $f_{\rm A} = \langle F_{\rm I}(\mathcal{H}_{\rm I})\rangle_{\rm A}$ as

$$\begin{aligned} \langle F_{\rm I}(\mathcal{H}_{\rm I})\rangle_{\rm A} &= \langle F_{\rm I}(\mathcal{H}_{\rm A} - \delta\mathcal{V})\rangle_{\rm A} \\ &= F_{\rm I}(\mathcal{H}_{\rm A}) - F_{\rm I}'(\mathcal{H}_{\rm A})\langle\delta\mathcal{V}\rangle_{\rm A} \\ &\quad + F_{\rm I}(\mathcal{H}_{\rm A})\int_\mu F_{\rm I}'(\mathcal{H}_{\rm A})\delta\mathcal{V} dq dp + O(h^2), \end{aligned} \tag{15}$$

where the asymptotic Hamiltonian $\mathcal{H}_{\mathrm{A}}$ is expanded as $\mathcal{H}_{\mathrm{A}} = \mathcal{H}_{\mathrm{I}} + \delta\mathcal{V}$ with

$$\delta\mathcal{V} = -\sum_{k=1}^{K}(V_k\delta m_k + h_k)\cos kq \tag{16}$$

small. We note that the last term of the right-hand-side of (15) is not of $O(h^2)$ at the critical point, but this change does not affect the following discussions since the term will be omitted. The third term comes from expansion of the normalization factor. To understand the third term, we remark that the normalized $F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}})$ can be written by $F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}}) = G(\mathcal{H}_{\mathrm{I}})/\int_\mu G(\mathcal{H}_{\mathrm{I}})\,dq\,dp$, where, for instance, the function of energy $G(E)$ is $G(E) = \exp(-E/T)$ if f_{I} is in canonical thermal equilibrium with temperature T. Using $\mathcal{H}_{\mathrm{A}} = \mathcal{H}_{\mathrm{I}} + \delta\mathcal{V}$ again in the first term of the right-hand-side of (15), we have

$$\begin{aligned} F_{\mathrm{I}}(\mathcal{H}_{\mathrm{A}}) &= F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}}) + F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\delta\mathcal{V} \\ &\quad - F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}})\int_\mu F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\delta\mathcal{V}dqdp + O(h^2). \end{aligned} \tag{17}$$

We may replace $\mathcal{H}_{\mathrm{A}}$ with $\mathcal{H}_{\mathrm{I}}$ in the third term of (15) by omitting $O(h^2)$ terms, and the replaced term cancels out with the last term of (17). Combining them, we have

$$f_{\mathrm{A}} = f_{\mathrm{I}} + F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\delta\mathcal{V} - F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{A}})\,\langle\delta\mathcal{V}\rangle_{\mathrm{A}} + O(h^2). \tag{18}$$

To clarify the physical interpretation of each term, we rewrite it as follows:

$$\begin{aligned} f_{\mathrm{A}} &= f_{\mathrm{I}} + F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})(\delta\mathcal{V} - \langle\delta\mathcal{V}\rangle_{\mathrm{I}}) \\ &\quad + [F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\langle\delta\mathcal{V}\rangle_{\mathrm{I}} - F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{A}})\langle\delta\mathcal{V}\rangle_{\mathrm{A}}] + O(h^2), \end{aligned} \tag{19}$$

where the first two terms of the right-hand side can be also obtained by the linear response theory [12], and the third one is the main nonlinear effect of order $o(h)$. This is the main expansion of this article.

Five remarks for (19) are in order: (1) The factors $\langle\delta\mathcal{V}\rangle_{\mathrm{I}}$ and $\langle\delta\mathcal{V}\rangle_{\mathrm{A}}$ in (19) are the origin of the nonclassical critical exponents as we will see later. (2) It will be shown that the third term is of higher order than the second term. (3) The expansion up to the second term is consistent with the linear response theory based on the Vlasov equation [9,10]. Essence of the Vlasov linear response theory is to input existence of the Casimir invariants, $\int_\mu s(f)\,dq\,dp$, for s smooth. Simple algebraic computations reveal that contribution from the term $-F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\langle\delta\mathcal{V}\rangle_{\mathrm{I}}$ keeps the Casimirs up to the linear order [12]. (4) We assumed $\delta\mathcal{V}$ small but did not assume m_k small up to here. (5) The right-hand side still depends on f_{A} through $\mathcal{H}_{\mathrm{A}}$, $\delta\mathcal{V}$ and the average $\langle\cdot\rangle_{\mathrm{A}}$. We will introduce self-consistent equations for the order parameters, whose expansions correspond to the Landau's equation (2).

To discuss the critical exponents, we consider a one-parameter family of the initial states f_{I} parameterized by τ continuously and set the critical point as $\tau = 0$. For instance, τ is the reduced temperature $(T - T_{\mathrm{c}})/T_{\mathrm{c}}$ if one considers a family of Boltzmann distributions, $F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}}) \propto \exp(-\mathcal{H}_{\mathrm{I}}/T)$. Another example is a family of Fermi-Dirac type distributions where $F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}}) \propto 1/\{\exp[(\mathcal{H}_{\mathrm{I}} - \mu)/T] + 1\}$. For a suitably fixed value of T, this family has a critical point of continuous transition at $\mu = \mu_{\mathrm{c}}$, and we may set $\tau = \mu - \mu_{\mathrm{c}}$ [12].

A. HMF model case

It might be instructive to derive the critical exponents from (19) for the HMF model before progressing to the general case. Let m_1 be the order parameter in f_{I} and $m_1 + \delta m_1$ in f_{A}. The perturbation $\delta\mathcal{V}$ is $-(\delta m_1 + h)\cos q$ in this case. The self-consistent equation in the asymptotic state is $m_1 + \delta m_1 = \int_\mu f_{\mathrm{A}}(q,p)\cos q\,dq\,dp$ and, by using the main expansion (19), it is expanded into

$$D^{(\mathrm{homo})}m_1 + Bm_1^3 + \cdots = 0 \tag{20}$$

for the spontaneous part corresponding to (3), and

$$D(\delta m_1 + h_1) + C(\delta m_1 + h_1) - h_1 = O\left(h_1^2\right) \tag{21}$$

for the response part corresponding to (4). Here

$$D = 1 + \int_\mu F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})(\cos q - \langle\cos q\rangle_{\mathrm{I}})\cos q\,dq\,dp, \tag{22}$$

$$C = \int_\mu [F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})\langle\cos q\rangle_{\mathrm{I}} - F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{A}})\langle\cos q\rangle_{\mathrm{A}}]\cos q\,dq\,dp, \tag{23}$$

and $D^{(\mathrm{homo})}$ is defined by forcedly setting $\mathcal{H}_{\mathrm{I}} = p^2/2$ and $\langle\cos q\rangle_{\mathrm{I}} = 0$ accordingly in (22). The functional B is obtained by expanding $F_{\mathrm{I}}(\mathcal{H}_{\mathrm{I}})$ with respect to m_1 and is assumed to be positive. The functional D is called the dispersion function, or the dielectric function in the literature of plasma, and the state f_{I} is stable if and only if $D > 0$ [21]. The functional C goes to zero in the limit of $h_1 \to 0$, in other words A $\to$ I, and the second term of (21) is of higher order than the first. Following the spirit of Landau theory, we compute the critical exponents by picking up the first two leading terms.

The dispersion function for homogeneous state, $D^{(\mathrm{homo})}$, is positive (resp. negative) in the Para (resp. Ferro) sides, and spontaneous magnetization in the Ferro side is $m_1 \propto \sqrt{-D^{(\mathrm{homo})}}$. In general, we may expect $|D^{(\mathrm{homo})}| \propto \tau$ around $\tau = 0$ and hence $\beta = 1/2$.

In the response part, the first two leading terms make

$$D(\delta m_1 + h_1) - h_1 = 0, \tag{24}$$

and the critical exponents $\gamma_\pm$ are determined by the convergent speed of D to zero. To discuss D in the two phases separately, we denote D in the Para and the Ferro sides by $D^{(\mathrm{Para})}$ and $D^{(\mathrm{Ferro})}$, respectively. In the Para side, $D^{(\mathrm{Para})} = D^{(\mathrm{homo})}$ and immediately $\gamma_+ = 2\beta$. In the Ferro side, we have nonzero $\langle\cos q\rangle_{\mathrm{I}}$, and this factor makes the convergence slower. This slow convergence is observed by introducing a new variable

$$\kappa = \sqrt{\frac{\mathcal{H}_{\mathrm{I}} - \mathcal{H}_{\mathrm{I}}(0,0)}{\Delta\mathcal{H}_{\mathrm{I}}}}, \quad \Delta\mathcal{H}_{\mathrm{I}} = \mathcal{H}_{\mathrm{I}}(\pi,0) - \mathcal{H}_{\mathrm{I}}(0,0) = 2m_1, \tag{25}$$

where $\kappa = 0$ at the energy minimum point, the origin, and $\kappa = 1$ on the separatrix. The system with the effective Hamiltonian $\mathcal{H}_{\mathrm{I}}$ has two fixed points: One is (0,0), which is the center, and the other is the saddle $(\pi,0)$, which is identical to $(-\pi,0)$. The separatrix $\{(q,p)|\kappa = 1\}$ is the iso-energy set, which consists of stable and unstable manifolds of the saddle and connects the two (identical) saddles. The separatrix width to momentum direction is of $O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) = O(\sqrt{m_1})$ from the definition of

$\Delta\mathcal{H}_{\rm I}$ and plays an important role to obtain the critical exponent γ_-. To observe it, we first note that [12]

$$1+\int_\mu F_{\rm I}'(\mathcal{H}_{\rm I})\cos^2 q\,dq\,dp = O(|D^{\rm (homo)}|) = O\big(m_1^2\big). \quad (26)$$

Next, we focus on the remaining part of D, namely, $\int_\mu F_{\rm I}'(\mathcal{H}_{\rm I})\langle\cos q\rangle_{\rm I}\cos q dq dp$. This term goes to zero as $\tau\to 0$ since $\langle\cos q\rangle_{\rm I}\to 0$, and hence inhomogeneous nature, in other words nonzero separatrix width, controls the convergent speed of the term to zero. We hence extract the convergent speed from the integral by scaling the separatrix width to a constant. Remembering that $\kappa=1$ represents the separatrix, we change the variable from p to κ and $dq\,dp\propto\sqrt{\Delta\mathcal{H}_{\rm I}}\,dq\,d\kappa$. Consequently, we have the estimation of $D^{\rm (Ferro)}=O(\sqrt{\Delta\mathcal{H}_{\rm I}})=O(\sqrt{m_1})$, since the scaled integral does not vanish at the critical point [12] and the term $\int_\mu F_{\rm I}'(\mathcal{H}_{\rm I})\langle\cos q\rangle_{\rm I}\cos q\,dq\,dp$ dominates the other. The exponent γ_- is, therefore, $\gamma_-=\beta/2$. We stress that the crucial scaling of this estimation is $O(p)=O(\kappa\sqrt{\Delta\mathcal{H}_{\rm I}})$ in the definition (25) to scale the separatrix width to a constant.

At the critical point, the dispersion function D vanishes, and the first two leading terms make

$$C(\delta m_1+h_1)-h_1=0. \quad (27)$$

Using $\langle\cos q\rangle_{\rm I}=0$ and the same variable transform from p to κ as (25) with replacing $\mathcal{H}_{\rm I}$ with $\mathcal{H}_{\rm A}$, we can estimate C as $C\propto\sqrt{\delta m_1+h_1}$. Thus, the critical exponent is $\delta=3/2$.

With the aid of above understanding, we reveal that the scaling relation is inevitable by generalizing the key scaling as $O(p)=O[\kappa(\Delta\mathcal{H}_{\rm I/A})^x]$ with $0<x<1$. The condition $x<1$ ensures that the discussed terms are larger than the omitted $O(h_1^2)$. The same computations with the HMF case give $\gamma_-=\beta x$ and $\delta=1+x$, which satisfy the scaling relation $\gamma_-=\beta(\delta-1)$. Thus, averaged terms of $\langle\delta\mathcal{V}\rangle_{\rm I}$ and $\langle\delta\mathcal{V}\rangle_{\rm A}$, which appears in the Vlasov (non)linear response theory, induces the nonclassical critical exponents and the scaling relation inevitably.

B. General case

Let us come back to the general case. Let $m=(m_1,\ldots,m_K)$ be the spontaneous order parameter vector in $f_{\rm I}$, and $\delta m=(\delta m_1,\ldots,\delta m_K)$ the response to the external force $h=(h_1,\ldots,h_K)$. As the HMF case, substituting the main expansion (19) into the self-consistent equation

$$m_k+\delta m_k=\int_\mu f_{\rm A}\cos kq\,dq\,dp, \quad (28)$$

we have

$$D_{kk}^{\rm (homo)}m_k-\varphi_k(m)=0 \quad (29)$$

for the spontaneous part, and

$$D(\Lambda\delta m+h)+\Lambda C(\Lambda\delta m+h)-h=O(h^2) \quad (30)$$

for the response part. Here $\Lambda={\rm diag}(V_1,\ldots,V_K)$, D and C are now matrices of size $K\times K$ with the (k,l) elements

$$D_{kl}=\delta_{kl}+V_k\int_\mu F_{\rm I}'(\mathcal{H}_{\rm I})\cos kq(\cos lq-\langle\cos lq\rangle_{\rm I})\,dq\,dp, \quad (31)$$

$$C_{kl}=\int_\mu \cos kq(F_{\rm I}'(\mathcal{H}_{\rm I})\langle\cos lq\rangle_{\rm I}-F_{\rm I}'(\mathcal{H}_{\rm A})\langle\cos lq\rangle_{\rm A})\,dq\,dp, \quad (32)$$

and $D^{\rm (homo)}$ is defined from D as the HMF case, that is,

$$(D^{\rm (homo)})_{kl}=\delta_{kl}\left[1+\pi V_k\int_\mu F_{\rm I}'(p^2/2)\,dp\right]. \quad (33)$$

The functions $\varphi_k(m)$ are polynomials consisting of monomials whose degrees are more than 1. The second term of (30) is of higher order than the first again.

It might be worth noting the concrete forms of matrix D both in statistical mechanics and in the Vlasov dynamics by setting the initial state as the canonical equilibrium, $F_{\rm I}(\mathcal{H}_{\rm I})=F_{\rm eq}(\mathcal{H}_{\rm I})\propto\exp(-\mathcal{H}_{\rm I}/T)$, which implies $F_{\rm I}'=-F_{\rm I}/T$. Let us denote the average over $F_{\rm eq}(\mathcal{H}_{\rm I})$ by $\langle\cdot\rangle_{\rm eq}$. From (31) the Vlasov dynamics gives

$$D_{kl}=\delta_{kl}-\frac{V_k}{T}(\langle\cos kq\,\cos lq\rangle_{\rm eq}-\langle\cos kq\langle\cos lq\rangle_{\rm I}\rangle_{\rm eq}). \quad (34)$$

On the other hand, expanding $f_{\rm A}\propto\exp(-\mathcal{H}_{\rm A}/T)$, the statistical mechanics gives

$$D_{kl}=\delta_{kl}-\frac{V_k}{T}(\langle\cos kq\,\cos lq\rangle_{\rm eq}-\langle\cos kq\rangle_{\rm eq}\langle\cos lq\rangle_{\rm eq}). \quad (35)$$

The two D matrices, and γ_+ accordingly, coincide for homogeneous initial states associated with $F_{\rm eq}(\mathcal{H}_{\rm I})\propto\exp(-p^2/2T)$, because $\langle\cos lq\rangle_{\rm I}=0$ and $\langle\cos lq\rangle_{\rm eq}=0$.

Before progressing to the critical exponents, we remark on the critical point. The diagonal elements of $D^{\rm (homo)}$ represent the dispersion functions for the Fourier mode k with the reference state homogeneous as the Para side. In other words, $D_{kk}^{\rm (homo)}>0$ implies that $m_k=0$ is stable. Assuming that $F_{\rm I}$ is a monotonically decreasing function of energy, we have the relation $V_k>V_l\Longrightarrow D_{kk}^{\rm (homo)}<D_{ll}^{\rm (homo)}$. We are focusing on the Para-Ferro phase transition, and hence V_1 must be positive and larger than $V_2,\ldots,V_K$ to make the mode $k=1$ unstable first. Thus, the critical point is determined by $D_{11}^{\rm (homo)}=0$, and around it, D_{11} is small but $D_{kk}=O(1)(2\leqslant k\leqslant K)$ in both of the Para and the Ferro sides. We remark that both the Vlasov dynamics and the statistical mechanics have the identical critical point for $F_{\rm I}=F_{\rm eq}$, since they have the identical matrix D in the homogeneous Para side as mentioned above.

Computation of the critical exponent β is rather complicated than the HMF case, but we can show that the leading term in φ_1 is of $O(m_1^3)$ as the HMF case. First, we can show that $O(m_k)\leqslant O(m_1^2)(2\leqslant k\leqslant K)$ (see the Appendix). Then, remember that the function $\varphi_1(m)$ is obtained by expanding $\int_\mu F_{\rm I}(\mathcal{H}_{\rm I})\cos q\,dq\,dp$, where m_k dependence comes from $\mathcal{H}_{\rm I}$ including the term of $-V_km_k\cos kq$. Thus, terms of $O(m_1^2)$ do not appear in $\varphi_1(m)$ since $\int\cos^3 q\,dq=0$. On the other hand, terms of $O(m_1^3)$ survive, and by the relation $O(m_k)\leqslant O(m_1^2)$, this is the leading order of φ_1. Scaling of the spontaneous magnetization is, therefore, $m_1\propto\sqrt{-D_{11}^{\rm (homo)}}$ and $\beta=1/2$ in general.

The linear response for off-critical is obtained by

$$D(\Lambda\delta m + h) - h = 0. \tag{36}$$

Thus, susceptibility matrix whose (k,l) elements are defined by

$$\chi_{kl} = \lim_{||h||\to 0} \frac{\partial(\delta m_k)}{\partial h_l} \tag{37}$$

is expressed as

$$\chi = \Lambda^{-1}D^{-1}(1-D). \tag{38}$$

In the Para side, the off-diagonal elements of D vanish due to $\mathcal{H}_{\mathrm{I}} = p^2/2$ and $\langle \cos lq\rangle_{\mathrm{I}} = 0$. The matrix D is hence estimated as

$$D^{(\mathrm{Para})} = \mathrm{diag}\big(O\big(D_{11}^{(\mathrm{homo})}\big), O(1), \ldots O(1)\big). \tag{39}$$

This estimation immediately gives $\gamma_+ = 2\beta$ for χ_{11}, and the other susceptibilities do not diverge. In the Ferro side, we have

$$D^{(\mathrm{Ferro})} = \begin{pmatrix} O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & \cdots & O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) \\ O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & O(1) & \cdots & O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) \\ \vdots & \vdots & \ddots & \vdots \\ O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & \cdots & O(1) \end{pmatrix} \tag{40}$$

with the factor $\Delta\mathcal{H}_{\mathrm{I}} = 2\sum_{k:\mathrm{odd}} V_k m_k$, which is dominated by $k = 1$ from the ordering $O(m_k) \leqslant O(m_1^2)$ mentioned previously. The estimation (40) is obtained as follows.

The ordering also suggests that the fixed point of $\mathcal{H}_{\mathrm{I}}$ are solely $(q,p) = (0,0)$ stable and $(\pi,0)$ unstable, and therefore, the first diagonal element is estimated by the same strategy with the HMF case. Each off-diagonal element is also dominated by $O(\sqrt{\Delta\mathcal{H}_{\mathrm{I}}})$ coming from the term having $\langle \cos lq\rangle_{\mathrm{I}}$, since the other term gives a contribution of higher order $O(m_1) = O(\Delta\mathcal{H}_{\mathrm{I}})$ from the expansion of $F_{\mathrm{I}}'(\mathcal{H}_{\mathrm{I}})$ with respect to small m.

The inverse matrix of $D^{(\mathrm{Ferro})}$ is

$$[D^{(\mathrm{Ferro})}]^{-1} = \begin{pmatrix} O(1/\sqrt{\Delta\mathcal{H}_{\mathrm{I}}}) & O(1) & \cdots & O(1) \\ O(1) & O(1) & \cdots & O(1) \\ \vdots & \vdots & \ddots & \vdots \\ O(1) & O(1) & \cdots & O(1) \end{pmatrix}. \tag{41}$$

Therefore, remembering $O(\Delta\mathcal{H}_{\mathrm{I}}) = O(m_1)$ and estimating $[(D^{(\mathrm{Ferro})})^{-1}](1 - D^{(\mathrm{Ferro})})$, the critical exponent for χ_{11} is $\gamma_- = \beta/2$, and the other elements do not diverge.

The unique divergence in the susceptibility matrix χ appears in χ_{11}, and we consider the response to the external force $h = (h_1, 0, \ldots, 0)$ at the critical point. The matrix D does not vanish even at the critical point, and hence we consider the equation

$$(D + \Lambda C)(\Lambda\delta m + h) - h = 0. \tag{42}$$

The matrix $D + \Lambda C$ can be estimated at the critical point as $D^{(\mathrm{Ferro})}$ (40), but replacing $\Delta\mathcal{H}_{\mathrm{I}}$ with $\Delta\mathcal{H}_{\mathrm{A}}$, where $\Delta\mathcal{H}_{\mathrm{A}} = 2\sum_{k:\mathrm{odd}}(V_k\delta m_k + h_k)$. We may expect that $V_1\delta m_1 + h_1$ dominates $\Delta\mathcal{H}_{\mathrm{A}}$ and $O(\Delta\mathcal{H}_{\mathrm{A}}) = O(V_1\delta m_1 + h_1)$, since the susceptibility χ_{11} diverges at the critical point but the others do not. Consequently we have $(V_1\delta m_1 + h_1)^{3/2} \propto h_1$, which implies $\delta m_1 \propto h_1^{2/3}$ and $\delta = 3/2$.

IV. SUMMARY AND DISCUSSIONS

We investigated critical exponents for the continuous Para-Ferro phase transitions in QSSs from one simple expanded expression of self-consistent equations for the order parameters like the Landau theory. The expression is obtained from recently proposed nonlinear response theory, and we successfully unified to derive the four critical exponents $\beta = 1/2, \gamma_+ = 2\beta, \gamma_- = \beta/2$ and $\delta = 3/2$ in the HMF model for reference families of QSSs including thermal equilibrium family. The unification is further extended into generalized mean-field 1D systems periodic spatially, and we obtained the same values for all the critical exponents, where $\gamma_\pm$ and δ are associated with χ_{11}. These critical exponents satisfy the scaling relation $\gamma_- = \beta(\delta - 1)$. This relation breaks for γ_+ defined in the Para side, but it might be reasonable since β is defined in the Ferro side only. We have also shown that the other elements of susceptibility matrix do not diverge even at the critical point.

We remark that the essential mechanism of the nonclassical critical exponents, γ_- and δ, is existence of averaged factor $\langle\delta\mathcal{V}\rangle_{\mathrm{A}}$ in (18). Then, from the key scaling of $O(p) = O(\kappa\sqrt{\Delta\mathcal{H}_{\mathrm{A}}})$, the factor gives contribution of order $\sqrt{m_1 + \delta m_1 + h_1}$ to the self-consistent equations, and this square root contribution yields the two nonclassical exponents. We stress that this key scaling sheds light on understanding the scaling relation $\gamma_- = \beta(\delta - 1)$ by generalizing the exponent $1/2$ to x. The generalized exponent x gives $\gamma_- = \beta x$ and $\delta = 1 + x$, and immediately the scaling relation irrespective of the value of β.

We have restricted ourselves to finite K, which is the number of Fourier modes in the interaction $V(q)$. If we may assume that the interaction $V(q)$ and the external part $H_{\mathrm{ext}}(q)$ are sufficiently smooth, the amplitudes of their Fourier modes, $|V_k|$ and $|h_k|$, converges to 0 rapidly enough. Then, we conjecture that the higher modes are negligible, and the critical exponents do not change even K is infinite.

The starting equation of the present study corresponds to the derivative of pseudo-free energy in the Landau theory. Constructing the pseudo-free energy might be done in a future work. We took averages over iso-action lines with the aid of an ergodic-like formula [11,22]. Spatially higher dimensional systems are not integrable in general, but the present theory could be extended by taking the averages over iso-energy surfaces if we may use an ergodic-like formula on the surfaces. The extended theory conserves the Casimirs within the linear order and therefore, can be applied to vaster class of Hamiltonian systems. We have considered the Hamiltonian external forces associated with the order parameters, following the conventional setting of response theory. Extension to non-Hamiltonian external force or random perturbation might be other future works.

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APPENDIX: PROOF OF THE ORDERING $O(m_k) \leqslant O(m_1^2) (k \geqslant 2)$

We show the ordering of spontaneous order parameters as $O(m_k) \leqslant O(m_1^2) (k = 2,\dots,K)$ around the critical point of Para-Ferro transition, which implies $|m_k| \ll 1$. We assume that the function F_{I} is expanded into the Taylor series. Using the small m_k, we expand the self-consistent equation

$$m_k = \int F_{\mathrm{I}}\left(p^2/2 - \sum_{l=1}^{K} V_l m_l \cos lq\right) \cos kq \, dq \, dp \tag{A1}$$

as

$$m_k = -\sum_{l=1}^{K} V_l m_l \int F_{\mathrm{I}}'(p^2/2) \cos kq \cos lq \, dq \, dp + \cdots . \tag{A2}$$

We write the expanded equation as

$$D_{kk}^{(\mathrm{homo})} m_k = \varphi_k (V_1 m_1, \dots, V_K m_K), \tag{A3}$$

where φ_k are series consisting of monomials whose degrees are more than 1. We will derive a contradiction by assuming that there exists $c_2 \in \{2,\dots,K\}$ such that $O(m_{c_2}) > O(m_1)$. The contradiction implies $O(m_{c_2}) \leqslant O(m_1)$ for any $c_2 \in \{2,\dots,K\}$. Substituting this relation into (A3), remembering $D_{kk}^{(\mathrm{homo})} = O(1)$ for any $k \geqslant 2$ in the vicinity of a critical point, and using that the degree of φ_k is more than 1, we conclude $O(m_k) = O(\varphi_k) \leqslant O(m_1^2)$.

Let us derive the contradiction. We focus on the equation for $k = c_2$. The left-hand-side of (A3) is of $O(m_{c_2})$, and hence the function φ_{c_2} must include monomials of the same order with m_{c_2}. We pick up one of them denoted by $\varphi_{c_2}^*$. Remembering $|m_l| \ll 1$ for any l and that the degree of $\varphi_{c_2}^*$ is more than 1, we find that $\varphi_{c_2}^*$ does not include m_{c_2}. Next, if $\varphi_{c_2}^*$ includes m_1, the same reasoning induces the relation $O(m_{c_2}) < O(m_1)$, but this breaks the assumption of $O(m_{c_2}) > O(m_1)$. Thus, we conclude that $\varphi_{c_2}^*$ includes neither m_{c_2} nor m_1. We choose m_{c_3} included in $\varphi_{c_2}^*$ such that $c_3 \in \{2,\dots,K\} \setminus \{c_2\}$ and satisfying $O(m_1) < O(m_{c_2}) < O(m_{c_3})$, and we shift the focusing equation to $k = c_3$. This discussion can repeat up to choosing c_K, but no next number c_{K+1} exists. The nonexistence suggests that there is no monomial in φ_{c_K} which is of the same order with m_{c_K}. The self-consistent equation for m_{c_K} is not satisfied, and a contradiction has been induced. ■

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