Fluctuations in the two-dimensional one-component plasma and associated fourth-order phase transition

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We study the distribution of the mean radial displacement of charges of a two-dimensional (2D) one-component plasma in the thermodynamic limit $N \to \infty$ at finite temperature $\beta > 0$. We compute explicitly the large deviation functions showing the emergence of a fourth-order phase transition as a consequence of a change of topology in the plasma distribution. This weak phase transition occurs exactly at the ground state of the plasma. These results have been compared with the integrable case (finite N) of plasma parameter $\beta q^2 = 2$. In this case the problem can be mapped to the stationary properties of 2D Dyson Brownian particles and to a non-Hermitian matrix model.

Introduction. The two-dimensional one-component plasma (2D-OCP) is one of the most basic statistical mechanics fluid (i.e., nonlattice) models in dimension greater than one. Very recently, Sandier and Serfaty [1] have rigorously established a connection between the Ginzburg-Landau model [2] for superconductors in the critical regime with vortices and the 2D-OCP. In this regime the interaction between vortices is well described by a one-component plasma in a quadratic confining potential. Heuristic arguments suggest that the very same approximation could be valid to describe more general vortex systems, most notably in superfluids or Bose-Einstein condensates [3,4].

The 2D-OCP (also known as *log gas*) is a system of classical pointlike particles of same charge q (one species of particles) immersed in a two-dimensional domain with a neutralising background of opposite charge. The interaction between the charges is $(1/2) \sum_{i \neq j} q^2 v(|\vec{r}_i - \vec{r}_j|)$, where the Coulomb potential is solution of the 2D Poisson equation, $v(\vec{r}) = -\log(|\vec{r}|/L)$. The Boltzmann-Gibbs canonical measure of a 2D-OCP of N particles at inverse temperature β in a quadratic potential is

$$\mathbb{P}_\beta(\{\vec{r}_k\}) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\beta H(\vec{r}_1,\dots,\vec{r}_N)}, \tag{1}$$

$$H(\{\vec{r}_k\}) = -\frac{q^2}{2} \sum_{i \neq j} \log\left(\frac{r_{ij}}{L}\right) + \frac{q^2 N}{2} \sum_k \left(\frac{r_k}{L}\right)^2, \tag{2}$$

where $\vec{r}_k = (x_k, y_k) \in \mathbb{R}^2$ is the position of the k-th particle ($k = 1, \dots, N$), $r_k = |\vec{r}_k|$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. The thermodynamic state of the system is characterised essentially by the two-dimensional plasma parameter $\gamma = \beta q^2$. Hereafter the length L that fixes the zero of the potential and the elementary charge q will be scaled to unity. (The plasma parameter γ is therefore identified with the inverse temperature β.)

Besides being an approximation of the Ginzburg-Landau model in the vortices phase, these log gases are interesting *per se*, being ubiquitous in many fields of physics and mathematics. At the special value $\beta = 2$ of the plasma parameter, the model is exactly integrable and deserves particular attention. In this case, Eq. (1) is the stationary distribution of a 2D

Dyson Brownian motion [5,6], namely a Brownian motion of N particles with logarithmic repulsion in the plane confined in a harmonic potential. Again at $\beta = 2$, with the identification $\mathbb{R}^2 \simeq \mathbb{C}$, Eq. (1) is the joint distribution of the eigenvalues $\lambda_k = x_k + iy_k$ of non-Hermitian random matrices with iid complex Gaussian entries, the Ginibre ensemble [7]. There has been extensive work on 2D-OCP systems in the mean-field theory; numerical studies are also available, while exact results can be derived in the integrable case $\beta = 2$ [8–17].

In this Rapid Communication we present a study on the large deviations (rare events) of 2D-OCP systems. We will consider a rather simple observable, namely the mean radial displacement $\Delta_N = N^{-1} \sum_k r_k$ of the charges of the plasma (the vortices of the Ginzburg-Landau model in the critical phase). Here we compute explicitly the large N limit probability law $\mathcal{P}_{N,\beta}(x) = \langle \delta(x - \Delta_N) \rangle$ of Δ_N and its rescaled Laplace transform $\widehat{\mathcal{P}}_{N,\beta}(s) = \langle e^{-\beta s N^2 \Delta_N} \rangle$ [see Eqs. (3)–(5) and (4)–(7) below] for all $\beta > 0$ (the angle brackets denote the canonical average). As a byproduct, at $\beta = 2$ these results provide large deviations functions for the Ginibre ensemble and 2D Dyson Brownian particles.

Results on the probability of rare events are popular in the one-dimensional case, in particular in the literature on Hermitian random matrices [18–25]. In most of the models analyzed the 2D-OCP is confined on the line and undergoes a phase transition, usually of third-order if the plasma is constrained by a hard wall (for an interesting review see [26]). Large deviations have been much less explored in two dimensions and few results are available [27–30]. Here we show explicitly that the statistical distribution of the mean displacement Δ_N in the plane unveils a *fourth-order phase transition* of the 2D-OCP related to a change of topology in the equilibrium distribution of the plasma. This phenomenon is absent in one dimension. We point out that the extraordinary weak singularity of $\mathcal{P}_{N,\beta}(x)$ [or $\widehat{\mathcal{P}}_{N,\beta}(s)$] occurs at the average value $x = \langle \Delta_N \rangle$ (at $s = 0$ in the Laplace domain), i.e., at the ground state of the 2D plasma. Therefore this phase transition may be detected by the anomalous behavior of the high-order cumulants of Δ_N.

Results. The general theory of large deviations of log gases [29,30] ensures that the Laplace transform $\widehat{\mathcal{P}}_{N,\beta}(s)$ of Δ_N has

a thermodynamic limit

$$J(s) = - \lim_{N \to \infty} \frac{1}{\beta N^2} \log \widehat{\mathcal{P}}_{N,\beta}(s) \quad (3)$$

for all $s \in \mathbb{R}$. Such a limit is the scaled *cumulant generating function* (cumulant GF) of Δ_N. Then, by the Gärtner-Ellis theorem [31,32], the probability law of Δ_N satisfies a large deviation principle[1] with *speed* βN^2, i.e., $\mathcal{P}_{N,\beta}(x) \approx e^{-\beta N^2 \Psi(x)}$, where the *rate function* $\Psi(x)$ is the Legendre-Fenchel transform of the cumulant GF (3):

$$\Psi(x) = -\lim_{N \to \infty} \frac{1}{\beta N^2} \log \mathcal{P}_{N,\beta}(x) = \inf_s [J(s) - sx]. \quad (4)$$

The challenge here is to compute explicitly the large deviation functions $J(s)$ and $\Psi(x)$. We will show that the cumulant GF has the following expression independent of β:

$$J(s) = \frac{1}{2} \text{arsinh}\left(\frac{s}{2}\right) - \frac{s^2}{4} + \frac{s}{48}[(s^2 + 10)\sqrt{s^2 + 4} - |s|^3]. \quad (5)$$

This is the central result of the paper. Before turning to the derivation, we make few observations and we discuss the consequences of Eq. (5). The cumulant GF is strictly concave and satisfies the normalization condition $J(0) = 0$. Notice that the large deviation function is not analytic. The *fourth* derivative of $J(s)$ is discontinuous at $s = 0$ with a finite jump $J^{(4)}(0^-) - J^{(4)}(0^+) = 1$; this is a consequence of a *fourth-order phase transition* in the equilibrium distribution of the 2D-OCP. Phase transitions of 2D-OCP constrained on a line, i.e., of eigenvalues of Hermitian random matrices, are well understood. In two dimensions, however, they are more gentle than for Hermitian matrices, where the singularities are of third order. This behavior corresponds to the physical intuition that the logarithmic repulsion has stronger effects in 1D than in 2D gases. Here the singularity emerges in the thermodynamic limit as a consequence of a change in topology (disk-annulus) of the support of the equilibrium distribution of the charges. This mechanism is absent in one dimension. We note that the phase transition would not occur in the absence the logarithmic interaction, since without it Eq. (2) is the Hamiltonian of a system of free particles, which cannot undergo a phase transition—this case is discussed in more detail in the last section. As a final remark, we stress that the phase transition occurs *at* the ground state ($s = 0$) and hence is not related to the atypical fluctuations (this is another departure from the one-dimensional scenario for continuous statistics).

At leading order in N, the first three cumulants of Δ_N can be obtained by differentiation of $J(s)$ at $s = 0$

$$\kappa_1(\Delta_N) = \frac{2}{3}; \quad \kappa_2(\Delta_N) = \frac{1}{2\beta N^2}; \quad \kappa_3(\Delta_N) = \frac{1}{2\beta^2 N^4}. \quad (6)$$

At $\beta = 2$ the first two cumulants (average and variance) agree with asymptotic formulas for the Ginibre matrix ensemble [35]. The nonzero third cumulant measures the deviation from the Gaussian behavior at leading order in N. Higher cumulants cannot be obtained by differentiation of $J(s)$

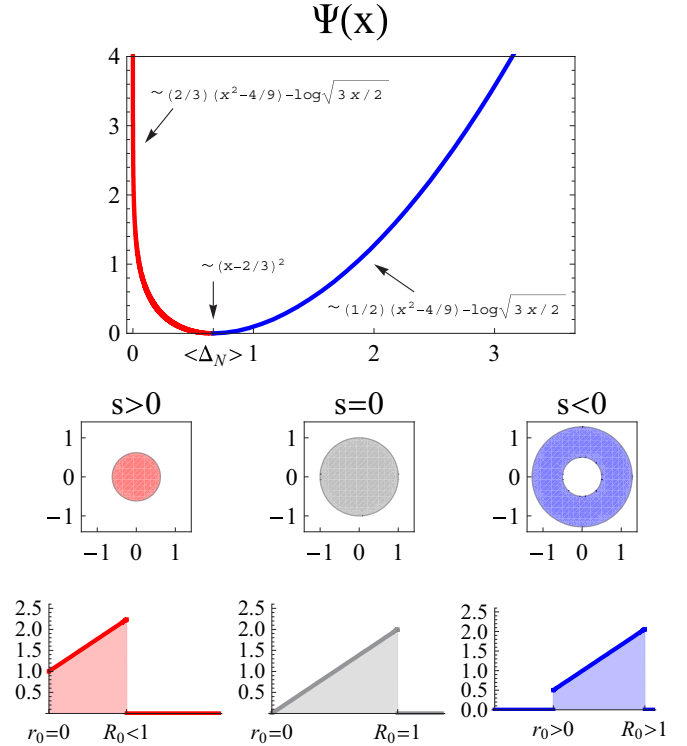FIG. 1. (Color online) Top: Rate function $\Psi(x)$ from Eq. (20) and its leading terms for $x \to 0$, $x \simeq 2/3$ and $x \to \infty$ [see Eq. (7)]. At $x = \langle \Delta_N \rangle \equiv 2/3$ the function is not analytic. Central: Support $\Sigma(s)$ of the equilibrium distribution $\mu_s^\star$ of the plasma for different values of s. From left to right $s = 1, 0, -0.5$. Bottom: The corresponding radial distributions $\int_0^{2\pi} d\mu_s^\star(|\vec{r}|, \varphi)$.

(we will elaborate more on this point later). Since $J(s)$ is everywhere differentiable, we can apply the Gärtner-Ellis theorem and recover the rate function $\Psi(x)$ using Eq. (4). The limiting behavior of $\mathcal{P}_{N,\beta}(x) \approx e^{-\beta N^2 \Psi(x)}$ may be summarized as follows:

$$\mathcal{P}_{N,\beta}(x) \approx \begin{cases} \left(\frac{2x}{3}\right)^{\beta N^2 / 2} e^{-(2/3)\beta N^2 (x^2 - 4/9)} & \text{for } x \ll 1 \\ \exp\left[-\beta N^2 \left(x - \frac{2}{3}\right)^2\right] & \text{for } x \simeq \frac{2}{3} \quad (7) \\ \left(\frac{2x}{3}\right)^{\beta N^2 / 2} e^{-(1/2)\beta N^2 (x^2 - 4/9)} & \text{for } x \gg 1. \end{cases}$$

The rate function $\Psi(x)$ is plotted in the top panel of Fig. 1. From the Gaussian form around $x \simeq 2/3$, one easily reads off the values for mean and variance: $\langle \Delta_N \rangle = 2/3$ and $\text{var}(\Delta_N) = 1/(2\beta N^2)$, according to Eq. (6). On the other hand, the rate function is not quadratic and this implies the polynomial and sub-Gaussian far tails of $\mathcal{P}_{N,\beta}(x)$ for $x \to 0$ and $x \to \infty$, respectively, in Eq. (7).

We stress that the exact computation of $J(s)$ is essential to unveil the phase transition. Standard computations usually concern with the quadratic approximation about the ground state of the 2D-OCP (Gaussian approximation of $J(s)$ about $s = 0$). However the fourth-order singularity at $s = 0$ does not emerge at the level of quadratic [i.e., $O(s^2)$] effects.

Derivation. In order to compute the cumulant GF (3), we cast the Laplace transform of Δ_N as a ratio of

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[1]For more references on the terminology see [33,34].

two partition functions $\widehat{\mathcal{P}}_{N,\beta}(s) = \frac{\mathcal{Z}_{N,\beta}(s)}{\mathcal{Z}_{N,\beta}(0)}$, where $\mathcal{Z}_{N,\beta}(s) = \int e^{-\beta H(\vec{r}_1,...,\vec{r}_N;s)}$ is the partition function of the 2D-OCP with modified energy

$$H(\vec{r}_1, \ldots, \vec{r}_N; s) = -\frac{1}{2} \sum_{i \neq j} \log r_{ij} + N \sum_k V_s(r_k), \quad (8)$$

$$V_s(r) = \frac{1}{2}r^2 + sr. \quad (9)$$

Clearly, at zero temperature $\beta \to \infty$ the only contribution to $\mathcal{Z}_{N,\beta}(s)$ comes the configurations of the system that minimize the energy $H(\vec{r}_1, \ldots, \vec{r}_N; s)$. At finite temperature $\beta < \infty$ in the large N limit, the picture is not very different in the sense that the 'typical' configurations of the system are still the minimizers of the energy and one can characterize the probability of 'atypical' configurations. For large N, we have $H(\vec{r}_1, \ldots, \vec{r}_N; s) = N^2(H_s[\mu_N] + o(1))$, where $\mu_N = N^{-1} \sum_i \delta_{\vec{r}_i}$ is the normalized distribution of the charges and $H_s[\cdot]$ is the mean-field energy functional:

$$H_s[\mu] = -\frac{1}{2} \iint_{\vec{r} \neq \vec{r}'} d\mu(\vec{r}) d\mu(\vec{r}') \log |\vec{r} - \vec{r}'| + \int d\mu(\vec{r}) V_s(|\vec{r}|). \quad (10)$$

These considerations justify a saddle-point evaluation of the large deviation function $J(s)$ as excess free energy

$$J(s) = -\lim_{N \to \infty} \frac{1}{\beta N^2} \log \left[\frac{\int e^{-\beta H(\vec{r}_1,...,\vec{r}_N;s)}}{\int e^{-\beta H(\vec{r}_1,...,\vec{r}_N;0)}} \right]$$

$$= H_s[\mu_s^\star] - H_0[\mu_0^\star], \quad (11)$$

where $\mu_s^\star$ ($\mu_0^\star$) is the minimizer of the energy functional $H_s[\cdot]$ ($H_0[\cdot]$):

$$H_s[\mu_s^\star] = \min \left\{ H_s[\mu] : \ \mu \geqslant 0, \int_{\mathbb{R}^2} d\mu(\vec{r}) = 1 \right\}. \quad (12)$$

Hence, the problem of computing $J(s)$ reduces to find the equilibrium configuration $\mu_s^\star$ of the charges of plasma in the large N limit. Find the configuration of the 2D-OCP that minimizes a mean-field energy functional like Eq. (10) is one of the central objectives in *potential theory* [36]. Fortunately, minimization problems such as Eq. (12) can be solved in closed form with a judicious use of classical electrostatics. We first observe that the problem is radially symmetric and the radial potential V_s is convex. Therefore, the equilibrium measure inherits the radial symmetry of the energy $d\mu_s^\star(\vec{r}) = (2\pi)^{-1} d\mu_s^\star(r) d\varphi$ and is supported in general on the annulus

$$\Sigma(s) = \{\vec{r} \in \mathbb{R}^2 : r_0(s) \leqslant r \leqslant R_0(s)\}, \quad (13)$$

where we have switched to polar coordinates $\vec{r} = (r, \varphi)$. At equilibrium, the Gauss law on a generic domain of the plane holds. For a centered disk of radius r, the Gauss law reads

$$V_s'(r)r = 2\pi \int_0^r d\mu_s^\star(r') r'. \quad (14)$$

Using the positivity and the normalization of $\mu_s^\star$, it is easy to find the conditions $V_s'(r_0) = 0$ and $V_s'(R_0)R_0 = 1$ on the inner and outer radii of the support $\Sigma(s)$. From Eq. (14), the

equilibrium distribution of the charge has the explicit form

$$d\mu_s^\star(\vec{r}) = \left(\frac{2|\vec{r}| + s}{2\pi |\vec{r}|} \right) 1_{\vec{r} \in \Sigma(s)} d\vec{r}, \quad (15)$$

$$r_0(s) = \max\{0, -s\}, \quad R_0(s) = \frac{1}{2}[\sqrt{s^2 + 4} - s]. \quad (16)$$

We see that the optimal distribution of the plasma experiences a change of topology driven by s: $\mu_s^\star$ is supported on a disk for $s \geqslant 0$ and on an annulus for $s < 0$ (see Fig. 1). The fourth-order singularity of $J(s)$ at $s = 0$ witnesses this sudden disk-to-annulus change of topology. In particular, for $s = 0$ we get the uniform measure on the unit disk: $d\mu_0^\star(\vec{r}) = \frac{1}{\pi} 1_{|\vec{r}| \leqslant 1} d\vec{r}$. This corresponds to the *circular law* [7,37–39], one of the most celebrated results on the empirical density of the eigenvalues of non-Hermitian random matrices. For a generic value $s \in \mathbb{R}$, the distribution $\mu_s^\star$ is the typical configuration of the 2D-OCP with a fixed value of Δ_N given by

$$x(s) = \int d\mu_s^\star(\vec{r})|\vec{r}| = \frac{(s^2 + 4)^{3/2} - 6s - |s|^3}{12}. \quad (17)$$

The energy of the equilibrium configuration at $s = 0$ (i.e., at the ground state) is $H_0[\mu_0^\star] = 3/8$, a value known as H-stability bound of the 2D-OCP [40,41]. For a generic value of s, the minimal energy attained at $\mu_s^\star$ is

$$H_s[\mu_s^\star] = \frac{1}{2} \left[\int d\mu_s^\star(\vec{r}) V_s(|\vec{r}|) + V_s(R_0) - \log R_0 \right]. \quad (18)$$

Evaluating Eq. (18), we get the cumulant GF $J(s)$ in Eq. (5) as excess energy (11).

Since $J(s)$ is strictly concave and differentiable, using the properties of the Legendre-Fenchel transform, we can write the thermodynamical relation $J(s) - \Psi(x) = sx$, where the conjugate variables x and s are related by

$$x(s) = J'(s) \quad \text{and} \quad s(x) = -\Psi'(x), \quad (19)$$

with $x(s)$ given in Eq. (17). These relations, supplemented with the conditions $J(0) = \Psi[x(0)] = 0$, provide the following simple expression for the rate function $\Psi(x)$:

$$\Psi(x) = -\int_{\frac{2}{3}}^x s(x') \, dx', \quad (20)$$

where $s(x)$ is the (real nonincreasing) inverse of Eq. (17) and we have used $x(0) = 2/3$. One may verify that the cumulant GF (5) can be computed also from Eq. (19). The asymptotic analysis of $\Psi(x)$ in Eq. (20) provides the limiting behavior summarized in Eq. (7).

The integrable case $\beta = 2$. As already mentioned, the value $\beta = 2$ is special. At this value of the plasma parameter the 2D-OCP canonical measure has a determinantal structure and the partition function is integrable at finite N. This fact has been largely exploited in random matrix theory. For $\beta = 2$, the Laplace transform of radially symmetric linear statistics has been computed exactly by Forrester [35]. By specializing his result to Δ_N we obtain the following finite N formula at (crucially) $\beta = 2$:

$$\widehat{\mathcal{P}}_{N,\beta=2}(s) = \prod_{\ell=1}^N \left[\frac{\int_0^\infty dt \, e^{-(t+2s\sqrt{Nt})} t^{\ell-1}}{\int_0^\infty dt \, e^{-t} t^{\ell-1}} \right]. \quad (21)$$
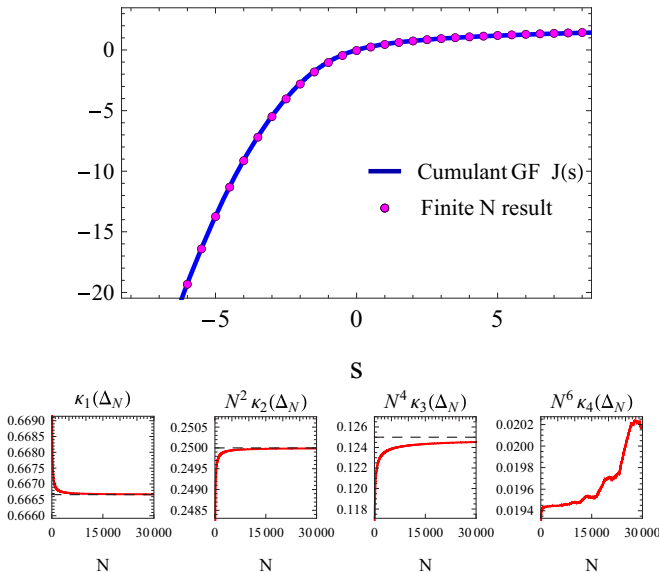
FIG. 2. (Color online) Top: The cumulant generating function $J(s)$ in Eq. (5) (solid line) compared with the exact finite N formula (21) (at $\beta = 2$) for $N = 16$ (dots). Already for such a small value of N, the large deviation function $J(s)$ matches the finite N expression of $-1/(2N^2) \log \widehat{\mathcal{P}}_{N,\beta=2}(s)$ with an error $\mathcal{O}(10^{-2})$ in the plotted region. Bottom: Rescaled cumulants $N^{2(r-1)} \kappa_m(\Delta_N)$ for $m = 1,2,3,4$ at increasing values of N for $\beta = 2$. The first three cumulants have a regular behavior in N and approach their leading order (6), while $N^6 \kappa_4$ behaves irregularly and does not converge for large N.

In Fig. 2 we show that our large N result (5) agrees with this finite N result even for moderate values of N. We stress however that the large deviation formula is valid for any value of $\beta > 0$. Moreover, the route presented here shows explicitly that, even at $\beta = 2$, the function $-1/(\beta N^2) \log \widehat{\mathcal{P}}_{N,\beta}(s)$ analytic at finite N [see Eq. (21)] develops a nonanalyticity as a result of the thermodynamic limit. To understand better the emergence of the singularity we have performed another numerical experiment. For $\beta = 2$ the (unordered!) collection of displacements $\{r_k\}$ of the charges is distributed as a collection of *independent* random variables $\{N^{-1/2} \xi_k\}$, where $\xi_k \geqslant 0$ has density $f_k(x) = (2/\Gamma(k)) x^{2k-1} \exp(-x^2)$ and moments $\langle \xi_k^m \rangle = \Gamma(k + m/2)/\Gamma(k)$ [42,43]. The cumulants κ_m ($m \geqslant 1$) are linear functionals for independent variables and hence we have $\kappa_m(\Delta_N) = N^{-\frac{3m}{2}} \sum_{k=1}^N \kappa_m(\xi_k)$. We have evaluated numerically this sum for increasing values of N and compared the output with our $N \gg 1$ results on the cumulants (6) (see Fig. 2). For $m = 1,2,3$, the sum asymptotically approaches Eq. (6), while the fourth cumulant $\kappa_4(\Delta_N)$ shows a bizarre behavior for large N. For $\beta = 2$, this is a further confirmation of the weak singularity of the large deviations functions.

Role of the logarithmic interaction. In order to convey the key ideas of this Rapid Communication, we conclude with a brief digression to clarify the role of the 2D Coulomb interaction in the mechanism of the phase transition. The computation of $J(s)$ led to study the minimization problem (12). For $s \geqslant 0$ the effective external potential $V_s(r)$ in Eq. (9) is convex and the distribution of charges $\mu_s^\star$ is supported on a disk. When s is negative the convexity of $V_s(r)$ is broken

and the plasma distribution concentrates on an annulus. One might think that this change in the potential is sufficient to modify the topology of the charge distribution and to induce a phase transition even in absence of interaction between the charges. This is incorrect, as we will demonstrate.

Removing the logarithmic interaction from Eq. (2) turns the plasma into a noninteracting gas with single-particle energy $H(\{\vec{r}_k\}) = (1/2) \sum_k r_k^2$ at inverse temperature $\beta > 0$.[2] Since the particles are independent, the single-particle distribution is Gaussian (the system is no more confined in a bounded region) and $\Delta_N = N^{-1} \sum_k r_k$ is a sum of independent and identically distributed displacements. Hence, an elementary computation shows that $\langle \Delta_N \rangle = \sqrt{\pi/(2\beta)}$, $\text{var}(\Delta_N) = (4 - \pi)/(2\beta N)$, and that the Laplace transform is $\widehat{\mathcal{P}}_{N,\beta}(s) = \langle e^{-\beta s N \Delta_N} \rangle = [1 - \sqrt{2\pi\beta} \, s \exp(-\beta s^2/2) \Phi(-\beta s)]^N$, where $\Phi(x)$ is the standard Gaussian cumulative function. (Note the different scaling in N and β dependence in the above expressions compared with the 2D-OCP.) Therefore, in absence of interaction the free energy of the system is analytic in s: noninteracting particles do not undergo phase transitions.

Conclusions. Many of the ideas presented here can be generalized to other problems. In particular, the method to compute large deviations formulas for 2D-OCP is applicable to a large class of radially symmetric observables $A_N = \sum_i f(|\vec{r}_i|)$ with $f(r)$ continuous. The problem of discontinuous statistics, like the charge fluctuations (i.e., the index problem in random matrices), is delicate and needs more refined tools (see [44] for recent developments).

We detect a novel and unusual type of weak phase transition due to a change of topology in the plasma distribution. Therefore this work naturally prompts the following question: is this fourth-order phase transition on the plane as ubiquitous and universal as the third-order transition for charged particles on the line? We mention that similar disk-annulus phase transitions have been detected for non-Hermitian non-Gaussian random matrices [45,46]. However, non-Gaussian ensembles do not have a 2D-OCP interpretation and it is not clear how to relate these works with our findings.

An interesting question is to look at the finite N corrections of the large deviations principle (3)–(4). At microscopic scales it has been shown [1,17,41] that, among all lattice configurations, the charges of the plasma prefer to arrange themselves in a triangular one (the very same of the *Abrikosov lattice* [47] of vortices in the Ginzburg-Landau model). Showing that the triangular lattice is the global minimizer of the 2D-OCP energy requires a careful analysis beyond the mean-field limit. Such a conjecture is still unsolved even in the integrable case $\beta = 2$.

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[2]Here we drop off the N scaling in front of the harmonic potential in order to keep the particles of the gas $r_k = \mathcal{O}(1)$.

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