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Covariance of fluid-turbulence theory

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Covariance of physical quantities in fluid-turbulence theory and their governing equations under generalized coordinate transformation is discussed. It is shown that the velocity fluctuation and its governing law have a covariance under far wider group of coordinate transformation than that of conventional Euclidean invariance, and, as a natural consequence, various correlations and their governing laws are shown to be formulated in covariant manners under this wider transformation group. In addition, it is also shown that the covariance of the Reynolds stress is tightly connected to the objectivity of the mean flow.

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I. INTRODUCTION

In continuum physics, *covariance* (or *form invariance*) of the constitutive relations under the coordinate transformation plays critically important role. As well as the material-frame-indifference principle, the principle of covariance imposes mathematical constraints on the constitutive relations, which provide us strong guidelines in constructing realistic material models. Some pioneers have shown that the covariance principle is also applicable to turbulence modeling, unlike the material-frame-indifference principle [1–3]. The first application of the covariance principle to fluid turbulence was done by Speziale, showing in a clear manner that the Reynolds stress and other higher-order correlations are all covariant under arbitrary time-dependent rotation and translation [3]. In the community of fluid turbulence, the transformation of this class is sometime referred to as *the Euclidean transformation*, and the covariance under this transformation group is termed *the Euclidean invariance*. After this pioneering work, various studies on the Euclidean invariance and its application to the first-order modeling have been published. Weis & Hutter and Hamba claimed the importance of the Euclidean invariance for *the algebraic Reynolds-stress model* (ARSM) [4,5]. The resultant ARSM based on the Euclidean invariance has shown the preferable results in the simulation of the rotating channel flow [6]. In addition to the Reynolds stress, Qiu *et al.* obtained the Euclidean-invariant algebraic heat flux model (AHFM) by extending the strategy of ARSM [7]. According to these pioneers, the covariance under the Euclidean transformation may now have a remarkable position at least in algebraic turbulence modeling.

The objective of the present work is to propose more generalized understandings of the covariance in turbulence physics. It will be shown that various correlations and their transport equations are covariant under a far more generalized class of transformation than conventional Euclidean transformation. Furthermore, an organic connection between the velocity fluctuation and the mean flow is revealed in terms of the covariance, which shows the fundamental importance of the covariance for the physical objectivity of the mean flow; the covariance should be carefully considered in objective and realistic descriptions of turbulence. Note that the author attempts not to propose some specific models using the covariance principle but to suggest some generalization and clarity of the covariance principle in fluid turbulence, which may lead us to more proper understandings of turbulence physics.

The composition of the present paper can be summarized as follows. In Sec. II some mathematical basics are reviewed with emphasis on the tensor analysis and its covariance under the coordinate transformation. In Sec. III transformation rules of turbulence quantities are discussed on the basis of the Reynolds decomposition, which leads to the new observation that various correlations possess a generalized covariance under far wider group of transformation than that of the Euclidean transformation. In Sec. IV the covariant formalism of dynamical equations for the instantaneous, mean, and fluctuation velocities are proposed. In Sec. V the covariant forms of the turbulence constitutive equations are naturally derived from the covariant fluctuation equation shown in Sec. IV. In addition, the importance of the general covariance of the Reynolds stress will be discussed in terms of the objectivity of the mean flow, using the covariant mean-flow equation obtained in Sec. IV.

II. MATHEMATICAL PRELIMINARIES

A. Coordinate representations

First we assume the physical space to be a three-dimensional space with a flat Riemann metric. Then we discriminate the inertial frame from the other frames of reference. For simplicity, let us here introduce an orthonormal-coordinate system $\{z^1,z^2,z^3\} = \{\mathbf{z}\}$ as an inertial frame of reference which is schematically shown by "A" in Fig. 1. Note that we employ the capital Roman letters for the indices of inertial-coordinate representation. For example, the coordinate variables and an arbitrary multicomponent quantity are represented as z^I and $M^{IJ\cdots}{}_{KL\cdots}(I,J,\ldots,K,L,\cdots = 1,2,3)$, respectively.

Let us introduce another class of coordinate system $\{{}^*\mathbf{z}\}$, which has a linear relation with the orthonormal inertial-coordinate system $\{\mathbf{z}\}$ as follows:

$$z^I = Q^I_{A^*}(t)\,{}^*z^{A^*} + Z^I(t). \tag{2.1}$$

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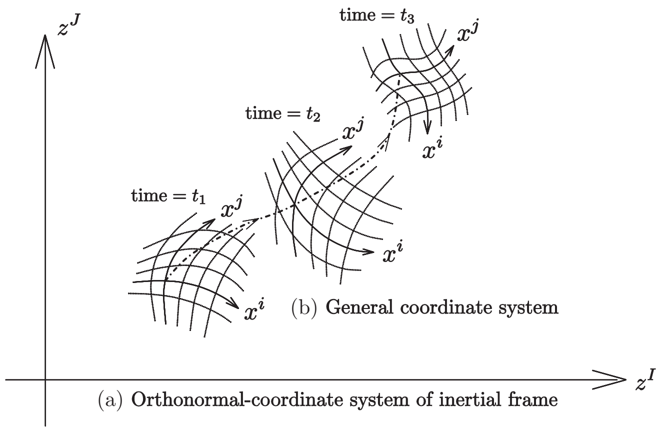

FIG. 1. General coordinate system; The general coordinate system $\{\mathbf{x}\}$ (b) is illustrated in contrast to an inertial reference frame (a) (In general, $\{\mathbf{z}\}$ is not necessarily orthonormal.) We define the class of general coordinate system as those freely moving as time passes.

If we put Q as a time-independent orthogonal matrix and $\mathbf{Z}(t)$ as linear function of t, (2.1) represents the Galilean transformation. If $\mathbf{Z}(t)$ is nonlinear in t, (2.1) is transformation to an accelerated frame. If Q is a time-dependent orthogonal matrix, (2.1) represents time-dependent rotation and translation where the trajectory of the rotation origin is given by $\mathbf{Z}(t)$. The most general case of (2.1) is obtained when both Q and $\mathbf{Z}$ are general functions of t, which may be called as *the affine transformation*. More general class of transformation can be introduced by generalizing (2.1); we introduce a curvilinear-coordinate system $\{\mathbf{x}\}$, which can move freely against the inertial frame $\{\mathbf{z}\}$ by providing a relation such as

$$x^i = x^i\,(z^1,z^2,z^3,t) = x^i(\mathbf{z},t), \tag{2.2}$$

where $i = 1,2,3$. Also we impose $\partial\mathbf{x}/\partial\mathbf{z} \neq 0$ where we use as the coordinate frame. Note that we do not consider the transformation of time parameter since we treat time as an independent parameter from physical space. Generally speaking, coordinate system $\{\mathbf{x}\}$ may have rotation and distortion nonuniform in time and space, which is depicted by (b) in Fig. 1. In this paper, we call a coordinate system such as $\{\mathbf{x}\}$ given by (2.2) *the general coordinate system*. We employ small Roman letters for the indices of the general-coordinate representations. In addition, we call the transformations between general coordinate systems *the general coordinate transformation*; i.e., the transformation from $\{\mathbf{x}\}$ to another general-coordinate system $\{\tilde{\mathbf{x}}\}$ is given by

$$\tilde{x}^{\tilde{a}} = \tilde{x}^{\tilde{a}}(x^1,x^2,x^3,t) = \tilde{x}^{\tilde{a}}(\mathbf{x},t), \tag{2.3}$$

where $\tilde{a} = 1,2,3$. In the later discussions, we rewrite $\tilde{x}^{\tilde{a}}$ as $x^{\tilde{a}}$ to simplify the notations, since we can enough distinguish the coordinate representations by indices. Likewise, arbitrary multicomponent quantity M can be written as $M^{ij\cdots}{}_{kl\cdots}$ and $M^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}$ in $\{\mathbf{x}\}$ and $\{\tilde{\mathbf{x}}\}$, respectively. This logic can be also applied to the transformation coefficients. For example, $\partial x^i/\partial x^{\tilde{a}}$ can be rewritten as $x^i{}_{,\tilde{a}}$, since we can recognize the $x^{\tilde{a}}$ differentiation only by the index $\tilde{a}$. Following these manners, in transformations between the general and inertial frames, we use $x^i{}_{,I}$ or $z^I{}_{,i}$ instead of $\partial x^i/\partial z^I$ or $\partial z^I/\partial x^i$.

Note that the conventional transformation groups such as the Galilean and Euclidean groups ($\mathcal{G}_a$ and $\mathcal{E}_u$) are both the subgroup of the general-coordinate-transformation group ($\mathcal{G}_e$):

$$\mathcal{G}_e \supset \mathcal{A}_f \supset \mathcal{E}_u \supset \mathcal{G}_a,$$

where $\mathcal{A}_f$ is the affine transformation group. Some more details about these group structures are explained in Appendix B.

B. Tensor analysis and general covariance

In continuum physics, the *tensor* plays especially important roles in the covariant formulation of the theory. We require an arbitrary tensor field C to satisfy the transformation rule as follows:

$$C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots} = x^{\tilde{a}}{}_{,i}\,x^{\tilde{b}}{}_{,j}\cdots x^k{}_{,\tilde{c}}\,x^l{}_{,\tilde{d}}\cdots C^{ij\cdots}{}_{kl\cdots}. \tag{2.4}$$

It is well known that sums and products of tensors are tensors, which may be rephrased as follows: *a polynomial of tensors is a tensor*. A tensor monomial is defined as a multicomponent quantity whose coordinate components are given by a product such as $A^{i\cdots}{}_{j\cdots}\,B^{k\cdots}{}_{l\cdots}\cdots$, where A, B, ... are tensors. In general, some pairs of covariant and contravariant indices may be contracted, so it would be better to rewrite the previous form as $A^{i\cdots m\cdots}{}_{j\cdots n\cdots}\,B^{k\cdots n\cdots}{}_{l\cdots m\cdots}\cdots$. The transformation rule of the monomial is given as follows:

$$\begin{aligned} &A^{\tilde{a}\cdots\tilde{e}\cdots}{}_{\tilde{b}\cdots\tilde{f}\cdots}\,B^{\tilde{c}\cdots\tilde{f}\cdots}{}_{\tilde{d}\cdots\tilde{e}\cdots}\cdots \\ &\quad= x^{\tilde{a}}_{,i}\cdots x^j{}_{,\tilde{b}}\cdots x^{\tilde{c}}_{,k}\cdots x^l{}_{,\tilde{d}}\cdots \\ &\quad\quad\times A^{i\cdots m\cdots}{}_{j\cdots n\cdots}\,B^{k\cdots n\cdots}{}_{l\cdots m\cdots}\cdots . \end{aligned}$$

We should remark that the form of monomial itself does not change through the coordinate transformation while the values of components may change, which also holds for a polynomial. We may denote the transformation of a tensor polynomial as

$$\begin{aligned} &{}^{P}F^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}[\tilde{\mathsf{A}},\tilde{\mathsf{B}},\cdots] \\ &\quad= x^{\tilde{a}}{}_{,i}\,x^{\tilde{b}}{}_{,j}\cdots x^k{}_{,\tilde{c}}\,x^l{}_{,\tilde{d}}\cdots{}^{P}F^{ij\cdots}{}_{kl\cdots}[\mathsf{A},\mathsf{B},\cdots], \end{aligned} \tag{2.5}$$

where ${}^{P}\mathsf{F}$ is a polynomial function of $\mathsf{A},\mathsf{B},\ldots$. In the present work, we call the invariance of algebraic structures of polynomial or equations under the general coordinate transformations *covariance under the general coordinate transformation*, *general covariance,* or simply *covariance* unless it creates any confusion. The "general covariance" here is discussed on the basis of 3+1 space-time concepts of nonrelativistic mechanics and never means the transformations of four-dimensional space-time manifold in the theory of general relativity. In terms of covariance, there is an important tensor theorem:

Theorem 1. A polynomial of tensors is generally covariant.

As a natural consequence, algebraic equations consisting of tensor polynomials are generally covariant. Because of the above theorem, tensors play distinct roles from the other multicomponent quantities in descriptions of covariant relations. In some pioneering works, the terminology *tensor* is also applied to Q in (2.1), which is actually $x^{\tilde{a}}{}_{,i}$ or $x^i{}_{,\tilde{a}}$ in general. However, these quantities are only transformation coefficients that never possess the features of tensor discussed above, so that we clearly discriminate tensors from transformation coefficients.

Tensor polynomial is not the only way to construct the covariant relations. For example, *the covariant derivative* of an arbitrary tensor C given by

$$\nabla_a C^{ij\cdots}{}_{kl\ldots} = C^{ij\cdots}{}_{kl\ldots,a} + \Gamma^i_{ma} C^{mj\cdots}{}_{kl\ldots} + \Gamma^j_{ma} C^{im\cdots}{}_{kl\ldots} + \cdots - \Gamma^n_{ka} C^{ij\cdots}{}_{nl\ldots} - \Gamma^n_{la} C^{ij\cdots}{}_{kn\ldots} - \cdots$$

is also a generally covariant operation, where $\boldsymbol{\Gamma}$ is the Christoffel symbol of the second kind given by

$$\Gamma^a_{ij} = \frac{1}{2} g^{ab}(g_{bj,i} + g_{bi,j} - g_{ij,b}),$$

where g is the metric tensor. The relation between the metric and the Christoffel symbol given above leads to a well-known identity: $g_{ij;k} = 0$. Note that neither the simple derivative $C^{ab\cdots}{}_{cd\cdots,j}$ nor $\boldsymbol{\Gamma}$ transforms as tensor. Especially the transformation rule of the Christoffel symbol is given by

$$\Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} = x^{\tilde{a}}{}_{,i}\, x^j{}_{,\tilde{b}}\, x^k{}_{,\tilde{c}}\, \Gamma^i_{jk} - x^{\tilde{a}}{}_{,jk}\, x^j{}_{,\tilde{b}}\, x^k{}_{,\tilde{c}}. \tag{2.6}$$

If necessary, we use the abbreviated form of the covariant derivative given by $\nabla_a C^{ij\cdots}{}_{kl\ldots} = C^{ij\cdots}{}_{kl\ldots;a}$.

In the later sections, we will treat the Reynolds decomposition of physical quantities based on the ensemble average. Let us see here some trivial but important theorems about the Reynolds decomposition of tensors. In the Reynolds decomposition, we decompose a physical quantity, say, f, into the ensemble average $\langle f\rangle$ and the fluctuation $f' \equiv f - \langle f\rangle$, where the angular bracket $\langle\cdots\rangle$ represents the ensemble average. Note that we apply the coordinate frames independently from the ensemble of realizations in the present work. By taking the ensemble average of (2.4), we have

$$\langle C^{ij\cdots}{}_{kl\ldots}\rangle \to \langle C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\ldots}\rangle = x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j} \cdots x^k{}_{,\tilde{c}}\, x^l{}_{,\tilde{d}} \cdots \langle C^{ij\cdots}{}_{kl\ldots}\rangle,$$

which leads to the following theorem:

Theorem 2. The ensemble average of an arbitrary tensor is a tensor.

The fluctuation C' given by the difference between two tensors C and $\langle\mathsf{C}\rangle$ is obviously a tensor from Theorem II B. Indeed, we have

$$\begin{aligned} C'^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\ldots} &= C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\ldots} - \langle C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\ldots}\rangle \\ &= x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j} \cdots x^k{}_{,\tilde{c}}\, x^l{}_{,\tilde{d}} \cdots (C^{ij\cdots}{}_{kl\ldots} - \langle C^{ij\cdots}{}_{kl\ldots}\rangle) \\ &= x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j} \cdots x^k{}_{,\tilde{c}}\, x^l{}_{,\tilde{d}} \cdots C'^{ij\cdots}{}_{kl\ldots}, \end{aligned}$$

which may be summarized as follows:

Theorem 3. The fluctuation of an arbitrary tensor is a tensor.

Another theorem is obtained by averaging (2.5):

$$\langle {}^p F^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\ldots}[\tilde{\mathsf{A}},\tilde{\mathsf{B}},\cdots]\rangle = x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j} \cdots x^k{}_{,\tilde{c}}\, x^l{}_{,\tilde{d}} \cdots \langle {}^p F^{ij\cdots}{}_{kl\ldots}[\mathsf{A},\mathsf{B},\cdots]\rangle, \tag{2.7}$$

where the algebraic structure of ${}^p\mathsf{F}$ including angular bracket conserves under the coordinate transformation. Thus we have the following;

Theorem 4. The average of a tensor polynomial is generally covariant.

So far we have discussed the tensor fields under the general-coordinate-transformation group $\mathcal{G}_e$. Note that some quantities obey (2.4) only under limited-transformation groups such as $\mathcal{A}_f$, $\mathcal{E}_u$, and $\mathcal{G}_a$, and we may call these quantities *the affine*, *Euclidean,* and *Galilean tensors,* respectively. Thus the pioneering works have been discussing the limited covariance under the Euclidean groups (we rephrase the "Euclidean invariance" the "*Euclidean covariance*" in the following discussions). The above theorems hold for tensors of these limited groups by replacing the words "tensor" and "covariant" with limited versions. For example, Theorem II B turns into the following: *a polynomial of affine/Euclidean/Galilean tensors is affine/Euclidean/Galilean covariant.*

C. Continuum quantities

A typical example of a tensor in continuum physics may be the stress; providing the contravariant components of the stress be σ^{ij}, its transformation rule is given by $\sigma^{\tilde{a}\tilde{b}} = x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j}\sigma^{ij}$. On the contrary, a velocity field, which plays the central role in continuum physics, does not obey the tensor rule (2.4). Instead, its transformation rule is given by

$$v^{\tilde{a}} = x^{\tilde{a}}{}_{,i}\, v^i + x^{\tilde{a}}{}_{,t}, \tag{2.8}$$

which is accompanied by an extra term (see Appendix A). Thus we need some special treatment in extracting the covariant properties of material motion. For example, the velocity gradient $v^i{}_{;j}$ transforms as

$$v^{\tilde{a}}{}_{;\tilde{b}} = x^{\tilde{a}}{}_{,i}\, x^j{}_{,\tilde{b}}\, v^i{}_{;j} + x^{\tilde{a}}{}_{,it}\, x^i{}_{,\tilde{b}} + \Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}}\, x^{\tilde{c}}{}_{,t}, \tag{2.9}$$

which again does not obey (2.4) and is not a tensor. Thus the symmetric part of it, namely, $v_{i;j} + v_{j;i}$, is not a tensor under the general-coordinate formalism. $s_{ij} = v_{i,j} + v_{j,i}$, the familiar form in classical fluid mechanics, holds only in the case of the Euclidean transformation; namely, $v_{i,j} + v_{j,i}$ is a Euclidean tensor. Instead, in the modern theory of continuum physics, the strain rate is defined by

$$\begin{aligned} s_{ij} &= \frac{\mathfrak{d}}{\mathfrak{d}t} g_{ij} = \frac{d}{dt} g_{ij} + v^b{}_{;i}\, g_{bj} + v^b{}_{;j}\, g_{ib} \\ &= g_{ij,t} + v_{i;j} + v_{j;i}, \end{aligned} \tag{2.10}$$

which is generally covariant (often defined as the half of the above in rheology, namely, $\frac{1}{2}\mathfrak{d}g_{ij}/\mathfrak{d}t$). The operation $\mathfrak{d}/\mathfrak{d}t$, which is called the convected derivative, is a generally covariant operation [8]. The operation of $\mathfrak{d}/\mathfrak{d}t$ for the general tensor C is given by

$$\frac{\mathfrak{d}}{\mathfrak{d}t} C^{ij\cdots}{}_{kl\ldots} = \frac{d}{dt} C^{ij\cdots}{}_{kl\ldots} - v^i{}_{;a} C^{aj\cdots}{}_{kl\ldots} - v^j{}_{;a} C^{ia\cdots}{}_{kl\ldots} - \cdots + v^b{}_{;k} C^{ij\cdots}{}_{bl\ldots} + v^b{}_{;l} C^{ij\cdots}{}_{kb\ldots} + \cdots, \tag{2.11}$$

where the operator d/dt called the Lagrangian derivative, which has been often used in ordinary fluid dynamics, is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^j \nabla_j. \tag{2.12}$$

It is well known that the Lagrangian derivative is Galilean covariant. Note that this is not covariant under $\mathcal{E}_u$ and wider groups, except when it operates on scalar fields.

III. TRANSFORMATION RULES OF TURBULENCE QUANTITIES

A. Velocity fluctuation

By taking the fluctuation of (2.8), we obtain

$$v'^{\tilde{a}} \equiv v^{\tilde{a}} - \langle v^{\tilde{a}} \rangle = x^{\tilde{a}}{}_{,i}\{v^i - \langle v^i \rangle\} = x^{\tilde{a}}{}_{,i}\, v'^i, \qquad (3.1)$$

which is consistent with the general tensor rule (2.4). Thus the velocity fluctuation $\mathbf{v}'$, which is the most fundamental quantity of turbulence, is a vector (one-rank tensor). As a natural consequence, we notice that various quantities constructed by the velocity fluctuation are generally covariant. For example, from Theorem II B, we can immediately conclude that moments of $\mathbf{v}'$ at arbitrary orders are generally covariant. Indeed, using (3.1) we obtain

$$\langle v'^{\tilde{a}}\, v'^{\tilde{b}}\, v'^{\tilde{c}} \cdots \rangle = x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j}\, x^{\tilde{c}}{}_{,k} \cdots \langle v'^i\, v'^j\, v'^k \cdots \rangle. \qquad (3.2)$$

In case of the second order (3.2) proves that the Reynolds stress is covariant. The turbulence energy $K \equiv \langle v'{}_j v'^j \rangle/2$ and its dissipation rate $\epsilon \equiv \nu \langle v'^{i;j} v'_{i;j} \rangle$ (ν; kinematic viscosity), both of which play important roles in case of incompressible Newtonian fluids, are also generally covariant, or, more strictly speaking, invariant.

B. Mean velocity

The transformation rule for the mean velocity field is obtained by taking the ensemble average of (2.8) as follows:

$$V^{\tilde{a}} = x^{\tilde{a}}{}_{,i}\, V^i + x^{\tilde{a}}{}_{,t}, \qquad (3.3)$$

where $\mathbf{V} = \langle \mathbf{v} \rangle$ is the mean velocity. Here we should remark that the mean velocity field $\mathbf{V}$ transforms in the same manner as the instantaneous velocity field $\mathbf{v}$ as (2.8). Owing to this similarity between (3.3) and (2.8), we can apply various effective techniques that have been developed in the ordinary continuum physics to the analysis of the mean flow. For example, as the counterpart of $\mathfrak{d}/\mathfrak{d}t$ given by (2.11), we can immediately derive the following derivative operation:

$$\frac{\mathfrak{D}}{\mathfrak{D}t} C^{ij\cdots}{}_{kl\cdots} = \frac{D}{Dt} C^{ij\cdots}{}_{kl\cdots} - C^{aj\cdots}{}_{kl\cdots} V^i{}_{;a} - C^{ia\cdots}{}_{kl\cdots} V^j{}_{;a} - \cdots + C^{ij\cdots}{}_{bl\cdots} V^b{}_{;k} + C^{ij\cdots}{}_{kb\cdots} V^b{}_{;l} + \cdots, \qquad (3.4)$$

where the derivative operation given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V^j \nabla_j$$

shows completely the same behavior as d/dt in terms of their coordinate transformation rule. As well as d/dt, D/Dt is Galilean-covariant operation.

Likewise, we can derive some useful derivative operations based on $\mathbf{V}$ by simply replacing $\mathbf{v}$ in derivative operations of continuum physics, such as the Jaumann or Truesdell derivatives, with $\mathbf{V}$. Here let us introduce an important counterpart of the strain rate defined by (2.10):

$$S_{ij} = \frac{\mathfrak{D}}{\mathfrak{D}t} g_{ij} = g_{ij,t} + V_{i;j} + V_{j;i}, \qquad (3.5)$$

which may be called as the strain rate of the mean flow. The covariance of S is obvious from the covariance of $\mathfrak{D}/\mathfrak{D}t$. One may call S as the averaged strain rate since it is also obtained by taking the ensemble average of s, where the covariance of S is also proved by Theorem II B. Likewise, simple replacement of $\mathbf{v}$ with $\mathbf{V}$ can yield physical quantities of the mean flow corresponding to those of instantaneous flow. In this paper, let us call this as *the* $\mathbf{v} - \mathbf{V}$ *correspondence*.

IV. GENERAL COVARIANCE OF DYNAMICAL EQUATIONS

In Sec. III A the general covariance of the velocity fluctuation has been proved in a rigorous manner, so that one may naturally expect the velocity-fluctuation equation to be also generally covariant. In this section, however, we will see that not only the fluctuation equation but also the equations for the instantaneous and mean flows can be written in generally covariant forms, despite the instantaneous and mean velocities, namely, $\mathbf{v}$ and $\mathbf{V}$, are not general vectors. The importance of generally covariant formulation of these three will be discussed in Sec. V.

A. Instantaneous-flow equation

In an inertial frame $\{\mathbf{z}\}$, the equation of instantaneous flow is given by

$$\frac{dv^I}{dt} = \frac{1}{\rho} \sigma^{IJ}{}_{;J} + f^I, \qquad (4.1)$$

where ρ is the mass density, $\boldsymbol{\sigma}$ is the stress and $\mathbf{f}$ is the external force per unit mass. It is well known that (4.1) has the covariance under the Galilean group. Although $\boldsymbol{\sigma}$ and $\mathbf{f}$ are covariant under the general-coordinate transformations, the acceleration term $d\mathbf{v}/dt$ has only covariance under the Galilean group. In this section, it will be shown that (4.1) can be rewritten in generally covariant form if we once define the inertial frame of reference.

Let us start from the transformation rule of the acceleration using (2.8). Taking the simple time derivative of (2.8) yields

$$v^{\tilde{a}}{}_{,t} = (x^{\tilde{a}}{}_{,it} + x^{\tilde{a}}{}_{,ij} x^j{}_{,t}) v^i + x^{\tilde{a}}{}_{,i} (v^i{}_{,t} + v^i{}_{,j} x^j{}_{,t}) + x^{\tilde{a}}{}_{,tt} + x^{\tilde{a}}{}_{,ti} x^i{}_{,t}.$$

Taking a partial spatial derivative of (2.8) and multiplying it by $v^{\tilde{b}} = x^{\tilde{b}}{}_{,j} v^j + x^{\tilde{b}}{}_{,t}$ yields

$$v^{\tilde{a}}{}_{,\tilde{b}}\, v^{\tilde{b}} = x^{\tilde{a}}{}_{,ij}\, v^i (v^j - x^j{}_{,t}) + x^{\tilde{a}}{}_{,i}\, v^i{}_{,j} (v^j - x^j{}_{,t}) + x^{\tilde{a}}{}_{,jt} (v^j - x^j{}_{,t}).$$

Combining (2.6) and (2.8) yields

$$\Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} v^{\tilde{b}} v^{\tilde{c}} = \left(x^{\tilde{a}}_{,i} \Gamma^i_{jk} - x^{\tilde{a}}{}_{,jk} \right) v^j v^k + 2\Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} x^{\tilde{b}}{}_{,i} x^{\tilde{c}}{}_{,t} v^i + \Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} x^{\tilde{b}}{}_{,t} x^{\tilde{c}}{}_{,t}.$$

By summing up these three, we obtain the transformation rule of acceleration $d\mathbf{v}/dt$ as follows:

$$\frac{dv^{\tilde{a}}}{dt} = x^{\tilde{a}}{}_{,i} \frac{dv^i}{dt} + \left(x^{\tilde{a}}{}_{,tt} + \Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} x^{\tilde{b}}{}_{,t} x^{\tilde{c}}{}_{,t} \right) + 2\left(x^{\tilde{a}}{}_{,ti} x^i{}_{,\tilde{b}} + \Gamma^{\tilde{a}}_{\tilde{b}\tilde{c}} x^{\tilde{c}}{}_{,t} \right) (v^{\tilde{b}} - x^{\tilde{b}}{}_{,t}).$$

This clearly shows that the acceleration of material does not transform as a general vector. However, we can reform this into a general vector by introducing an inertial frame of reference. Providing we identify an inertial-coordinate system $\{\mathbf{z}\}$ (which is referred to as *the basic-inertial frame* in this article), we have

$$\frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\mathbf{z}\}] = x^i{}_{,I}\frac{dv^I}{dt}, \tag{4.2}$$

where $\boldsymbol{\alpha}[\mathbf{v};\{\mathbf{z}\}]$ is given by

$$\alpha^i[\mathbf{v};\{\mathbf{z}\}] = 2\big(x^i{}_{,tI}z^I{}_{,j} + \Gamma^i_{jk}x^k{}_{,t}\big)(v^j - x^j{}_{,t}) + \big(x^i{}_{,tt} + \Gamma^i_{jk}x^j{}_{,t}x^k{}_{,t}\big). \tag{4.3}$$

In totally the same manner, we obtain the counterpart in another general-coordinate frame $\{\tilde{\mathbf{x}}\}$ as

$$\frac{dv^{\tilde{a}}}{dt} - \alpha^{\tilde{a}}[\mathbf{v};\{\mathbf{z}\}] = x^{\tilde{a}}{}_{,I}\frac{dv^I}{dt}. \tag{4.4}$$

Transforming (4.2) as $dv^I/dt = z^I{}_{,i}(dv^i/dt - \alpha^i)$ and substituting it into the right-hand side of (4.4) yields

$$\begin{aligned}\frac{dv^{\tilde{a}}}{dt} - \alpha^{\tilde{a}}[\mathbf{v};\{\mathbf{z}\}] &= x^{\tilde{a}}{}_{,I}\,z^I{}_{,i}\left(\frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\mathbf{z}\}]\right)\\ &= x^{\tilde{a}}{}_{,i}\left(\frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\mathbf{z}\}]\right).\end{aligned} \tag{4.5}$$

Thus $d\mathbf{v}/dt - \boldsymbol{\alpha}$ transforms in a covariant manner. Note that there exists a subgroup of coordinate transformation where $\boldsymbol{\alpha}$ vanishes. Let one of such coordinate frames be $\{\tilde{\mathbf{z}}\}$, for which we have $\alpha^{\tilde{A}}[\mathbf{v};\{\mathbf{z}\}] = 0$. From (4.3) such a coordinate variable $\tilde{\mathbf{z}}(\mathbf{z},t)$ is determined by the following set of equations:

$$z^{\tilde{A}}{}_{,tt} + \Gamma^{\tilde{A}}_{\tilde{B}\tilde{C}}\,z^{\tilde{B}}{}_{,t}\,z^{\tilde{C}}{}_{,t} = 0,$$

$$z^{\tilde{A}}{}_{,tI}\,z^I{}_{,\tilde{B}} + \Gamma^{\tilde{A}}_{\tilde{B}\tilde{C}}\,z^{\tilde{C}}{}_{,t} = 0.$$

These are interpreted as the equations of motion in $\{\tilde{\mathbf{z}}\}$ of every point fixed to $\{\mathbf{z}\}$; the first equation means that every fixed point has a linear uniform motion in $\{\tilde{\mathbf{z}}\}$, while the second means that the velocity of a fixed point is obtained by the parallel shift of others. Thus the frame $\{\tilde{\mathbf{z}}\}$ is another inertial frame, and the subgroup such that $\boldsymbol{\alpha}$ vanishes is the Galilean group, and hence we notice that $\boldsymbol{\alpha}$ is related to the inertial force. Indeed, by substituting (2.1) into (4.3), we obtain $\boldsymbol{\alpha}$ in a Euclidean coordinate system $\{{}^*\mathbf{z}\}$ as follows:

$$\begin{aligned}\alpha^{A^*} = &-Q^{A^*}_I\,\ddot{Z}^I - 2\,{}^F\Omega^{A^*}{}_{B^*}\,v^{B^*}\\ &- {}^F\Omega^{A^*}{}_{B^*}\,{}^F\Omega^{B^*}{}_{C^*}\,z^{C^*} - {}^F\dot{\Omega}^{A^*}{}_{B^*}z^{B^*},\end{aligned} \tag{4.6}$$

where ${}^F\boldsymbol{\Omega}$ represents the angular velocity of the frame rotation (see Appendix C for more details). Note that ${}^F\boldsymbol{\Omega}$ does not transform as a general tensor. The first term of (4.6) originates from the acceleration of the center of $\{{}^*\mathbf{z}\}$, and the second and third terms are the Coriolis and centrifugal forces. Under larger transformation groups $\mathcal{G}_e$ and $\mathcal{A}_f$, $\boldsymbol{\alpha}$ includes more complicated terms such as caused by the deformation of the coordinate frame. In this sense $\boldsymbol{\alpha}$ may be understood as a generalization of the inertial force. A brief derivation of (4.6) is given in Appendix D.

In the following discussions we write $\boldsymbol{\alpha}$ without $\{\mathbf{z}\}$. It is easily proved in Appendix G that $\boldsymbol{\alpha}$ is invariant under an arbitrary change of the basic-inertial frame $\{\mathbf{z}\}$ to another, so that we are not stuck on only one basic-inertial frame anymore.

Now let us return to the equation of motion. Since $\boldsymbol{\alpha} = \mathbf{0}$ in inertial frames, (4.1) can be rewritten as

$$\frac{dv^I}{dt} - \alpha^I[\mathbf{v}] = \frac{1}{\rho}\sigma^{IJ}{}_{;J} + f^I. \tag{4.7}$$

It is well known that $\boldsymbol{\sigma}$ and $\mathbf{f}$ transform as the general tensor and vector for ordinary material in nonrelativistic frameworks. Thus, by multiplying (4.7) by $x^i{}_{,I}$ and using the identity (4.5), we immediately obtain the equation of motion in an arbitrary general coordinate system $\{\mathbf{x}\}$ as follows:

$$\frac{dv^i}{dt} - \alpha^i[\mathbf{v}] = \frac{1}{\rho}\sigma^{ij}{}_{;j} + f^i, \tag{4.8}$$

which is the generally covariant dynamical equation of the instantaneous flow. Comparing (4.7) and (4.8), it is obvious that the equation in the inertial frame is included in the generally covariant representation (4.8).

B. Mean-flow equation

Here we attempt to derive the mean-flow equation which is simply obtained by taking the ensemble average of (4.8). Averaging the Lagrangian-derivative term yields

$$\begin{aligned}\left\langle\frac{dv^i}{dt}\right\rangle &= \left\langle\left(\frac{D}{Dt} + v'^j\nabla_j\right)(V^i + v'^i)\right\rangle\\ &= \frac{DV^i}{Dt} + \langle v'^i{}_{;j}\,v'^j\rangle = \frac{DV^i}{Dt} + \langle v'^i v'^j\rangle_{;j}.\end{aligned}$$

Owing to the linear dependence of $\boldsymbol{\alpha}$ on $\mathbf{v}$, the average of $\boldsymbol{\alpha}$ is achieved simply by replacing $\mathbf{v}$ by $\mathbf{V}$:

$$\begin{aligned}\langle\alpha^i[\mathbf{v}]\rangle &= 2\big(x^i{}_{,tI}z^I{}_{,j} + \Gamma^i_{jk}x^k{}_{,t}\big)(V^j - x^j{}_{,t})\\ &\quad + \big(x^i{}_{,tt} + \Gamma^i_{jk}x^j{}_{,t}x^k{}_{,t}\big)\\ &= \alpha^i[\mathbf{V}].\end{aligned}$$

Thus the ensemble average of (4.8) yields

$$\frac{DV^i}{Dt} - \alpha^i[\mathbf{V}] = \frac{1}{\rho}\langle\sigma^{ij}\rangle_{;j} + \langle f^i\rangle - R^{ij}{}_{;j}. \tag{4.9}$$

Considering the similarity to (4.8), one may soon notice the general covariance of (4.9). $\langle\boldsymbol{\sigma}\rangle$ and $\langle\mathbf{f}\rangle$ are general tensor and vector from Theorem II B. In addition, owing to the $\mathbf{v}-\mathbf{V}$ correspondence, the left-hand side of (4.9) transforms in the same manner as (4.5), namely,

$$\frac{DV^{\tilde{a}}}{Dt} - \alpha^{\tilde{a}}[\mathbf{V}] = x^{\tilde{a}}{}_{,i}\left(\frac{DV^i}{Dt} - \alpha^i[\mathbf{V}]\right), \tag{4.10}$$

which is also directly proved by averaging (4.5).

C. Velocity-fluctuation equation

The velocity-fluctuation equation is obtained by subtracting (4.9) from (4.8), which yields

$$\begin{aligned}\frac{Dv'^i}{Dt} &+ V^i{}_{;j}v'^j + v'^i{}_{;j}\,v'^j - 2\big(x^i{}_{,It}\,z^I{}_{,j} + \Gamma^i_{jk}\,x^k{}_{,t}\big)\\ &= \frac{1}{\rho}\sigma'^{ij}{}_{;j} + f'^i + R^{ij}{}_{;j}.\end{aligned} \tag{4.11}$$

Note that both $\boldsymbol{\sigma}'$ and $\mathbf{f}'$ are a tensor and vector because of Theorem II B. In order to transform the fourth term on the left-hand side of (4.11) we utilize the transformation rule for the velocity gradient (2.9); applying this to the transformation of the mean-velocity gradient from $\{\mathbf{z}\}$ to $\{\mathbf{x}\}$ yields

$$V^i{}_{;j} = x^i{}_{,I}\, z^J{}_{,j}\, V^I{}_{;J} + x^i{}_{,It}\, z^I{}_{,j} + \Gamma^i_{jk}\, x^k{}_{,t},$$
$$\Leftrightarrow x^i{}_{,It}\, z^I{}_{,j} + \Gamma^i_{jk}\, x^k{}_{,t} = V^i{}_{;j} - x^i{}_{,I}\, z^J{}_{,j}\, V^I{}_{;J}.$$

By substituting this into (4.11) we obtain

$$\frac{Dv'^i}{Dt} - V^i{}_{;j} v'^j + v'^i{}_{;j}\, v'^j - 2x^i{}_{,I}\, z^J{}_{,j}\, V^I{}_{;J}\, v'^j = \frac{1}{\rho}\sigma'^{ij}{}_{;j} + f'^i + R^{ij}{}_{;j}.$$

The first and second terms are combined together and rewritten as the convected derivative of the velocity fluctuation. Let us introduce a two-rank tensor in general coordinate frame $\{\mathbf{x}\}$ given by

$$\Sigma^i{}_j \equiv x^i{}_{,I}\, z^J{}_{,j}\, V^I{}_{;J}. \tag{4.12}$$

$\boldsymbol{\Sigma}$ is an objective measure of deviation of the mean-flow motion from the inertial motion and behaves as a general tensor (see Appendix E). Then we reach a covariant equation of the velocity fluctuation:

$$\frac{\mathfrak{D}v'^i}{\mathfrak{D}t} + v'^i{}_{;j}\, v'^j + 2\Sigma^i{}_j\, v'^j = \frac{1}{\rho}\sigma'^{ij}{}_{;j} + f'^i + R^{ij}{}_{;j}. \tag{4.13}$$

The symmetric part of $\Sigma_{ij} = g_{ik}\Sigma^k{}_j$ is rewritten as

$$\begin{aligned}\Sigma_{ij} + \Sigma_{ji} &= z^I{}_{,i}\, z^I{}_{,j}(V_{I;J} + V_{J;I}) \\ &= z^I{}_{,i}\, z^I{}_{,j}\big(g_{IJ,t} + V^K g_{IJ;K} + V^K{}_{;I}\, g_{KJ} + V^K{}_{;J}\, g_{IK}\big) \\ &= z^I{}_{,i}\, z^I{}_{,j}\frac{\mathfrak{D}}{\mathfrak{D}t} g_{IJ} = \frac{\mathfrak{D}}{\mathfrak{D}t} g_{ij} = S_{ij},\end{aligned} \tag{4.14}$$

where we used $g_{IJ,t} = 0$ and $g_{IJ;K} = 0$. The antisymmetric part is the generalized absolute vorticity of the mean flow (see Appendix F), which is written in the present work as

$$\Theta_{ij} = \Sigma_{ij} - \Sigma_{ji}. \tag{4.15}$$

Thus (4.13) is also written as

$$\frac{\mathfrak{D}v'^i}{\mathfrak{D}t} = -\big(S^i_j + \Theta^i{}_j\big)\, v'^j - v'^i{}_{;j}\, v'^j + R^{ij}{}_{;j} + \frac{1}{\rho}\sigma'^{ij}{}_{;j} + f'^i. \tag{4.16}$$

Since both (4.13) and (4.16) are written as a tensor-polynomial equation, these are generally covariant equations for Theorem II B. Note that one can choose another time derivative instead of $\mathfrak{D}/\mathfrak{D}t$. This can be conducted by adding generally covariant terms to both sides of (4.13) or (4.16), which again results in the generally covariant equations.

V. DISCUSSIONS

A. General covariance of turbulence constitutive relations

Since (4.13) and (4.16) are tensor-polynomial equations, any moment equations constructed from (4.13) or (4.16) are represented as the averaged tensor-polynomial equations, which are generally covariant for Theorem II B. Indeed, by multiplying (4.16) by v'^j, adding an $i - j$ transposed equation to it, and taking the ensemble average of the result, we obtain the following differential equation for the Reynolds stress:

$$\begin{aligned}\frac{\mathfrak{D}}{\mathfrak{D}t}R^{ij} &= -\big(S^i_k R^{jk} + S^j_k R^{ik}\big) - (\Theta^i{}_k R^{jk} + \Theta^j{}_k R^{ik}) \\ &\quad - T^{ijk}{}_{;k} + \frac{1}{\rho}\langle \sigma'^{ik}{}_{;k} v'^j + \sigma'^{jk}{}_{;k} v'^i\rangle \\ &\quad - \langle f'^i v'^j + f'^j v'^i\rangle,\end{aligned} \tag{5.1}$$

where T^{ijk} is the triple moment $\langle v'^i v'^j v'^k\rangle$. Obviously (5.1) is a generally covariant equation. By multiplying (4.16) by v'^j and v'^k, summing up the cyclic permutations of (i,j,k), and taking the ensemble average of the result, we obtain the triple-moment equation as follows:

$$\begin{aligned}\frac{\mathfrak{D}}{\mathfrak{D}t}T^{ijk} &= -\big(S^i_a T^{ajk} + S^j_a T^{iak} + S^k_a T^{ija}\big) \\ &\quad - \big(\Theta^i{}_a T^{ajk} + \Theta^j{}_a T^{iak} + \Theta^k{}_a T^{ija}\big) \\ &\quad - \langle v'^i v'^j v'^k v'^a\rangle_{;a} \\ &\quad + \frac{1}{\rho}\langle \sigma'^{ia}{}_{;a} v'^j v'^k + \sigma'^{ja}{}_{;a} v'^k v'^i + \sigma'^{ka}{}_{;a} v'^i v'^j\rangle \\ &\quad - \langle f'^i v'^j v'^k + f'^j v'^k v'^i + f'^k v'^i v'^j\rangle \\ &\quad + R^{ia}{}_{;a} R^{jk} + R^{ja}{}_{;a} R^{ki} + R^{ka}{}_{;a} R^{ij},\end{aligned} \tag{5.2}$$

which is again obviously covariant. Likewise, we can obtain equations for moments of arbitrary orders in generally covariant forms. Let us see here the force-free Navier-Stokes equation for simplicity, where $\mathbf{f} = \mathbf{0}$. The stress tensor is given by

$$\sigma^{ij} = \rho(-pg^{ij} + \nu s^{ij}),$$

where p is the pressure divided by the constant mass density, and ν is the kinematic viscosity. The velocity-fluctuation equation (4.13) is rewritten as

$$\frac{\mathfrak{D}v'^i}{\mathfrak{D}t} + v'^i{}_{;j}\, v'^j + 2\Sigma^i{}_j\, v'^j = -p'^{;i} + \nu\triangle v'^i + R^{ij}{}_{;j}.$$

By taking the covariant divergence of both sides, we obtain a generally covariant equation for the pressure fluctuation as follows:

$$\triangle p' = -2\Sigma^i{}_j\, v'^j{}_{;i} - v'^i{}_{;j}\, v'^j{}_{;i} + R^{ij}{}_{;ij},$$

which determines p' in a covariant manner. Indeed p' is a scalar (zero-rank tensor) because of Theorem II B. Now (5.1) turns into

$$\begin{aligned}\frac{\mathfrak{D}}{\mathfrak{D}t}R^{ij} &= -\big(S^i_k R^{jk} + S^j_k R^{ik}\big) - (\Theta^i{}_k R^{jk} + \Theta^j{}_k R^{ik}) \\ &\quad - \epsilon^{ij} + \phi^{ij} - t^{ijk}{}_{;k},\end{aligned} \tag{5.3}$$

where the dissipation $\boldsymbol{\epsilon}$, the redistribution $\boldsymbol{\phi}$, and the flux t are given, respectively, by

$$\begin{aligned}\epsilon^{ij} &= 2\nu\langle v'^{i;k} v'^j{}_{;k}\rangle, \\ \phi^{ij} &= \langle p'\,(v'^{i;j} + v'^{j;i})\rangle, \\ t^{ijk} &= \langle p' g^{ik} v'^j + p' g^{jk} v'^i\rangle + \langle v'^i v'^j v'^k\rangle - \nu R^{ij;k},\end{aligned}$$

all of which are obviously covariant. Thus (5.3) provides the Reynolds stress equation in generally covariant form. A simple example of modeling of (5.3) is shown in Appendix H, where each term should be modeled in covariant form. The equations for turbulence energy and its dissipation rate are obtained, respectively, as

$$\frac{\mathfrak{D}K}{\mathfrak{D}t} = -\frac{1}{2}R^{ij}S_{ij} - \epsilon + \left\langle \left(\frac{1}{2} g_{ab} v'^a v'^b + p' \right) v'^j \right\rangle_{;j} + \nu \triangle K, \tag{5.4}$$

$$\begin{aligned}\frac{\mathfrak{D}\epsilon}{\mathfrak{D}t} &= -2\nu \langle v'^i{}_{;j} v'_{i;k} v'^{j;k} \rangle - 2\nu^2 \langle v'^{i;jk} v'_{i;jk} \rangle \\ &\quad - \nu \left(S^i_j + \Theta^i{}_j \right) \langle v'_{i;k} v'^{j;k} + v'_{k;i} v'^{k;j} \rangle \\ &\quad - \nu (S_{ij;k} + \Theta_{ij;k}) \langle v'^j v'^{i;k} \rangle \\ &\quad - (\nu \langle v'^{a;b} v'_{a;b} v'^j \rangle + 2\nu \langle p'_{,a} v'^{j;a} \rangle)_{;j} + \nu \triangle \epsilon,\end{aligned} \tag{5.5}$$

both of which are generally covariant (or, more strictly speaking, invariant). Note that $\mathfrak{D}/\mathfrak{D}t$ operations on scalars coincide with those of D/Dt, so that one can replace $\mathfrak{D}/\mathfrak{D}t$ in the above two equations with D/Dt, which may be more frequently used in conventional notations.

Note that equations for various correlations are written in generally covariant forms due to the general covariance of the fluctuation equation. In this sense the covariance of the fluctuation and its equation is much more fundamental feature than that of turbulence constitutive relations such as (5.1)–(5.5). The very reason for the general covariance of the constitutive relations is finally attributed to the general covariance of (4.16).

B. Covariance and physical objectivity

Generally speaking, change of the reference frame does not change the physical state of nature. Since the instantaneous flow remains in the same physical state under arbitrary coordinate transformation (2.8), we may say that *the instantaneous flow has the physical objectivity*. If Eq. (4.8) properly predicts the instantaneous flow, it must be consistent with the physical objectivity. Let us investigate this consistency through the following steps, providing we solve the instantaneous flow in two coordinate systems independently; namely, we solve (4.8) and obtain v^i in a frame $\{\mathbf{x}\}$ on the one hand, and also solve

$$\frac{dv^{\tilde{a}}}{dt} - \alpha^{\tilde{a}}[\mathbf{v}] = \frac{1}{\rho}\sigma^{\tilde{a}\tilde{b}}{}_{;\tilde{b}} + f^{\tilde{a}}$$

and obtain $v^{\tilde{a}}$ in another frame $\{\tilde{\mathbf{x}}\}$ on the other hand. As long as the instantaneous flow is physically objective, the above two solutions v^i and $v^{\tilde{a}}$ must satisfy its proper transformation rule (2.8). Since we have shown (4.5) from (2.8), we have (2.8) ⇒ (4.5); namely, (4.5) must hold for the above two solutions v^i and $v^{\tilde{a}}$, otherwise these two represent different flows. This can be confirmed by using the transformation rule of $\boldsymbol{\sigma}$ and $\mathbf{f}$ as follows:

$$\begin{aligned}\frac{dv^{\tilde{a}}}{dt} - \alpha^{\tilde{a}}[\mathbf{v}] &= \frac{1}{\rho}\sigma^{\tilde{a}\tilde{b}}{}_{;\tilde{b}} + f^{\tilde{a}} = x^{\tilde{a}}{}_{,i}\left(\frac{1}{\rho}\sigma^{ij}{}_{;j} + f^i\right) \\ &= x^{\tilde{a}}{}_{,i}\left(\frac{dv^i}{dt} - \alpha^i[\mathbf{v}]\right).\end{aligned}$$

Namely, (4.5), a necessary condition for the objectivity, can be reproduced as long as $\boldsymbol{\sigma}$ and $\mathbf{f}$ transform as a tensor and vector. As the contraposition of this logic, we reach the following statement: *if the constitutive model for either $\boldsymbol{\sigma}$ or $\mathbf{f}$ breaks the general covariance, the velocities calculated in different coordinate frames represent different flows,* where the objectivity of the flow breaks down.

Due to the **v**-**V** correspondence, the objectivity of the mean flow can be discussed in the same manner. If (3.3) holds, namely, $V^{\tilde{a}} = x^{\tilde{a}}{}_{,i}\, V^i + x^{\tilde{a}}{}_{,t}$, V^i and $V^{\tilde{a}}$ observed in the different frames $\{\mathbf{x}\}$ and $\{\tilde{\mathbf{x}}\}$ represent the very same state of the mean flow, which may be termed as *the physical objectivity of the mean flow*. Besides, we have (3.3) ⇒ (4.10), so that (4.10) is necessary for the objectivity of the mean flow. The Reynolds stress need transform in covariant manner for (4.10). On the contrary, if one constructs a model of the Reynolds stress, say, ${}^{M}\mathsf{R}$, which breaks the general covariance, i.e., ${}^{M}R^{\tilde{a}\tilde{b}} \neq x^{\tilde{a}}{}_{,i}\, x^{\tilde{b}}{}_{,j}\, {}^{M}R^{ij}$, we have

$$\begin{aligned}\frac{DV^{\tilde{a}}}{Dt} - \alpha^{\tilde{a}}[\mathbf{V}] &= \frac{1}{\rho}\langle\sigma^{\tilde{a}\tilde{b}}\rangle_{;\tilde{b}} - {}^{M}R^{\tilde{a}\tilde{b}}{}_{;\tilde{b}} \\ &\neq x^{\tilde{a}}{}_{,i}\left(\frac{1}{\rho}\langle\sigma^{ij}\rangle_{;j} - {}^{M}R^{ij}{}_{;j}\right) \\ &= x^{\tilde{a}}{}_{,i}\left(\frac{DV^i}{Dt} - \alpha^i[\mathbf{V}]\right),\end{aligned} \tag{5.6}$$

which contradicts the identity (4.10), and (3.3) breaks down, namely, $V^{\tilde{a}} \neq x^{\tilde{a}}{}_{,i}\, V^i + x^{\tilde{a}}{}_{,t}$, which means that the mean-velocity fields calculated in $\{\mathbf{x}\}$ and $\{\tilde{\mathbf{x}}\}$ represent different mean flows. Thus the general covariance of the Reynolds stress is very needed in the physical objectivity of the mean flow.

VI. CONCLUSIONS

Covariance of turbulence quantities and equations has been discussed on the basis of the general coordinate transformations, which form a far wider transformation group than the Euclidean-transformation group. The author obtained the following as our conclusions:

(1) The velocity fluctuation, the most fundamental quantity in turbulence physics, is a vector (one-rank tensor) under the general-coordinate-transformation group. In addition, its governing equation is generally covariant. Thus the general covariance should be considered as one of the most fundamental features of turbulence. As natural consequences, various correlations and their dynamical equations are formulated in a generally covariant manner.

(2) The governing equations for the instantaneous and mean flows can be rewritten in generally covariant forms, which can illustrate how the objectivity of the instantaneous and mean flows under the change of frames are guaranteed. Due to the **v**-**V** correspondence, the objectivity of the mean flow can be discussed as well as the instantaneous flow. In the same manner that the instantaneous-flow objectivity requires the constitutive equations for stress and external force to be generally covariant, the general covariance of the Reynolds stress is needed for the physical objectivity of the mean flow.

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APPENDIX A: TRANSFORMATION RULE OF THE VELOCITY FIELD

Let an element of continuum be P whose position at time t is given by ${}^{P}x^i(t)$ in a coordinate system $\{\mathbf{x}\}$. The velocity of P in this coordinate representation is given by

$$ {}^{P}v^i(t) = \frac{d}{dt}\,{}^{P}x^i(t). $$

We introduce another coordinate system $\{\tilde{\mathbf{x}}\}$ whose relation with $\{\mathbf{x}\}$ is given by $x^{\tilde{a}} = x^{\tilde{a}}(\mathbf{x},t)$. In the new coordinate system, the position of P is given by

$$ {}^{P}x^{\tilde{a}}(t) = x^{\tilde{a}}({}^{P}\mathbf{x}(t),t). $$

Thus the velocity in the coordinate system $\{\tilde{\mathbf{x}}\}$ is given by

$$ {}^{P}x^{\tilde{a}}(t) = \frac{d}{dt}\,{}^{P}x^{\tilde{a}}(t) = \frac{d}{dt}x^{\tilde{a}}({}^{P}\mathbf{x}(t),t) = \frac{\partial x^{\tilde{a}}}{\partial x^i}({}^{P}\mathbf{x}(t),t)\,{}^{P}v^i(t) + \frac{\partial x^{\tilde{a}}}{\partial t}({}^{P}\mathbf{x}(t),t). $$

Here we should notice that the last term on the right-hand side is obtained by substituting $\mathbf{x} = {}^{P}\mathbf{x}(t)$ into the derivative function $\partial\tilde{x}^a/\partial t\,(\mathbf{x},t)$. The element velocity ${}^{P}\mathbf{v}(t)$ is given by $\mathbf{v}\,({}^{P}\mathbf{x}(t),t)$, so that the above is reduced to

$$ v^{\tilde{a}}({}^{P}\tilde{\mathbf{x}}(t),t) = \frac{\partial x^{\tilde{a}}}{\partial x^i}({}^{P}\mathbf{x}(t),t)v^i({}^{P}\mathbf{x}(t),t) + \frac{\partial x^{\tilde{a}}}{\partial t}({}^{P}\mathbf{x}(t),t). $$

This relation holds for arbitrary element of continuum. Thus, by replacing $({}^{P}\mathbf{x}(t),t)$ and $({}^{P}\tilde{\mathbf{x}}(t),t)$ with $(\mathbf{x},t)$ and $(\tilde{\mathbf{x}},t)$, we obtain

$$ v^{\tilde{a}}(\tilde{\mathbf{x}},t) = \frac{\partial x^{\tilde{a}}}{\partial x^i}(\mathbf{x},t)\; v^i\,(\mathbf{x},t) + \frac{\partial x^{\tilde{a}}}{\partial t}(\mathbf{x},t)\,, $$

which is rewritten as $v^{\tilde{a}} = x^{\tilde{a}}{}_{,i}\, v^i + x^{\tilde{a}}{}_{,t}$ in our abbreviated notation.

APPENDIX B: GROUP STRUCTURES OF $\mathcal{G}_e$, $\mathcal{A}_f$, $\mathcal{E}_u$, AND $\mathcal{G}_a$

In our general-coordinate transformation (2.3), we do not treat the transformation of the time parameter, which may be written as

$$ \tilde{t} = \tilde{t}(x^1,x^2,x^3,t) = t. \tag{B1} $$

Let us introduce the four-dimensional representation to make later discussions simple: $X^1 = x^1$, $X^2 = x^2$, $X^3 = x^3$, and $X^4 = t$. In this notation (2.3) and (B1) may be combined as follows:

$$ \mathfrak{g}_e : \{X^1,X^2,X^3,X^4\} \to \{\tilde{X}^1,\tilde{X}^2,\tilde{X}^3,\tilde{X}^4\}, \tag{B2} $$

where $\tilde{X}^4 = X^4$. Thus $\mathcal{G}_e$ is the set of mappings such as $\mathfrak{g}_e$. Using the nondegenerateness condition, we obtain

$$ \frac{\partial\tilde{\mathbf{X}}}{\partial\mathbf{X}} = \begin{vmatrix} \frac{\partial\tilde{X}^1}{\partial X^1} & \frac{\partial\tilde{X}^1}{\partial X^2} & \frac{\partial\tilde{X}^1}{\partial X^3} & \frac{\partial\tilde{X}^1}{\partial X^4} \\ \frac{\partial\tilde{X}^2}{\partial X^1} & \frac{\partial\tilde{X}^2}{\partial X^2} & \frac{\partial\tilde{X}^2}{\partial X^3} & \frac{\partial\tilde{X}^2}{\partial X^4} \\ \frac{\partial\tilde{X}^3}{\partial X^1} & \frac{\partial\tilde{X}^3}{\partial X^2} & \frac{\partial\tilde{X}^3}{\partial X^3} & \frac{\partial\tilde{X}^3}{\partial X^4} \\ \frac{\partial\tilde{X}^4}{\partial X^1} & \frac{\partial\tilde{X}^4}{\partial X^2} & \frac{\partial\tilde{X}^4}{\partial X^3} & \frac{\partial\tilde{X}^4}{\partial X^4} \end{vmatrix} = \begin{vmatrix} \frac{\partial\tilde{x}^1}{\partial x^1} & \frac{\partial\tilde{x}^1}{\partial x^2} & \frac{\partial\tilde{x}^1}{\partial x^3} & \frac{\partial\tilde{x}^1}{\partial t} \\ \frac{\partial\tilde{x}^2}{\partial x^1} & \frac{\partial\tilde{x}^2}{\partial x^2} & \frac{\partial\tilde{x}^2}{\partial x^3} & \frac{\partial\tilde{x}^2}{\partial t} \\ \frac{\partial\tilde{x}^3}{\partial x^1} & \frac{\partial\tilde{x}^3}{\partial x^2} & \frac{\partial\tilde{x}^3}{\partial x^3} & \frac{\partial\tilde{x}^3}{\partial t} \\ & & & \frac{\partial\tilde{t}}{\partial t} \end{vmatrix} = \frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\frac{\partial\tilde{t}}{\partial t} = \frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}} \times 1 \neq 0. \tag{B3} $$

Thus the inverse mapping $\mathfrak{g}_e^{-1}$ exists (*Existence of the inverse element*). The identity element is obtained as an identity mapping (*Existence of the identity element*):

$$ \imath : \{X^1,X^2,X^3,X^4\} \to \{X^1,X^2,X^3,X^4\}. $$

The group operation is defined as the composition of two mappings. Let us think of multiple mappings such as

$$ \{\mathbf{X}\} \overset{\mathfrak{g}'_e}{\to} \{\mathbf{X}'\} \overset{\mathfrak{g}''_e}{\to} \{\mathbf{X}''\} \overset{\mathfrak{g}'''_e}{\to} \{\mathbf{X}'''\}. \tag{B4} $$

We define an operation μ between two elements $\mathfrak{g}'_e$ and $\mathfrak{g}''_e$ as a composite mapping from $\mathbf{X}$ to $\mathbf{X}''$, which may be written as

$$ \mu(\mathfrak{g}''_e,\mathfrak{g}'_e) : \mathbf{X}'' = \mathbf{X}''\left(\mathbf{X}'(\mathbf{X})\right) = \mathbf{X}'' \circ \mathbf{X}'\,(\mathbf{X}). \tag{B5} $$

Note that $\mu(\mathfrak{g}''_e,\mathfrak{g}'_e) \in \mathcal{G}_e$ (*Requirement for the closure*). Generally speaking, compositions of one-to-one mappings are associative. Thus the operation μ is also associative (*Requirement for the associativity*). Indeed, we have

$$ (\mathbf{X}''' \circ \mathbf{X}'') \circ \mathbf{X}'(\mathbf{X}) = \mathbf{X}''' \circ (\mathbf{X}'' \circ \mathbf{X}')(\mathbf{X}), \tag{B6} $$

which reduces to

$$ \mu(\mu(\mathfrak{g}'''_e,\mathfrak{g}''_e),\mathfrak{g}'_e) = \mu(\mathfrak{g}'''_e,\mu(\mathfrak{g}''_e,\mathfrak{g}'_e)). \tag{B7} $$

Thus the set and operation $(\mathcal{G}_e,\mu)$ satisfies the requirements for *identity element*, *inverse element*, *closure,* and *associativity*, which offer the proof that $\mathcal{G}_e$ has a group structure.

Next let us see $\mathcal{A}_f$: a set of transformations between coordinate systems defined by (2.1). A transformation $\mathfrak{a}_f$ ($\in \mathcal{A}_f$) between $\{{}^*\mathbf{z}\}$ and $\{{}^{**}\mathbf{z}\}$, namely,

$$ \mathfrak{a}_f : \{{}^*\mathbf{z}\} \to \{{}^{**}\mathbf{z}\}, $$

is in general given by the following linear transformation:

$$ z^{B^{**}} = Q^{B^{**}}_{A^*}(t)\, z^{A^*} + Z^{B^{**}}(t), \tag{B8} $$

where Q is again an arbitrary time-dependent regular matrix. Obviously, (B8) is a branch of (2.3) so that $\mathcal{A}_f$ is a subset of $\mathcal{G}_e$. For multiple transformations

$$ \{{}^*\mathbf{z}\} \overset{\mathfrak{a}'_f}{\to} \{{}^{**}\mathbf{z}\} \overset{\mathfrak{a}''_f}{\to} \{{}^{***}\mathbf{z}\}, $$

the operation μ of two affine transformations is given by the following composition:

$$ z^{C^{***}} = Q^{C^{***}}_{B^{**}}(t)\left[Q^{B^{**}}_{A^*}(t)\, z^{A^*} + Z^{B^{**}}(t)\right] + Z^{C^{***}}(t) = Q^{C^{***}}_{A^*}(t)\, z^{A^*} + Q^{C^{***}}_{B^{**}}(t)\, Z^{B^{**}}(t) + Z^{C^{***}}(t), \tag{B9} $$

which is obviously a branch of (B8); thus we have $\mu(\mathfrak{a}'_f \circ \mathfrak{a}''_f) \in \mathcal{A}_f$, namely, the closure condition $\mu : \mathcal{A}_f \times \mathcal{A}_f \to \mathcal{A}_f$

is obtained. $\mathcal{A}_f$ fulfills the identity and inverse requirements since $\mathcal{A}_f \subset \mathcal{G}_e$. The group operation μ is of course associative. Considering the consistency with the four requirements, we notice that $(\mathcal{A}_f,\mu)$ has also a group structure.

By imposing the orthogonality to $Q^{B^{**}}_{A^*}(t)$, we obtain the Euclidean transformation whose subset is $\mathcal{E}_u$ ($\subset \mathcal{A}_f$):

$$Q^{C^{***}}_{A^*}(t) = Q^{C^{***}}_{B^{**}}(t)\, Q^{B^{**}}_{A^*}(t),$$

in the second line in (B9) and is again a time-dependent orthogonal matrix, so the composition of any two Euclidean transformations is again another Euclidean transformation; namely, the closure condition $\mu : \mathcal{E}_u \times \mathcal{E}_u \to \mathcal{E}_u$ is obtained. The other three requirements are obviously satisfied. Thus $(\mathcal{E}_u,\mu)$ has again a group structure.

By imposing the time independence of $Q^{B^{**}}_{A^*}$ and linear time dependence of $Z^{B^{**}}(t)$ in (B8), we obtain a Galilean transformation whose set is $\mathcal{G}_a$. The composition of two Galilean transformations becomes another Galilean transformation for the time independence of $Q^{C^{***}}_{A^*}$ and the linear time dependence of $Q^{C^{***}}_{B^{**}}\, Z^{B^{**}}(t) + Z^{C^{***}}(t)$ in the second line of (B9); namely, the closure condition $\mu : \mathcal{G}_a \times \mathcal{G}_a \to \mathcal{G}_a$ is obtained. The other three requirements are again obviously satisfied. Thus $(\mathcal{G}_a,\mu)$ has also a group structure.

As is seen above, all the transformation groups $\mathcal{G}_e$, $\mathcal{A}_f$, $\mathcal{E}_u$ and $\mathcal{G}_a$ ($\mathcal{G}_e \supset \mathcal{A}_f \supset \mathcal{E}_u \supset \mathcal{G}_a$) have the group structures through the same group operation μ.

APPENDIX C: ANGULAR VELOCITY OF THE ROTATING FRAME

Here we review the Euclidean transformation in detail. Let us introduce a coordinate system $\{{}^*\mathbf{z}\}$ rotating and translating against an orthonormal inertial frame $\{\mathbf{z}\}$ by the following relation:

$$z^I = Q^I_{A^*}(t)\, z^{A^*} + Z^I(t). \tag{C1}$$

Here we use indices such as A^* as the signs for the orthonormal-coordinate representations. Q is a time-dependent orthogonal matrix, and $\mathbf{Z}$ is the origin of $\{{}^*\mathbf{z}\}$. Note that $Q^I_{A^*} = z^I{}_{,A^*}$. Its inverse matrix is given by $Q^{A^*}{}_I = z^{A^*}{}_{,I}$. The transformation rule of a metric tensor reduces to

$$g_{IJ} = g_{A^*B^*}\, Q^{A^*}_I\, Q^{B^*}_J = \text{const}, \tag{C2}$$

where we still write metric as g for retaining the tensor notation despite its being represented as a unit matrix; i.e., (C2) provides the orthogonality condition of Q. By differentiating both sides of (C2), we obtain

$$0 = \dot{g}_{IJ} = g_{A^*B^*}\, \dot{Q}^{A^*}_I\, Q^{B^*}_J + g_{A^*B^*}\, Q^{A^*}_I\, \dot{Q}^{B^*}_J,$$

which yields

$$g_{A^*B^*}\, \dot{Q}^{A^*}_I\, Q^{B^*}_J = -g_{A^*B^*}\, \dot{Q}^{A^*}_J\, Q^{B^*}_I,$$

where we used the symmetry of the metric tensor under the exchange of its indices. This means that the matrix $g_{A^*B^*}\, \dot{Q}^{A^*}_I\, Q^{B^*}_J$ is antisymmetric under the exchange of indices I and J. Thus we define an antisymmetric matrix A as

$$A_{IJ} = g_{A^*B^*}\, \dot{Q}^{A^*}_I\, Q^{B^*}_J.$$

By multiplying the above by the contravariant metric, we obtain

$$A^I{}_J = g^{IL} A_{LJ} = g^{IL} g_{KL}\, Q^{A^*}_J\, \dot{Q}^K_{A^*} = \delta^I_K\, Q^{A^*}_J\, \dot{Q}^K_{A^*} = Q^{A^*}_J\, \dot{Q}^I_{A^*},$$

which yields the dynamical equation of Q as follows:

$$\dot{Q}^I_{A^*} = A^I{}_J\, Q^J_{A^*}. \tag{C3}$$

Let us review the physical interpretation of the above by introducing a point fixed to the rotating coordinate system, whose coordinate value in $\{\mathbf{z}\}$ is given by

$$a^I = Q^I_{A^*}(t)\, a^{A^*} + Z^I(t).$$

The velocity of the point is obtained by taking the time derivative of the above as follows:

$$\dot{a}^I = \dot{Q}^I_{A^*}(t)\, a^{A^*} + \dot{Z}^I(t) = A^I{}_J(a^J - Z^J) + \dot{Z}^I,$$

which clearly means that the motion of the fixed point is a superposition of the rotation around $\mathbf{Z}$ and its translation. Thus A can be interpreted as *the angular velocity* of the frame $\{{}^*\mathbf{z}\}$, and we rewrite this as $\mathsf{A} = {}^{F}\mathbf{\Omega}$. Following this we rewrite (C3) as

$$\dot{Q}^I_{A^*} = {}^{F}\Omega^I{}_J\, Q^J_{A^*}. \tag{C4}$$

The rotational transformation is completely prescribed by (C1), (C4), and the trajectory of the origin $\mathbf{Z}(t)$. On the other hand, the equation for the inverse matrix $Q^{A^*}_I$ is obtained by taking the time derivative of the identity $\delta^{A^*}_{B^*} = Q^{A^*}_I\, Q^I_{B^*}$, namely,

$$0 = \dot{Q}^{A^*}_J\, Q^J_{B^*} + Q^{A^*}_I\, \dot{Q}^I_{B^*},$$

which reduces to

$$\begin{aligned} \dot{Q}^{A^*}_J\, Q^J_{B^*} &= -Q^{A^*}_I\, \dot{Q}^I_{B^*} = -Q^{A^*}_I \left({}^{F}\Omega^I{}_J\, Q^J_{B^*}\right) \\ &\equiv -{}^{F}\Omega^{A^*}{}_{B^*} \\ \Leftrightarrow\ \dot{Q}^{A^*}_I &= -{}^{F}\Omega^{A^*}{}_{B^*}\, Q^{B^*}_I, \end{aligned} \tag{C5}$$

where ${}^{F}\Omega^{A^*}{}_{B^*}$ represents the frame rotation observed from the rotating frame $\{{}^*\mathbf{z}\}$.

Note that ${}^{F}\mathbf{\Omega}$ does not transform as tensor for its definition, namely,

$${}^{F}\Omega^I{}_J = \dot{Q}^I_{A^*}\, Q^{A^*}_J,$$

which clearly depends on the frame $\{{}^*\mathbf{z}\}$. Indeed, for another rotating frame $\{{}^{**}\mathbf{z}\}$ one may obtain

$${}^{\bar{F}}\Omega^I{}_J = \dot{Q}^I_{B^{**}}\, Q^{B^{**}}_J,$$

which is never obtained from the Euclidean transformation of the previous.

APPENDIX D: PROOF OF EQ. (4.6)

Straight substitution of (2.1) into (4.3) yields

$$\alpha^{A^*} = 2z^{A^*}{}_{,tI}\, z^I{}_{,B^*}(v^{B^*} - z^{B^*}{}_{,t}) + z^{A^*}{}_{,tt}. \tag{D1}$$

Note that $z^I{}_{,A^*} = Q^I_{A^*}$, $z^{A^*}{}_{,I} = Q^I_{A^*}$, and $z^{A^*}{}_{,tI} = \dot{Q}^{A^*}_I$. In addition, $z^{B^*}{}_{,t}$ and $z^{A^*}{}_{,tt}$ are calculated, respectively, as

$$\begin{aligned} z^{B^*}{}_{,t} &= -{}^{F}\Omega^{B^*}{}_{C^*}\, Q^{C^*}_I (z^I - Z^I) - Q^{B^*}_I \dot{Z}^I \\ &= -{}^{F}\Omega^{B^*}{}_{C^*}\, z^{C^*} - Q^{B^*}_I \dot{Z}^I, \end{aligned}$$

$$\begin{aligned} z^{A^*}{}_{,tt} &= -{}^F\dot{\Omega}^{A^*}{}_{B^*}\, Q_I^{B^*}(z^I - Z^I) \\ &\quad + {}^F\Omega^{A^*}{}_{B^*}\, {}^F\Omega^{B^*}{}_{C^*}\, Q_I^{C^*}(z^I - Z^I) \\ &\quad + 2\,{}^F\Omega^{A^*}{}_{B^*}\, Q_I^{B^*}\, \dot{Z}^I - Q_I^{A^*}\, \ddot{Z}^I \\ &= -{}^F\dot{\Omega}^{A^*}{}_{B^*}\, z^{B^*} + {}^F\Omega^{A^*}{}_{B^*}\, {}^F\Omega^{B^*}{}_{C^*}\, z^{C^*} \\ &\quad + 2\,{}^F\Omega^{A^*}{}_{B^*}\, Q_I^{B^*}\, \dot{Z}^I - Q_I^{A^*}\, \ddot{Z}^I. \end{aligned}$$

Thus (D1) is reduced to

$$\begin{aligned} \alpha^{A^*} &= -2\,{}^F\Omega^{A^*}{}_{B^*}(v^{B^*} + {}^F\Omega^{B^*}{}_{C^*}\, z^{C^*} + Q_I^{B^*}\, \dot{Z}^I) \\ &\quad - {}^F\dot{\Omega}^{A^*}{}_{B^*}\, z^{B^*} + {}^F\Omega^{A^*}{}_{B^*}\, {}^F\Omega^{B^*}{}_{C^*}\, z^{C^*} \\ &\quad + 2\,{}^F\dot{\Omega}^{A^*}{}_{B^*}\, Q^{B^*}{}_I\, \dot{Z}^I - Q_I^{A^*}\, \ddot{Z}^I \\ &= -Q_I^{A^*}\, \ddot{Z}^I - 2\,{}^F\Omega^{A^*}{}_{B^*}\, v^{B^*} \\ &\quad - {}^F\Omega^{A^*}{}_{B^*}\, {}^F\Omega^{B^*}{}_{C^*}\, z^{C^*} - {}^F\dot{\Omega}^{A^*}{}_{B^*}\, z^{B^*}, \end{aligned} \tag{D2}$$

which provides the proof for (4.6).

APPENDIX E: TRANSFORMATION RULE OF Σ

The transformation rule of $\mathbf{\Sigma}$ can be derived by the following steps. In an arbitrary coordinate system $\{\mathbf{x}\}$, its definition is given by (4.12). Thus, in another coordinate system $\{\tilde{\mathbf{x}}\}$, $\mathbf{\Sigma}$ is given by

$$\Sigma^{\tilde{a}}{}_{\tilde{b}} = x^{\tilde{a}}{}_{,I}\, z^J{}_{,\tilde{b}}\, V^I{}_{;J}.$$

Thus we easily reach the following transformation rule:

$$\Sigma^{\tilde{a}}{}_{\tilde{b}} = x^{\tilde{a}}{}_{,i}\, x^j{}_{,\tilde{b}}\, x^i{}_{,I}\, z^J{}_{,j} V^I{}_{;J} = x^{\tilde{a}}{}_{,i}\, x^j{}_{,\tilde{b}}\, \Sigma^i{}_j,$$

which is consistent with the general tensor rule (2.4). Covariant and contravariant components of $\mathbf{\Sigma}$ are obtained, respectively, as follows:

$$\begin{aligned} \Sigma_{ij} &\equiv g_{ik}\Sigma^k{}_j = g_{ik}\, x^k{}_{,I}\, z^J{}_{,j}\, V^I{}_{;J} \\ &= g_{IK}\, z^K{}_{,i}\, z^J{}_{,j}\, V^I{}_{;J} = z^K{}_{,i}\, z^J{}_{,j}\, V_{K;J}, \\ \Sigma^{ij} &\equiv g^{jk}\, \Sigma^i{}_k = g^{jk}\, x^i{}_{,I}\, z^J{}_{,k}\, V^I{}_{;J} \\ &= x^i{}_{,I}\, x^j{}_{,A}\, g^{JA}\, V^I{}_{;A} = x^i{}_{,I}\, x^j{}_{,A}\, V^{I;A}, \end{aligned}$$

where we used identities $g_{ik}\, x^k{}_{,I} = g_{IK}\, z^K{}_{,i}$ and $g^{jk}\, z^J{}_{,k} = x^j{}_{,A}\, g^{JA}$. It is noticeable that we have to identify the inertial frame of reference for the first place. Since the law of fluids treated in the present work is based on nonrelativistic framework, the inertial frame has a special meaning compared to any other frame of references. In this context, $\mathbf{\Sigma}$ is not only a velocity gradient in the inertial frame $\nabla\mathbf{V}$, but also an objective measure of how much the mean flow deviates from the inertial motion.

APPENDIX F: GENERALIZATION OF THE ABSOLUTE VORTICITY

In a general coordinate system $\{\mathbf{x}\}$, the explicit form of $\mathbf{\Theta}$ defined by (4.15) is written as follows:

$$\begin{aligned} \Theta_{ab} &= z^I{}_{,a}\, z^J{}_{,b}(V_{I;J} - V_{J;I}) \\ &= V_{a;b} - V_{b;a} - (g_{ac} z^J{}_{,b} - g_{bc} z^J{}_{,a}) x^c{}_{,Jt} \\ &\quad + (z^I{}_{,a} z^J{}_{,b} - z^I{}_{,b} z^J{}_{,a}) \Gamma_{I.JK} z^K{}_{,t} \\ &\quad + (g_{ac} z^J{}_{,b} - g_{bc} z^J{}_{,a}) x^c{}_{,IJ}\, z^I{}_{,t}. \end{aligned} \tag{F1}$$

Although this looks a bit complex, the expression in the Euclidean coordinate system is reduced to a simpler form since the second-order derivatives in the third and fourth terms vanish:

$$\begin{aligned} \Theta_{A^*B^*} &= V_{A^*,B^*} - V_{B^*,A^*} \\ &\quad - (g_{A^*C^*}\, z^J{}_{,B^*} - g_{B^*C^*}\, z^J{}_{,A^*}) z^{C^*}{}_{,Jt}, \end{aligned}$$

where the second term is reduced to

$$\begin{aligned} &-(g_{A^*C^*}\, z^J{}_{,B^*} - g_{B^*C^*}\, z^J{}_{,A^*}) z^{C^*}{}_{,Jt} \\ &\quad = -\big(g_{A^*C^*}\, Q^J_{B^*} - g_{B^*C^*}\, Q^J_{A^*}\big)\dot{Q}^{C^*}_J \\ &\quad = \big(g_{A^*C^*}\, \dot{Q}^J_{B^*} - g_{B^*C^*}\, \dot{Q}^J_{A^*}\big) Q^{C^*}_J \\ &\quad = \big(g_{A^*C^*}\, {}^F\Omega^J{}_K\, Q^K_{B^*} - g_{B^*C^*}\, {}^F\Omega^J{}_K\, Q^K_{A^*}\big) Q^{C^*}_J \\ &\quad = Q^I_{A^*} g_{IJ}\, {}^F\Omega^J{}_K\, Q^K_{B^*} - Q^I_{B^*} g_{IJ}\, {}^F\Omega^J{}_K\, Q^K_{A^*} \\ &\quad = Q^I_{A^*}\, {}^F\Omega_{IK}\, Q^K_{B^*} - Q^I_{B^*}\, {}^F\Omega_{IK}\, Q^K_{A^*} \\ &\quad = Q^I_{A^*}\, Q^K_{B^*}\, {}^F\Omega_{IK} - Q^I_{B^*}\, Q^K_{A^*}\, {}^F\Omega_{IK} \\ &\quad = 2\, {}^F\Omega_{A^*B^*}. \end{aligned}$$

Thus the tensor $\mathbf{\Theta}$, which is defined by (4.15) in general, is given in more familiar form as follows:

$$\Theta_{A^*B^*} = V_{A^*,B^*} - V_{B^*,A^*} + 2\,{}^F\Omega_{A^*B^*}, \tag{F2}$$

which coincides with what is usually called the absolute vorticity, which is often defined as half of the above in conventional notation; for instance, Weis and Hutter employed $\bar{W}_{A^*B^*} = \frac{1}{2}(V_{A^*,B^*} - V_{B^*,A^*}) + {}^F\Omega_{A^*B^*}$ [4]. Under the simple rotational coordinate transformation, we can globally define the frame rotation ${}^F\mathbf{\Omega}$, and thus the absolute vorticity can be defined by (F2). Equation (4.15) is exactly the generalization of the absolute vorticity in the general coordinate system, which has nonuniform rotation and deformation. The second term of (F1) corresponds to the nonuniform frame rotation.

APPENDIX G: GALILEAN TRANSFORMATION OF THE BASIC-INERTIAL FRAME

Replacing $\tilde{\mathbf{x}}$ in (4.2) with $\tilde{\mathbf{z}}$ yields

$$\frac{dv^{\tilde{A}}}{dt} = z^{\tilde{A}}{}_{,I}\frac{dv^I}{dt}. \tag{G1}$$

Replacing $\{\mathbf{z}\}$ in (4.2) with $\{\tilde{\mathbf{z}}\}$ yields

$$\frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\tilde{\mathbf{z}}\}] = x^i{}_{,\tilde{A}}\frac{dv^{\tilde{A}}}{dt}. \tag{G2}$$

Using (G1), (G2), and (4.2) yields

$$\begin{aligned} \frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\tilde{\mathbf{z}}\}] &= x^i{}_{,\tilde{A}}\, z^{\tilde{A}}{}_{,I}\frac{dv^I}{dt} = x^i{}_{,I}\frac{dv^I}{dt} \\ &= \frac{dv^i}{dt} - \alpha^i[\mathbf{v};\{\mathbf{z}\}], \end{aligned}$$

which reduces to

$$\alpha^i[\mathbf{v};\{\mathbf{z}\}] = \alpha^i[\mathbf{v};\{\tilde{\mathbf{z}}\}].$$

Thus the inertial force $\boldsymbol{\alpha}$ is invariant under the Galilean transformation of its basic-inertial frame.

APPENDIX H: SIMPLE EXAMPLE OF A CLOSURE MODEL

Let us see here a simple example of the Reynolds-stress model based on our covariant equation (5.3). The natural generalization of the model proposed by Launder *et al.* [9] can be given by the following modeled terms:

$$\epsilon^{ij} = \frac{2}{3}\epsilon g^{ij},$$

$$\phi^{ij} = -C_R\left(R^{ij} - \frac{2}{3}Kg^{ij}\right) + C_{IP}\left\{\left(S^i_k R^{jk} + S^j_k R^{ik}\right) + \left(\Theta^i{}_k R^{jk} + \Theta^j{}_k R^{ik}\right) - \frac{2}{3}\mathsf{S}\cdot\mathsf{R}g^{ij}\right\},$$

$$t^{ijk} = -C_{TR}\frac{K}{\epsilon}\left(R^{ij;a}R^k_a + R^{ki;a}R^j_a + R^{jk;a}R^i_a\right) - \nu R^{ij;k},$$

which yield a model equation for the Reynolds stress as follows:

$$\begin{aligned}\frac{\mathfrak{D}}{\mathfrak{D}t}R^{ij} =& -(1 - C_{IP})\left(S^i_k R^{jk} + S^j_k R^{ik}\right)\\ &-(1 - C_{IP})\left(\Theta^i{}_k R^{jk} + \Theta^j{}_k R^{ik}\right)\\ &-\frac{2}{3}\epsilon g^{ij} - C_{IP}\frac{2}{3}\mathsf{S}\cdot\mathsf{R}g^{ij} - C_R\frac{\epsilon}{K}\left(R^{ij} - \frac{2}{3}Kg^{ij}\right)\\ &+\left\{C_{TR}\frac{K}{\epsilon}\left(R^{ij;a}R^k_a + R^{ki;a}R^j_a + R^{jk;a}R^i_a\right)\right\}_{;k}\\ &+\nu\triangle R^{ij}, \end{aligned} \tag{H1}$$

where C_R, C_{IP}, and C_{TR} are constants. Note that due to the covariance of both K and ϵ (H1) is consistent with general covariance. In order to construct the total closure model, we have to model Eq. (5.5), namely, the equation for ϵ which is apparently covariant. Covariant modeling of (5.5) is needed for the covariance of the Reynolds stress through (H1).

Let us see an analogy of covariant model (H1) with a visco-elastic model. Substituting $\mathsf{R} = \frac{2}{3}\mathsf{g} - \mathsf{P}$ into (H1) yields

$$\begin{aligned}\frac{2}{3}\frac{\mathfrak{D}K}{\mathfrak{D}t}g^{ij} - \frac{2}{3}KS^{ij} - \frac{\mathfrak{D}}{\mathfrak{D}t}P^{ij} =& (1 - C_{IP})\left(S^i_k P^{jk} + S^j_k P^{ik}\right)\\ &+(1 - C_{IP})\left(\Theta^i{}_k P^{jk} + \Theta^j{}_k P^{ik}\right)\\ &-\frac{2}{3}\epsilon g^{ij} + C_{IP}\frac{2}{3}\mathsf{S}\cdot\mathsf{P}\\ &+C_R\frac{\epsilon}{K}P^{ij} + \text{divergence term}, \end{aligned} \tag{H2}$$

where $\mathsf{P} \equiv \frac{2}{3}K\mathsf{g} - \mathsf{R}$ can be understood as the deviatoric stress. In the derivation of the second term of the left-hand side we used $\mathfrak{D}g^{ij}/\mathfrak{D}t = -S^{ij}$ [8]. By subtracting the terms proportional to g from both sides of (H2) and transforming it, we obtain

$$\begin{aligned}&\left(1 + C_R^{-1}\frac{K}{\epsilon}\frac{\mathfrak{D}}{\mathfrak{D}t}\right)P^{ij} + C_R^{-1}(1 - C_{IP})\frac{K}{\epsilon}\left(S^i_k P^{jk} + S^j_k P^{ik}\right)\\ &\quad + C_R^{-1}(1 - C_{IP})\frac{K}{\epsilon}\left(\Theta^i{}_k P^{jk} + \Theta^j{}_k P^{ik}\right)\\ &= \frac{2}{3}(1 - 2C_{IP})C_R^{-1}\frac{K^2}{\epsilon}S^{ij} + \left(1 - \frac{2}{3}C_{IP}\right)C_R^{-1}\frac{K}{\epsilon}\mathsf{S}\cdot\mathsf{P}g^{ij}\\ &\quad + \text{divergence term}. \end{aligned} \tag{H3}$$

It is interesting to compare (H3) with the model equation for a nonlinear visco-elastic material introduced by Oldroyd, which can be rewritten in our notation as

$$\left(1 + \gamma_0\frac{\mathfrak{d}}{\mathfrak{d}t}\right)p^{ij} - \kappa\left(s^i_k p^{jk} + s^j_k p^{ik}\right) = \eta\left(1 + \zeta_0\frac{\mathfrak{d}}{\mathfrak{d}t}\right)s^{ij} - 2\eta\nu s^i_k s^{jk}, \tag{H4}$$

where p is visco-elastic stress, and γ_0, ζ_0 κ, η, and ν are all material constants [8]. In particular, γ_0 characterizes the relaxation effect caused by the visco-elasticity, and η is the molecular viscosity, whose counterparts appear in (H3) as $C_R^{-1}K/\epsilon$ and $\frac{2}{3}(1 - 2C_{IP})C_R^{-1}K^2/\epsilon$. Equation (H3) may provide us more clear explanation of the visco-elastic characters observed in turbulent flow than the conventional formalism. Not only the forms, but also the general covariance, of both equations we should note as having an important similarity between (H3) and (H4). On the other hand, we also see a remarkable difference between these two; (H3) includes the absolute vorticity while (H4) does not. This clearly shows the breaking of the material-frame indifference in a turbulence constitutive equation.

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