Mixtures of relativistic gases in gravitational fields: Combined Chapman-Enskog and Grad method and the Onsager relations

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In this work we study an r-species mixture of gases within the relativistic kinetic theory point of view. We use the relativistic covariant Boltzmann equation and incorporate the Schwarzschild metric. The method of solution of the Boltzmann equation is a combination of the Chapman-Enskog and Grad representations. The thermodynamic four-fluxes are expressed as functions of the thermodynamic forces so the generalized expressions for the Navier-Stokes, Fick, and Fourier laws are obtained. The constitutive equations for the diffusion and heat four-fluxes of the mixture are functions of thermal and diffusion generalized forces which depend on the acceleration and the gravitational potential gradient. While this dependence is of relativistic nature for the thermal force, this is not the case for the diffusion forces. We show also that the matrix of diffusion coefficients is symmetric, implying that the thermal-diffusion equals the diffusion-thermal effect, proving the Onsager reciprocity relations. The entropy four-flow of the mixture is also expressed in terms of the thermal and diffusion generalized forces, so its dependence on the acceleration and gravitational potential gradient is also determined.

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I. INTRODUCTION

The relativistic kinetic theory of gases is a subject that began in 1911 when Jüttner [1] proposed a relativistic version of the velocity distribution function which corresponds to the Maxwellian distribution function in the nonrelativistic limiting case. Later, several more studies were made, but for brevity's sake we mention Refs. [2,3], where several applications of the relativistic kinetic theory of gases are discussed.

This work represents a continuation of the study of the properties of relativistic gases using the Boltzmann equation in gravitational fields; this subarea has not yet been studied in depth. Here we quote some works [4–8] on this topic which have been recently published.

The method used in this paper to solve the covariant Boltzmann equation is a combination of the Chapman-Enskog and Grad methods [9,10]. It consists essentially in doing an expansion of the distribution function for each species which is the solution of the Boltzmann equation up to first correction. Such a procedure is like in the Chapman-Enskog method. Then we impose that such an expansion must be compatible with the solution of the Boltzmann equation given by the method of Grad [11]. In order to keep the linear regime we truncate the Grad distribution function for each species up to linear terms of the nonequilibrium pressure, pressure deviator tensor, diffusion, and heat four-fluxes. Therefore, we obtain a linearized Boltzmann equation that is written in terms of the local thermodynamic variables and fluxes: diffusion, heat, nonequilibrium pressure, and pressure deviator tensor. The next step is to get from that linearized Boltzmann equation a set of linear algebraic system for the fluxes. We generate one equation for each thermodynamic flux through the multiplication of a dynamical function of the particles by the linearized Boltzmann equation and then the integration over the momentum space. Hence, we find the constitutive equations for the fluxes in terms of gradients of the local thermodynamic variables and of a gravitational potential that arises from the Schwarzschild metric. The laws of Navier-Stokes for the nonequilibrium pressure and pressure deviator tensor are obtained as well as the generalized Fourier and Fick laws for the heat and diffusion four-fluxes.

It will be shown that there appears a generalized diffusion force that has dependence not only on the concentration and pressure gradients but also on a contribution of the four-acceleration and the gravitational potential gradient. The contributions of four-acceleration and potential gradient also appear as combined forces for the heat flux and they were analyzed separately by Eckart [12] and Tolman [13,14]. In the case of Eckart, for a relativistic gas in equilibrium and in the absence of gravitational fields, the temperature gradient is counterbalanced by an acceleration. On the other hand, in the case of Tolman for a relativistic gas in equilibrium and in the absence of an acceleration, the temperature gradient is counterbalanced by a gravitational potential gradient.

In order to show that the Onsager reciprocity relations hold we manipulate the constitutive equations for the heat and diffusion fluxes. The demonstration is general in the sense that the interaction of the particles are supposed to maintain the microscopic reversibility principle.

The structure of this paper is as follows. We define the problem in Sec. II and establish the Boltzmann equation and the definitions for both the thermodynamic variables and fluxes. In Sec. III, we use a method of solution of the Boltzmann equation that is a combination of the Grad and Chapman-Enskog ones, the solution is truncated up to first order so we obtain linear expressions. Such a process will lead us to an algebraic system of equations for the thermodynamic fluxes that, when it is properly solved, expresses the thermodynamic coefficients for an arbitrary intermolecular interaction. In Sec. IV, we show that the Onsager reciprocity relations hold for an arbitrary intermolecular interaction. Furthermore, we show that the laws of Fourier and Fick are expressed in terms of generalized

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thermal and diffusion forces in the presence of gravitational fields. To give a more general representation, we show in Sec. V that the entropy four-flow of the mixture is a function of the generalized thermal and diffusion forces. Section VI is devoted to the calculation of the constitutive equation for a relativistic Newtonian fluid, i.e., the Navier-Stokes law. Ultimately, in Sec. VII, we discuss the obtained results.

II. BACKGROUND

In this section we will define the problem of an r-species nonreacting mixture in a Riemannian space with metric tensor $g^{\mu\nu}$. The particles are supposed not to have internal degrees of freedom. Each of these particles of the constituent $a = 1,\dots,r$ have mass m_a and are characterized by the space-time coordinates $x^\mu = (ct,\mathbf{x})$ and the momentum $p_a^\mu = (p_a^0,\mathbf{p}_a)$. The mass-shell condition, i.e., $g_{\mu\nu}p_a^\mu p_a^\nu = m_a^2c^2$, implies the following relationships for the contravariant and covariant temporal components:

$$p_a^0 = \left(p_{a0} - g_{0i}p_a^i\right)/g_{00},\qquad p_{a0} = \sqrt{g_{00}m_a^2c^2 + (g_{0i}g_{0j} - g_{00}g_{ij})p_a^ip_a^j}, \tag{1}$$

respectively. The analysis is developed within the tenets of the general relativity, and we adopt the Schwarzschild metric $g^{\mu\nu}$ in which the line element reads [15]:

$$ds^2 = \left(1 - \frac{2GM}{c^2R}\right)(dx^0)^2 - \frac{1}{\left(1 - \frac{2GM}{c^2R}\right)}dR^2 - R^2(d\theta^2 + \sin^2\theta d\psi^2), \tag{2}$$

in terms of the spherical coordinates $\{R,\theta,\psi,ct = x^0\}$. Above, M is the total mass of the spherical source and G is the gravitational constant. Here we shall use the isotropic Schwarzschild metric, which reads

$$ds^2 = g_0(r)(dx^0)^2 - g_1(r)\delta_{ij}dx^idx^j, \tag{3}$$

$$g_0(r) = \frac{\left(1 - \frac{GM}{2c^2r}\right)^2}{\left(1 + \frac{GM}{2c^2r}\right)^2},\quad g_1(r) = \left(1 + \frac{GM}{2c^2r}\right)^4. \tag{4}$$

Along the calculation we will use a relativistic parameter $\zeta_a = \frac{m_ac^2}{kT}$, where c is the speed of light, k the Boltzmann constant, and T the local temperature, assumed as an invariant. This parameter is convenient because it tell us how relativistic is the system, for example, $\zeta_a \gg 1$ corresponds to a nonrelativistic limit. On the other hand, $\zeta_a \ll 1$ belongs to an ultrarelativistic limit.

The most fundamental equation in the kinetic theory is the the Boltzmann equation; such an equation can be obtained with two hypotheses as a basis. The first one is that particles collide elastically and only collisions of pairs are taken into account. The second one implies a description of the system with a one-particle distribution function, whereby collisions represent a process that does not depend on what has occurred in the past with the particles. This is also known in the literature as the molecular chaos hypothesis. In our case, the Boltzmann equation reads [3]:

$$p_a^\mu\frac{\partial f_a}{\partial x^\mu} - \Gamma^i_{\mu\nu}p_a^\mu p_a^\nu\frac{\partial f_a}{\partial p_a^i} = \sum_{b=1}^{r}\int(f_a'f_b' - f_af_b)F_{ba}\sigma_{ab}\,d\Omega\sqrt{-g}\frac{d^3p_b}{p_{b0}}, \tag{5}$$

for the a species. Here the Latin subindex denotes the species; note that we have one equation with the same structure of (5) for each component of the mixture $a = 1,\dots,r$. The distribution function $f_a(x^\mu,p_a^\mu)$ has a statistical meaning; indeed, the quantity $f_a(x^\mu,p_a^\mu)d^3x\,d^3p_a$ at time t is the number of particles of the constituent a in the volume element between $\mathbf{x}$, $\mathbf{x} + d^3x$ and $\mathbf{p}_a$, $\mathbf{p}_a + d^3p_a$. In Eq. (5) the Christoffel symbols $\Gamma^i_{\mu\nu}$ and the invariant flux $F_{ba} = \sqrt{(p_a^\mu p_{b\mu})^2 - m_a^2m_b^2c^4}$ also appear, which plays the role of the relative velocity of the nonrelativistic Boltzmann equation. We have also the invariant differential elastic cross section $\sigma_{ab}d\Omega$ for collisions of species a and b, where $d\Omega$ is the corresponding solid angle element. Integrals are made with the invariant differential element $\sqrt{-g}\frac{d^3p_b}{p_{b0}}$, where $\sqrt{-g} = \det[g^{\mu\nu}]$. In Eq. (5), quantities denoted with a prime are evaluated with the momentum of the particles after a binary collision occurs, i.e., $f_a' \equiv f(\mathbf{x},\mathbf{p}_a',t)$ and so on. The binary collision is characterized by the energy-momentum conservation law $p_a^\mu + p_b^\mu = p_a'^\mu + p_b'^\mu$.

Without solving Boltzmann's equation we can obtain two important results. The first arises from the H-theorem and the definition of the thermodynamic variables. A situation of local equilibrium means that the entropy four-flow production [see Eq. (29)] vanishes at equilibrium. The solution of the collisional term of the Boltzmann equation—when it is equal to zero—is the well-known local equilibrium distribution function, which reads

$$f_a^{(0)} = \frac{\mathsf{n}_a}{4\pi kTm_a^2cK_2(\zeta_a)}\exp\left(-\frac{U_\mu p_a^\mu}{kT}\right). \tag{6}$$

Here n_a is the local number of particles of species a, and the modified Bessel function of second kind is represented by

$$K_n(\zeta) = \left(\frac{\zeta}{2}\right)^n\frac{\Gamma(1/2)}{\Gamma(n+1/2)}\int_1^\infty e^{-\zeta y}(y^2-1)^{n-1/2}dy, \tag{7}$$

and U_μ—with $U^\mu U_\mu = c^2$—is the hydrodynamical four-velocity. The set of local hydrodynamic variables that describes the local equilibrium is $\{\mathsf{n}_1,\dots,\mathsf{n}_r,U_\mu,T\}$. For the calculations that will be performed, it is convenient to evaluate Eq. (6) in a comoving frame, that is, $U^\mu = (c/\sqrt{g_0},\mathbf{0})$, yielding

$$f_a^{(0)} = \frac{\mathsf{n}_a}{4\pi kTm_a^2cK_2(\zeta_a)}\exp\left(-\frac{c\sqrt{m_a^2c^2 + g_1|\mathbf{p}_a|^2}}{kT}\right). \tag{8}$$

The second important result that arises from the Boltzmann equation is the obtention of the balance equations, and for this purpose we proceed as follows. We multiply the Boltzmann equation (5) by the collisional invariants, that is, microscopic dynamical quantities that are conserved between collisions, i.e., $\Psi_a + \Psi_b = \Psi_a' + \Psi_b'$, and integrate the resulting equation over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$. The collisional invariants Ψ_a can take the value of the mass and the energy momentum of the colliding particles. To obtain the particle four-flow balance equation for the a species we take $\Psi_a = c$ and integrate; this process leads to the conservation law

$$N^\mu_{a;\mu} = 0. \tag{9}$$

Here the semicolon denotes a covariant derivative and we have defined

$$N^\mu_a = c\int p^\mu_a f_a \sqrt{-g}\frac{d^3 p_a}{p_{a0}} \tag{10}$$

as the particle four-flow of species a. We now introduce a general decomposition of N^μ_a in terms of the hydrodynamic four-velocity as

$$N^\mu_a = \mathsf{n}_a U^\mu + \mathsf{J}^\mu_a, \quad \text{where} \quad \mathsf{n}_a = \frac{N^\mu_a U_\mu}{c^2} \tag{11}$$

denotes the partial particle number density. The quantity J^μ_a is a spacelike vector defined as

$$\mathsf{J}^\mu_a = \Delta^\mu_\nu c\int p^\nu_a f_a \frac{d^3 p_a}{p_{a0}} \tag{12}$$

and holds the property $\mathsf{J}^\mu_a U_\mu = 0$. Above, we have introduced the projector

$$\Delta^{\mu\nu} = g^{\mu\nu} - \frac{1}{c^2}U^\mu U^\nu, \tag{13}$$

which has the property $\Delta^{\mu\nu}U_\nu = 0$. Equation (12) is the corresponding diffusion four-flux of species a of the mixture and by taking the sum of (11) over all the components we easily note that

$$N^\mu = \sum_{a=1}^r N^\mu_a = \mathsf{n}U^\mu, \quad \mathsf{n} = \sum_{a=1}^r \mathsf{n}_a, \quad \sum_{a=1}^r \mathsf{J}^\mu_a = 0, \tag{14}$$

where the last equation implies that there exist only $(r-1)$ partial diffusion fluxes that are linearly independent for a mixture of r constituents.

On the other hand, to obtain the balance equation for the energy momentum of the a species defined by

$$T^{\mu\nu}_a = c\int p^\mu_a p^\nu_a f_a \sqrt{-g}\frac{d^3 p_a}{p_{a0}}, \tag{15}$$

we multiply the Boltzmann equation (5) by the collisional invariant $\Psi_a = cp^\mu_a$ and integrate the resulting equation over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$. This process yields

$$T^{\mu\nu}_{a;\nu} = P^\mu_a, \tag{16}$$

where the production term P^μ_a is given by

$$P^\mu_a = \sum_{b=1}^r c\int \left(p'^\mu_a - p^\mu_a\right) f_a f_b F_{ba}\sigma_{ab} d\Omega \times \sqrt{-g}\frac{d^3 p_b}{p_{b0}}\sqrt{-g}\frac{d^3 p_a}{p_{a0}}. \tag{17}$$

Note that this equation does not represent a conservation law, but if we sum Eq. (16) over all species we obtain

$$T^{\mu\nu}_{;\nu} = \sum_{a=1}^r P^\mu_a = 0, \tag{18}$$

which represents a conservation equation for the energy-momentum tensor of the mixture $T^{\mu\nu} = \sum_{a=1}^r T^{\mu\nu}_a$. By following the decomposition of Eckart (see, e.g., Refs. [16–18]), the energy-momentum tensor of the a species can be written as

$$T^{\mu\nu}_a = \frac{\mathsf{n}_a \mathsf{e}_a}{c^2}U^\mu U^\nu + \frac{1}{c^2}U^\mu\left(\mathsf{q}^\nu_a + \mathsf{h}_a \mathsf{J}^\nu_a\right) + \frac{1}{c^2}U^\nu\left(\mathsf{q}^\mu_a + \mathsf{h}_a\mathsf{J}^\mu_a\right) - (\mathsf{p}_a + \varpi_a)\Delta^{\mu\nu} + \mathsf{p}^{\langle\mu\nu\rangle}_a, \tag{19}$$

where several definitions are to be made. First we can list the local equilibrium quantities: energy per particle e_a, hydrostatic pressure p_a, and the enthalpy per particle $\mathsf{h}_a = \mathsf{e}_a + \mathsf{p}_a/\mathsf{n}_a$. Next, the nonequilibrium quantities are dynamical pressure ϖ_a, heat four-flux q^μ_a, and pressure deviator tensor $\mathsf{p}^{\langle\mu\nu\rangle}_a$. They are given in terms of the following projections of the energy-momentum tensor of the a species:

$$\mathsf{q}^\mu_a + \mathsf{h}_a\mathsf{J}^\mu_a = \Delta^\mu_\sigma T^{\sigma\nu}_a U_\nu, \quad \mathsf{e}_a = \frac{1}{\mathsf{n}_a c^2}U_\mu T^{\mu\nu}_a U_\nu, \tag{20}$$

$$\mathsf{p}^{\langle\mu\nu\rangle}_a = \left(\Delta^\mu_\sigma\Delta^\nu_\tau - \tfrac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right)T^{\sigma\tau}_a, \tag{21}$$

$$\mathsf{p}_a + \varpi_a = -\tfrac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}_a. \tag{22}$$

The corresponding quantities for the mixture are

$$\mathsf{e} = \sum_{a=1}^r \frac{\mathsf{n}_a}{\mathsf{n}}\mathsf{e}_a, \quad \mathsf{p} = \sum_{a=1}^r \mathsf{p}_a, \quad \varpi = \sum_{a=1}^r \varpi_a, \tag{23}$$

$$\mathsf{h} = \sum_{a=1}^r \frac{\mathsf{n}_a}{\mathsf{n}}\mathsf{h}_a, \quad \mathsf{p}^{\langle\mu\nu\rangle} = \sum_{a=1}^r \mathsf{p}^{\langle\mu\nu\rangle}_a, \tag{24}$$

$$\mathsf{q}^\mu = \sum_{a=1}^r \left(\mathsf{q}^\mu_a + \mathsf{h}_a\mathsf{J}^\mu_a\right), \tag{25}$$

so the energy-momentum tensor of the mixture is written as

$$T^{\mu\nu} = \frac{\mathsf{n}\mathsf{e}}{c^2}U^\mu U^\nu + \frac{1}{c^2}(U^\mu\mathsf{q}^\nu + U^\nu\mathsf{q}^\mu) - (\mathsf{p}+\varpi)\Delta^{\mu\nu} + \mathsf{p}^{\langle\mu\nu\rangle}. \tag{26}$$

Note that the heat four-flux q^μ Eq. (25) of the mixture has two contributions; this is in accordance with the linear irreversible thermodynamics [19], where one term is related with the partial heat flux and another with the transport of energy driven by diffusion.

Another quantity which is important in the analysis of mixtures of relativistic gases is the entropy four-flow of the mixture S^μ, defined by

$$S^\mu = -kc\sum_{a=1}^r \int p^\mu_a f_a \ln(\mathsf{b}_a f_a)\sqrt{-g}\frac{d^3 p_a}{p_{a0}}, \tag{27}$$

where b_a is a constant which has inverse units of f_a. Its balance equation is obtained through the multiplication of the Boltzmann equation (5) by $-kc\ln(\mathsf{b}_a f_a)$, the subsequent integration over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$, and the sum over all species, yielding

$$S^{\mu}_{;\mu} = \sigma \geqslant 0, \tag{28}$$

$$\sigma = \frac{ck}{4}\sum_{a=1}^{r}\sum_{b=1}^{r}\int f_a f_b \ln\frac{f_a' f_b'}{f_a f_b}\left(\frac{f_a' f_b'}{f_a f_b} - 1\right) \times F_{ba}\sigma_{ab} d\Omega \sqrt{-g}\frac{d^3 p_b}{p_{b0}}\sqrt{-g}\frac{d^3 p_a}{p_{a0}}. \tag{29}$$

The quantity σ is the entropy four-flow production of the mixture, which is always positive semidefinite, thanks to the relationship $(x-1)\ln x \geqslant 0$ valid $\forall x > 0$. The entropy four-flow of the mixture is decomposed according to:

$$S^{\mu} = \mathsf{n}\mathsf{s}U^{\mu} + \Phi^{\mu}, \quad \mathsf{s} = \frac{1}{c^2\mathsf{n}}S^{\mu}U_{\mu}, \quad \Phi^{\mu} = \Delta^{\mu}_{\nu}S^{\nu}, \tag{30}$$

where the quantity s is identified as the entropy per particle of the mixture and Φ^{μ} its entropy flux. The entropy per particle of species a is given by

$$\mathsf{s}_a = -\frac{kU_{\mu}}{c\mathsf{n}_a}\int p^{\mu}_a f_a \ln(\mathsf{b}_a f_a)\sqrt{-g}\frac{d^3 p_a}{p_{a0}}, \tag{31}$$

so we have $\mathsf{n}\mathsf{s} = \sum_{a=1}^{r}\mathsf{n}_a\mathsf{s}_a$.

In the kinetic theory of relativistic gases there exist two decompositions that are often used: the Eckart and the Landau-Lifshitz (see, e.g., Refs. [2,3]). The difference between the decompositions is that the heat flux appears in the particle four-flow but not in the energy-momentum tensor in the Landau-Lifshitz decomposition, contrary to the Eckart one. One can take both decompositions for the determination of the constitutive equations and the results are the same. However, there are situations where one should apply only one of the decompositions, which is in the case of using BGK models of the Boltzmann collision operator. The model equations of the Boltzmann equation normally considered in the relativistic kinetic theory are due to Marle and Anderson and Witting (see, e.g., Refs. [2,3]). For the Marle model one should take the Eckart decomposition, while for the Anderson and Witting model the Landau-Lifshitz decomposition should be used.

The main problem in the kinetic theory is to find a solution of the Boltzmann equation (5), because, as we have seen, all the above definitions can be evaluated by integrating functions that involve $f_a(x^{\mu}, p^{\mu}_a)$. The equilibrium quantities can be evaluated with the local equilibrium distribution function (6) and read:

$$\mathsf{e}_a = m_a c^2\left(G_a - \frac{1}{\zeta_a}\right), \tag{32}$$

$$\mathsf{p}_a = \mathsf{n}_a kT, \quad \mathsf{h}_a = m_a c^2 G_a, \tag{33}$$

$$\mathsf{s}_a = k\left\{\ln\left[\frac{4\pi m_a^2 ckT K_2(\zeta_a)}{\mathsf{n}_a\mathsf{b}_a}\right] + \zeta_a G_a - 1\right\}. \tag{34}$$

The chemical potential of species a is introduced through the Gibbs function per particle, namely $\mu_a = \mathsf{e}_a - T\mathsf{s}_a + \mathsf{p}_a/\mathsf{n}_a$, and by taking into account the above expressions we get

$$\mu_a = kT\ln\frac{e\mathsf{n}_a\mathsf{b}_a}{4\pi m_a^2 ckT K_2(\zeta_a)}. \tag{35}$$

In next sections, we will use a method that allows us to obtain expressions for the diffusion fluxes J^{μ}_a, heat flux q^{μ}, nonequilibrium pressure ϖ, pressure deviator tensor $\mathsf{p}^{\langle\mu\nu\rangle}$, and entropy flux Φ^{μ}. Furthermore, we will show the dependence of J^{μ}_a and q^{μ} in terms of the gravitational potential and demonstrate the validity of the Onsager reciprocity relations.

III. COMBINED CHAPMAN-ENSKOG AND GRAD METHOD

In this section we will use a method to extract thermodynamic information from the Boltzmann equation [9,10] that combines the features of the Chapman-Enskog [20] and Grad's moments methods [11]. This method has mainly two advantages. The first is that we do not need a solution of the integrodifferential Boltzmann equation as in the Chapman-Enskog method. The second is that we do not need the field equations for the moments as in the Grad method.

First, we describe how the moment Grad method is constructed. The central idea is to expand $f_a(x^{\mu}, p^{\mu}_a)$ around the local equilibrium distribution function in a series of an orthonormal set. In this case, we have $13r+1$ unknown variables (fields) that are described with the quantities $\{\mathsf{n}_a, U^{\mu}, \mathsf{J}^{\mu}_a, T, \varpi_a, \mathsf{q}^{\mu}_a, \mathsf{p}^{\langle\mu\nu\rangle}_a\}$ (see Ref. [11]). Such an expansion reads

$$f_a = f_a^{(0)}\left[1 + \mathcal{A}^{\mu}_a p_{a\mu} + \mathcal{A}^{\mu\nu}_a p_{a\mu}p_{a\nu}\right], \tag{36}$$

where $f_a^{(0)}$ is the local equilibrium distribution function (Jüttner distribution) described by Eq. (6). In Eq. (36) the unknown tensorial coefficients $\{\mathcal{A}^{\mu}_a, \mathcal{A}^{\mu\nu}_a\}$ are calculated by solving an algebraic system constructed with the help of the definitions of the particle four-flow N^{μ}_a and the energy-momentum tensor $T^{\mu\nu}_a$. The details of such a calculation are long and it is not necessary to do them here; they can be reviewed in Refs. [3,21]. As a result of such developments the distribution function f_a will depend on linear terms of the thermodynamic fluxes, namely

$$f_a = f_a^{(0)}\left\{1 - \frac{\mathsf{J}_{a\mu}}{\mathsf{p}_a}p^{\mu}_a + \frac{\mathsf{q}_{a\mu}}{T\mathsf{p}_a}\frac{p^{\mu}_a}{\mathsf{c}^a_p}\left[\zeta_a G_a - \frac{U_{\nu}p^{\nu}_a}{kT}\right] + \frac{\mathsf{p}_{a\langle\mu\nu\rangle}}{2\mathsf{p}_a}\frac{\zeta_a}{m_a\mathsf{h}_a}p^{\mu}_a p^{\nu}_a + \frac{\varpi_a}{\mathsf{p}_a}\frac{\partial\ln\zeta_a}{\partial\ln\mathsf{c}^a_v}\left[\frac{U_{\mu}U_{\nu}p^{\mu}_a p^{\nu}_a}{k^2T^2} - \frac{3(\mathsf{c}^a_p + \mathsf{h}_a/T)}{\mathsf{c}^a_v}\frac{U_{\mu}p^{\mu}_a}{kT} - \frac{\mathsf{c}^a_v\zeta_a^2 + 3(\mathsf{c}^a_p - \mathsf{h}_a^2/kT^2)}{\mathsf{c}^a_v}\right]\right\}. \tag{37}$$

Here we have introduced the abbreviation $G_a = K_3(\zeta_a)/K_2(\zeta_a)$ and the partial specific heats per particle $\mathsf{c}^a_v = k(\zeta_a^2 + 5G_a\zeta_a - G_a^2\zeta_a^2 - 1)$ and $\mathsf{c}^a_p = \mathsf{c}^a_v + k$ at constant volume and pressure, respectively. Then, following the combined Chapman-Enskog–Grad method [9], the expansion (37) must be compatible with the truncated Chapman-Enskog series up to first order, that is, $f_a = f_a^{(0)}(1+\phi_a)$, where ϕ_a is the first correction to the distribution function $f_a^{(0)}$.

Now we can proceed to linearize the Boltzmann equation as follows. We substitute $f_a = f_a^{(0)}(1+\phi_a)$ into the left-hand side of the Boltzmann equation (5) and keep the linear terms. This process is technically the same as that developed in the Chapman-Enskog method. We also use the so-called functional hypothesis, namely $f_a = f_a(x^\mu, p_a^\mu|\mathsf{n}_a, U^\mu, T)$, leading to

$$p_a^\mu \frac{\partial f_a^{(0)}}{\partial x^\mu} - \Gamma^i_{\mu\nu} p_a^\mu p_a^\mu \frac{\partial f_a^{(0)}}{\partial p^\mu} = f_a^{(0)}\left\{\frac{p_a^\mu}{\mathsf{n}_a}\frac{\partial \mathsf{n}_a}{\partial x^\mu} + \frac{p_a^\mu}{T}\left[1-\zeta_a G_a + \frac{p_a^\lambda U_\lambda}{kT}\right]\frac{\partial T}{\partial x^\mu} - \frac{1}{kT}p_a^\mu p_a^i \frac{\partial U_i}{\partial x^\mu} - \frac{c^2}{2kT}\frac{p_a^k p_a^i p_a^j \delta_{ij}\delta_{kl}}{U^\tau p_{a\tau}}\frac{dg_1}{dr}\frac{x^l}{r} + \frac{c^2}{kT}g_1\Gamma^i_{\mu\nu}\frac{p_a^\mu p_a^\nu p_a^j \delta_{ij}}{U^\tau p_{a\tau}}\right\}. \tag{38}$$

On the other hand, we substitute the Grad function Eq. (37) in the collisional term (right-hand side) of the Boltzmann equation (5) and keep only the linear terms. This process yields

$$\begin{aligned}
&\sum_{b=1}^r \int (f_a' f_b' - f_a f_b) F_{ba}\sigma_{ab}\, d\Omega \sqrt{-g}\frac{d^3 p_b}{p_{b0}} \\
&= -\sum_{b=1}^r \Bigg\{\mathcal{I}_{ab}[p_b^\mu]\frac{\mathsf{J}_{b\mu}}{\mathsf{p}_b} + \mathcal{I}_{ab}[p_a^\mu]\frac{\mathsf{J}_{a\mu}}{\mathsf{p}_a} - \mathcal{I}_{ab}\left[\frac{p_b^\mu}{\mathsf{c}_p^b}\left(\zeta_b G_b - \frac{U_\nu p_b^\nu}{kT}\right)\right]\frac{\mathsf{q}_{b\mu}}{T\mathsf{p}_b} - \mathcal{I}_{ab}\left[\frac{p_a^\mu}{\mathsf{c}_p^a}\left(\zeta_a G_a - \frac{U_\nu p_a^\nu}{kT}\right)\right]\frac{\mathsf{q}_{a\mu}}{T\mathsf{p}_a} \\
&\quad - \mathcal{I}_{ab}\left[\frac{\zeta_b}{m_b \mathsf{h}_b}p_b^\mu p_b^\nu\right]\frac{\mathsf{p}_{b\langle\mu\nu\rangle}}{2\mathsf{p}_b} - \mathcal{I}_{ab}\left[\frac{\zeta_a}{m_a \mathsf{h}_a}p_a^\mu p_a^\nu\right]\frac{\mathsf{p}_{a\langle\mu\nu\rangle}}{2\mathsf{p}_a} - \mathcal{I}_{ab}\left[\frac{\partial \ln \zeta_b}{\partial \ln \mathsf{c}_v^b}\left(\frac{U_\mu U_\nu p_b^\mu p_b^\nu}{k^2T^2} - \frac{3(\mathsf{c}_p^b + \mathsf{h}_b/T)}{\mathsf{c}_v^b}\frac{U_\mu p_b^\mu}{kT}\right)\right]\frac{\varpi_b}{\mathsf{p}_b} \\
&\quad - \mathcal{I}_{ab}\left[\frac{\partial \ln \zeta_a}{\partial \ln \mathsf{c}_v^a}\left(\frac{U_\mu U_\nu p_a^\mu p_a^\nu}{k^2T^2} - \frac{3(\mathsf{c}_p^a + \mathsf{h}_a/T)}{\mathsf{c}_v^a}\frac{U_\mu p_a^\mu}{kT}\right)\right]\frac{\varpi_a}{\mathsf{p}_a}\Bigg\}.
\end{aligned} \tag{39}$$

Here we have introduced the collision operators

$$\mathcal{I}_{ab}[\varphi_a] = \int f_a^{(0)} f_b^{(0)} (\varphi_a' - \varphi_a) F_{ab}\sigma_{ab} d\Omega \sqrt{-g}\frac{d^3 p_b}{p_{b0}} \tag{40}$$

for any function that depends on the momentum four-vector $\varphi_a(p_a^\mu)$. Note that Eq. (40) implies that we can write for an arbitrary function $\psi_b(p_b^\mu)$

$$\int \psi_b \mathcal{I}_{ab}[\varphi_a]\sqrt{-g}\frac{d^3 p_a}{p_{a0}} = \int \varphi_a \mathcal{I}_{ab}[\psi_b]\sqrt{-g}\frac{d^3 p_a}{p_{a0}}, \tag{41}$$

thanks to the symmetry properties of the collision operator.

By collecting the above information the linearized Boltzmann equation in the combined Chapman-Enskog–Grad method becomes

$$\begin{aligned}
&f_a^{(0)}\left\{\frac{p_a^\mu}{\mathsf{n}_a}\frac{\partial \mathsf{n}_a}{\partial x^\mu} + \frac{p_a^\mu}{T}\left[1-\zeta_a G_a + \frac{p_a^\lambda U_\lambda}{kT}\right]\frac{\partial T}{\partial x^\mu} - \frac{1}{kT}p_a^\mu p_a^i \frac{\partial U_i}{\partial x^\mu} - \frac{c^2}{2kT}\frac{p_a^k p_a^i p_a^j \delta_{ij}\delta_{kl}}{U^\tau p_{a\tau}}\frac{dg_1}{dr}\frac{x^l}{r} + \frac{c^2}{kT}g_1\Gamma^i_{\mu\nu}\frac{p_a^\mu p_a^\nu p_a^j \delta_{ij}}{U^\tau p_{a\tau}}\right\} \\
&= -\sum_{b=1}^r \Bigg\{\mathcal{I}_{ab}[p_b^\mu]\frac{\mathsf{J}_{b\mu}}{\mathsf{p}_b} + \mathcal{I}_{ab}[p_a^\mu]\frac{\mathsf{J}_{a\mu}}{\mathsf{p}_a} - \mathcal{I}_{ab}\left[\frac{p_b^\mu}{\mathsf{c}_p^b}\left(\zeta_b G_b - \frac{U_\nu p_b^\nu}{kT}\right)\right]\frac{\mathsf{q}_{b\mu}}{T\mathsf{p}_b} - \mathcal{I}_{ab}\left[\frac{p_a^\mu}{\mathsf{c}_p^a}\left(\zeta_a G_a - \frac{U_\nu p_a^\nu}{kT}\right)\right]\frac{\mathsf{q}_{a\mu}}{T\mathsf{p}_a} \\
&\quad - \mathcal{I}_{ab}\left[\frac{\zeta_b}{m_b \mathsf{h}_b}p_b^\mu p_b^\nu\right]\frac{\mathsf{p}_{b\langle\mu\nu\rangle}}{2\mathsf{p}_b} - \mathcal{I}_{ab}\left[\frac{\zeta_a}{m_a \mathsf{h}_a}p_a^\mu p_a^\nu\right]\frac{\mathsf{p}_{a\langle\mu\nu\rangle}}{2\mathsf{p}_a} - \mathcal{I}_{ab}\left[\frac{\partial \ln \zeta_b}{\partial \ln \mathsf{c}_v^b}\left(\frac{U_\mu U_\nu p_b^\mu p_b^\nu}{k^2T^2} - \frac{3(\mathsf{c}_p^b + \mathsf{h}_b/T)}{\mathsf{c}_v^b}\frac{U_\mu p_b^\mu}{kT}\right)\right]\frac{\varpi_b}{\mathsf{p}_b} \\
&\quad - \mathcal{I}_{ab}\left[\frac{\partial \ln \zeta_a}{\partial \ln \mathsf{c}_v^a}\left(\frac{U_\mu U_\nu p_a^\mu p_a^\nu}{k^2T^2} - \frac{3(\mathsf{c}_p^a + \mathsf{h}_a/T)}{\mathsf{c}_v^a}\frac{U_\mu p_a^\mu}{kT}\right)\right]\frac{\varpi_a}{\mathsf{p}_a}\Bigg\},
\end{aligned} \tag{42}$$

due to (38) and (39).

In the next sections we will use (42) in order to determine the constitutive equations for the diffusion fluxes J_a^μ, heat flux q^μ, nonequilibrium pressure ϖ, and pressure deviator tensor $\mathsf{p}^{\langle\mu\nu\rangle}$.

IV. FICK AND FOURIER LAWS

Now we will obtain a system of linear equations for the determination of the the diffusion fluxes J_a^μ and the heat flux of the mixture q^μ. The solution of such a system will represent the form of the linear fluxes in terms of the thermodynamic forces. The integral functions for the transport coefficients and therefore the Onsager reciprocity relations will be analyzed in the next subsection.

To obtain the first one of the looked set of equations, we multiply Eq. (42) by $c\Delta^\mu_\nu p_a^\nu/\mathsf{n}_a$ and integrate over $\sqrt{-g}\frac{d^3 p_a}{p_{a0}}$.

The integrals used for this process can be consulted in the appendix. The resulting equation is

$$-\frac{1}{\mathsf{n}_a}\nabla^\mu \mathsf{p}_a + \frac{\mathsf{h}_a}{c^2}\Delta^{\mu i}\left[U^\nu\frac{\partial U_i}{\partial x^\nu} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial\Phi}{\partial x^i}\right] = \sum_{b=1}^{r}\left(\mathcal{A}_{ab}\mathsf{J}_b^\mu - \mathcal{F}_{ab}\mathsf{q}_b^\mu\right), \qquad (43)$$

where $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ is the gradient operator and $\Phi = -\frac{GM}{r}$ is the gravitational potential. In Eq. (43) we have introduced the matrices $\mathcal{A}_{ab}$ and $\mathcal{F}_{ab}$. We can split $\mathcal{A}_{ab}$ for different indices $\{a,b\}$,

$$\mathcal{A}_{ab} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a\mathsf{n}_b kT}\int p_{a\mu}\mathcal{I}_{ab}[p_{b\nu}]\sqrt{-g}\frac{d^3p_a}{p_{a0}}, \quad a\neq b, \qquad (44)$$

and for equal indices $\{a,b=a\}$,

$$\mathcal{A}_{aa} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a^2 kT}\left[\sum_{b=1}^{r}\int p_{a\mu}\mathcal{I}_{ab}[p_{a\nu}] + \int p_{a\mu}\mathcal{I}_{aa}[p_{a\nu}]\right]\sqrt{-g}\frac{d^3p_a}{p_{a0}}. \qquad (45)$$

The matrix $\mathcal{F}_{ab}$ introduced in Eq. (43) is written by doing the same splitting, and for unlike indices $\{a,b\}$ we have

$$\mathcal{F}_{ab} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a\mathsf{n}_b kT^2}\int p_{a\mu}\mathcal{I}_{ab}\left[\frac{\zeta_b}{\mathsf{c}_p^b}\left(G_b - \frac{U_\tau p_b^\tau}{m_b c^2}\right)p_{b\nu}\right] \times\sqrt{-g}\frac{d^3p_a}{p_{a0}}, \quad a\neq b, \qquad (46)$$

and for like indices $\{a,b=a\}$,

$$\mathcal{F}_{aa} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a^2 kT^2}\left\{\sum_{b=1}^{r}\int p_{a\mu}\mathcal{I}_{ab}\left[\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\tau p_a^\tau}{m_a c^2}\right)p_{a\nu}\right] + \int p_{a\mu}\mathcal{I}_{aa}\left[\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\tau p_a^\tau}{m_a c^2}\right)p_{a\nu}\right]\right\}\sqrt{-g}\frac{d^3p_a}{p_{a0}}. \qquad (47)$$

Next we look for a second equation which is independent from Eq. (43). Hence, we multiply the linearized Boltzmann equation (42) by $\Delta_\nu^\mu\frac{c\zeta_a}{\mathsf{c}_p^a\mathsf{n}_a T}(G_a - \frac{U_\sigma p_a^\sigma}{m_a c^2})p_a^\nu$ and integrate over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$; for this long process we use also the integrals that appear in the appendix. The result becomes

$$\frac{1}{T}\left\{\nabla^\mu T - \frac{T}{c^2}\Delta^{\mu i}\left[U^\nu\frac{\partial U_i}{\partial x^\nu} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial\Phi}{\partial x^i}\right]\right\} = \sum_{b=1}^{r}\left(\mathcal{F}_{ba}\mathsf{J}_b^\mu - \mathcal{H}_{ab}\mathsf{q}_b^\mu\right), \qquad (48)$$

where another matrix $\mathcal{H}_{ab}$ is defined. As with the others operators, we split $\mathcal{H}_{ab}$ into the part for unlike indices $\{a,b\}$,

$$\mathcal{H}_{ab} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a\mathsf{n}_b kT^3}\int\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\sigma p_a^\sigma}{m_a c^2}\right)p_{a\mu}\mathcal{I}_{ab} \times\left[\frac{\zeta_b}{\mathsf{c}_p^b}\left(G_b - \frac{U_\epsilon p_b^\epsilon}{m_b c^2}\right)p_{b\nu}\right]\sqrt{-g}\frac{d^3p_a}{p_{a0}}, \quad a\neq b, \qquad (49)$$

and the corresponding for same indices,

$$\mathcal{H}_{aa} = -\frac{c\Delta^{\mu\nu}}{3\mathsf{n}_a^2 kT^3}\left\{\sum_{b=1}^{r}\int\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\sigma p_a^\sigma}{m_a c^2}\right)p_{a\mu} \times\mathcal{I}_{ab}\left[\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\epsilon p_a^\epsilon}{m_a c^2}\right)p_{a\nu}\right] + \int\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\sigma p_a^\sigma}{m_a c^2}\right)p_{a\mu} \times\mathcal{I}_{aa}\left[\frac{\zeta_a}{\mathsf{c}_p^a}\left(G_a - \frac{U_\epsilon p_a^\epsilon}{m_a c^2}\right)p_{a\nu}\right]\right\}\sqrt{-g}\frac{d^3p_a}{p_{a0}}. \qquad (50)$$

Hence, we have obtained the desired system of algebraic equations, namely (43) and (48), which are an independent set of linear equations for the determination of the diffusion J_a^μ and heat q_a^μ fluxes.

A. Onsager reciprocity relations

In this section we show that the Onsager reciprocity relations hold for the system under consideration. The idea is to verify if the matrix associated with the diffusion coefficients is symmetric and therefore the so-called cross effects are equal as it is described from one of the hypotheses of the linear irreversible thermodynamics [19]. One cross effect for our system is the contribution to diffusion due to the temperature gradient; this is often called the "Soret" effect. The other cross effect is the contribution to the heat flux due to the chemical potential gradient or a concentration gradient. When it is due to the latter, it is called the "Dufour" effect. This demonstration is general in the sense that no interaction between the particles is established, but, of course, the microscopic reversibility principle is called for the collisional term of the Boltzmann equation (5).

Let us now write the thermodynamic forces in order to identify clearly the Soret and Dufour effects in terms of the temperature and chemical potential gradients.

First, we define a generalized thermal force as

$$\nabla^\mu\mathcal{T} = \nabla^\mu T - \frac{T}{c^2}\Delta^{\mu i}\left[U^\nu\frac{\partial U_i}{\partial x^\nu} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial\Phi}{\partial x^i}\right], \qquad (51)$$

where the first term contains a temperature gradient while the second one—whose nature is strictly relativistic due to the factor T/c^2—is proportional to the four-acceleration and the gravitational potential gradient. The term due to the four-acceleration was proposed by Eckart [12] while the one due to the gravitational potential gradient was proposed by Tolman [13,14]. If we think in a relativistic gas in equilibrium, we can conjecture the following two aspects: (i) in the absence of a gravitational potential gradient, the temperature gradient must be counterbalanced by an acceleration, and (ii) in the absence of an acceleration, the temperature gradient must be counterbalanced by a gravitational potential gradient. Now Eq. (48) can be written in terms of the thermal force as

$$\frac{1}{T}\nabla^\mu\mathcal{T} = \sum_{b=1}^{r-1}(\mathcal{F}_{ba} - \mathcal{F}_{ra})\mathsf{J}_b^\mu - \sum_{b=1}^{r}\mathcal{H}_{ab}\mathsf{q}_b^\mu. \qquad (52)$$

Above we have considered the constraint $\sum_{a=1}^{r} \mathsf{J}_a^\mu = 0$ which implies that there exist only $r-1$ linearly independent diffusion fluxes.

Next we recall that the chemical potential of species a is defined through the Gibbs function per particle ($\mu_a = \mathsf{e}_a - T\mathsf{s}_a + \mathsf{p}_a/\mathsf{n}_a$). So the following important relationship holds for its gradients:

$$\nabla^\mu\left(\frac{\mu_a}{T}\right) = \frac{1}{\mathsf{n}_a T}\nabla^\mu \mathsf{p}_a - \frac{\mathsf{h}_a}{T^2}\nabla^\mu T. \tag{53}$$

Therefore, the substitution of Eq. (53) into (43) yields

$$-T\nabla^\mu\left(\frac{\mu_a}{T}\right) - \frac{\mathsf{h}_a}{T}\nabla^\mu \mathcal{T} = \sum_{b=1}^{r}\left(\mathcal{A}_{ab}\mathsf{J}_b^\mu - \mathcal{F}_{ab}\mathsf{q}_b^\mu\right). \tag{54}$$

Moreover, by considering that there exist $r-1$ independent diffusion fluxes, we can take the rth component of Eq. (54) and subtract it from (54) itself, yielding

$$-T\nabla^\mu\left(\frac{\mu_a-\mu_r}{T}\right) - \frac{\mathsf{h}_a-\mathsf{h}_r}{T}\nabla^\mu \mathcal{T} = \sum_{b=1}^{r-1}(\mathcal{A}_{ab}-\mathcal{A}_{rb}-\mathcal{A}_{ar}+\mathcal{A}_{rr})\mathsf{J}_b^\mu - \sum_{b=1}^{r}(\mathcal{F}_{ab}-\mathcal{F}_{rb})\mathsf{q}_b^\mu. \tag{55}$$

Now we can proceed to solve the system of linear equations formed by Eqs. (52) and (55). First we solve Eq. (52) for q_b^μ, yielding

$$\mathsf{q}_c^\mu = \sum_{d=1}^{r}(\mathcal{H}^{-1})_{cd}\left\{-\frac{1}{T}\nabla^\mu \mathcal{T}\right\} + \sum_{d=1}^{r}\sum_{b=1}^{r-1}(\mathcal{H}^{-1})_{cd}(\mathcal{F}_{bd}-\mathcal{F}_{rd})\mathsf{J}_b^\mu, \tag{56}$$

where $(\mathcal{H}^{-1})_{cd}$ is the inverse matrix of $\mathcal{H}_{cd}$ so $(\mathcal{H}^{-1})_{cd}\mathcal{H}_{da} = \delta_{ca}$ is the identity matrix. Then we insert Eq. (56) into Eq. (55) and solve for J_a^μ,

$$\mathsf{J}_a^\mu = -T\sum_{b=1}^{r-1}\mathcal{D}'_{ab}\nabla^\mu\left(\frac{\mu_b-\mu_r}{T}\right) - \frac{\mathcal{D}_a}{T}\nabla^\mu \mathcal{T}. \tag{57}$$

Here we identify the above equation as the generalized Fick law, where the coefficients $\mathcal{D}'_{ab}$ and $\mathcal{D}_a$ are related with the diffusion and thermal-diffusion (Soret) effects, respectively. The inverse of the diffusion matrix reads

$$(\mathcal{D}'^{-1})_{ab} = \mathcal{A}_{ab}-\mathcal{A}_{rb}-\mathcal{A}_{ar}+\mathcal{A}_{rr} - \sum_{c=1}^{r}\sum_{d=1}^{r}(\mathcal{F}_{ac}-\mathcal{F}_{rc})(\mathcal{H}^{-1})_{cd}(\mathcal{F}_{bd}-\mathcal{F}_{rd}), \tag{58}$$

while the thermal-diffusion coefficients are given by

$$\mathcal{D}_a = \sum_{b=1}^{r-1}\mathcal{D}'_{ab}\left\{\mathsf{h}_b-\mathsf{h}_r+\sum_{c=1}^{r}\sum_{d=1}^{r}(\mathcal{F}_{bc}-\mathcal{F}_{rc})(\mathcal{H}^{-1})_{cd}\right\}. \tag{59}$$

Now we have to obtain the total heat flux as a function of the temperature and chemical potential gradients. For this end, we rewrite the total heat four-flux (25) as

$$\mathsf{q}^\mu = \sum_{a=1}^{r}\mathsf{q}_a^\mu + \sum_{a=1}^{r-1}(\mathsf{h}_a-\mathsf{h}_r)\mathsf{J}_a^\mu, \tag{60}$$

and substitute in it the expressions found for q_a^μ and J_a^μ, i.e., Eqs. (56) and (57). Hence it follows the Fourier law

$$\mathsf{q}^\mu = -\frac{\lambda'}{T}\nabla^\mu\mathcal{T} - T\sum_{a=1}^{r-1}\mathcal{D}'_a\nabla^\mu\left(\frac{\mu_a-\mu_r}{T}\right), \tag{61}$$

where we have introduced the thermal conductivity coefficient

$$\lambda' = \sum_{a=1}^{r}\sum_{b=1}^{r}(\mathcal{H}^{-1})_{ab} + \sum_{b=1}^{r-1}\mathcal{D}_b\bigg[\mathsf{h}_b-\mathsf{h}_r + \sum_{a=1}^{r}\sum_{c=1}^{r}(\mathcal{H}^{-1})_{ac}(\mathcal{F}_{bc}-\mathcal{F}_{rc})\bigg], \tag{62}$$

and the diffusion-thermal coefficient

$$\mathcal{D}'_a = \sum_{b=1}^{r-1}\mathcal{D}'_{ba}\left[\mathsf{h}_b-\mathsf{h}_r+\sum_{c=1}^{r}\sum_{d=1}^{r}(\mathcal{H}^{-1})_{cd}(\mathcal{F}_{bd}-\mathcal{F}_{rd})\right]. \tag{63}$$

Ultimately, we make a close inspection of the matrices $\mathcal{A}_{ab}$, $\mathcal{F}_{ab}$, and $\mathcal{H}_{ab}$, which are given as functions of the collision operators $\mathcal{I}_{ab}$. From (44), (46), and (49), we may infer that only $\mathcal{A}_{ab}$ and $\mathcal{H}_{ab}$ are symmetric matrices, while $\mathcal{F}_{ab}$ is nonsymmetric. Hence we may conclude from (58) that the matrix related with the diffusion coefficients are symmetric, i.e., $\mathcal{D}'_{ab} = \mathcal{D}'_{ba}$. Moreover, for the coefficients of cross effects—namely the Soret $\mathcal{D}_a$ and Dufour $\mathcal{D}'_a$—we note from the symmetry of $\mathcal{H}_{ab}$ and $\mathcal{D}_{ab}$ that (59) and (63) are equivalent, so $\mathcal{D}_a = \mathcal{D}'_a$. The relationships $\mathcal{D}'_{ab} = \mathcal{D}'_{ba}$ and $\mathcal{D}_a = \mathcal{D}'_a$ imply a demonstration of the validity of the Onsager reciprocity relations. Note that as in another's demonstrations [22], it appears to be an ultimate macroscopic effect that can only be proved because of the symmetries that belong to the collisional term of the Boltzmann equation (5) given from the H-theorem, i.e., the microscopic reversibility principle.

B. Thermal and diffusion forces

It is usual in the theory of fluid mixtures to express the diffusion fluxes and the heat flux of the mixture in terms of the generalized thermal and diffusion forces. The thermal force for a relativistic fluid was introduced in the last section [see (51)]. On the other hand, we follow [7] and define the generalized diffusion force of species a as

$$\mathsf{d}_a^\mu = \nabla^\mu \mathsf{x}_a + (\mathsf{x}_a-1)\nabla^\mu \ln \mathsf{p} - \frac{\mathsf{n}_a\mathsf{h}_a-\mathsf{nh}}{\mathsf{p}c^2}\Delta^{\mu j} \times\left[U^\tau\frac{\partial U_j}{\partial x^\tau} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial\Phi}{\partial x^j}\right], \tag{64}$$

where $\mathsf{x}_a = \mathsf{p}_a/\mathsf{p} = \mathsf{n}_a/\mathsf{n}$ is the concentration of species a. We can identify four contributions to the generalized diffusion force: a concentration gradient, a pressure gradient, a term proportional to the four-acceleration, and the gradient of the gravitational potential. Here it is important to emphasize that contrary to what happens with the thermal force, the terms with the four-acceleration and gradient of gravitational potential are not of a strictly relativistic nature. Indeed, $(\mathsf{n}_a\mathsf{h}_a - \mathsf{nh})/c^2\mathsf{p} = (\mathsf{n}_a m_a G_a - \sum_{b=1}^r \mathsf{n}_b m_b G_b)/\mathsf{p}$ and $G_b \to 1$ for $\zeta_b \gg 1$. This equation has a very important feature because it represents the generalization of the diffusion force originally written for the nonrelativistic case [20,23]. More discussions about this point can be found in Ref. [7]. Here as in the nonrelativistic case, exist only $r-1$ linearly independent generalized diffusion forces due to the relationship $\sum_{a=1}^r \mathsf{d}_a^\mu = 0$.

Now we can proceed to express the vectorial fluxes in terms of the generalized thermal and diffusive forces. To do so, we use the momentum density balance equation (see Ref. [7])

$$\frac{\partial \mathsf{p}}{\partial x^i} - \frac{\mathsf{nh}}{c^2}\left[U^\nu\frac{\partial U_i}{\partial x^\nu} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial \Phi}{\partial x^i}\right] = 0, \tag{65}$$

and the gradient of the chemical potential written as

$$\begin{aligned} -T\nabla^\mu\left(\frac{\mu_a - \mu_r}{T}\right) &= -\frac{\mathsf{p}}{\mathsf{n}_a}(\nabla^\mu \mathsf{x}_a + \mathsf{x}_a \nabla^\mu \ln \mathsf{p}) \\ &\quad + \frac{\mathsf{p}}{\mathsf{n}_r}(\nabla^\mu \mathsf{x}_r + \mathsf{x}_r\nabla^\mu \ln \mathsf{p}) + \frac{\mathsf{h}_a - \mathsf{h}_r}{T}\nabla^\mu T. \end{aligned} \tag{66}$$

After some rearrangements, the expressions for (57) and (61) become

$$\mathsf{J}_a^\mu = \sum_{b=1}^{r-1}\widetilde{\mathcal{D}}_{ab}\mathsf{d}_b^\mu + \frac{\widetilde{\mathcal{D}}_a}{T}\nabla^\mu \mathcal{T}, \tag{67}$$

$$\mathsf{q}^\mu = \frac{\widetilde{\lambda}}{T}\nabla^\mu\mathcal{T} + \sum_{a=1}^{r-1}\widetilde{\mathcal{D}}'_a\mathsf{d}_a^\mu. \tag{68}$$

In these representations for the generalized thermal and diffusion forces, the transport coefficients read:

$$\widetilde{\mathcal{D}}_{ab} = -\sum_{c=1}^{r-1}\mathcal{D}'_{ac}\frac{\mathsf{p}}{\mathsf{n}_b}\left(\delta_{bc} + \frac{\mathsf{n}_b}{\mathsf{n}_r}\right), \tag{69}$$

$$\widetilde{\mathcal{D}}_a = -\sum_{b=1}^{r-1}\sum_{c=1}^{r}\sum_{d=1}^{r}\mathcal{D}'_{ab}(\mathcal{F}_{bc} - \mathcal{F}_{rc})(\mathcal{H}^{-1})_{cd}, \tag{70}$$

$$\widetilde{\lambda} = -\sum_{a=1}^{r}\sum_{b=1}^{r}\left[(\mathcal{H}^{-1})_{ab} + \sum_{c=1}^{r-1}\mathcal{D}_c(\mathcal{H}^{-1})_{ab}(\mathcal{F}_{cb} - \mathcal{F}_{rb})\right], \tag{71}$$

$$\widetilde{\mathcal{D}}'_a = -\sum_{b=1}^{r-1}\mathcal{D}_a\frac{\mathsf{p}}{\mathsf{n}_a}\left(\delta_{ab} + \frac{\mathsf{n}_a}{\mathsf{n}_r}\right). \tag{72}$$

At this point it is worth pausing to make two comments. First, by looking the expression for the generalized diffusion force Eq. (64), we note that it depends on (i) a concentration gradient, which tends to reduce the nonhomogeneity of the mixture; (ii) a pressure gradient, where heavy particles tend to diffuse to places with high pressures, e.g., in centrifuges; (iii) an acceleration, which acts on different masses; and (iv) a gravitational potential gradient. Second, let us suppose a mixture in which the generalized thermal force vanishes, the pressure is constant, and there is no acceleration. We can think also that there is no diffusive flux, implying a pseudoequilibrium state. Such a situation is very interesting because of its physical implications, that is, the gradient of concentration has to be counterbalanced by the gravitational potential gradient.

To complete this section, we point out that the thermal conductivity coefficient λ in a mixture is defined as the ratio of the heat flux to the temperature gradient. This occurs when there is no diffusion i.e., when $\mathsf{J}_a^\mu = 0$. From (56) and (60), we have

$$\lambda = -\sum_{a=1}^{r}\sum_{b=1}^{r}(\mathcal{H}^{-1})_{ab}. \tag{73}$$

Furthermore, in the absence of a temperature gradient the constitutive equation for the diffusion fluxes (67) are proportional only to the generalized diffusion forces and $\widetilde{\mathcal{D}}_{ab}$ is identified as the matrix of the diffusion coefficients.

V. ENTROPY FLUX OF THE MIXTURE

In this section we will show that the entropy four-flow for the system under consideration takes the form as predicted by linear irreversible thermodynamics. According to (27) and third part of (30) the entropy flux of the mixture is given by

$$\Phi^\mu = -kc\Delta_\nu^\mu\sum_{a=1}^{r}\int p_a^\nu f_a \ln(\mathsf{b}_a f_a)\sqrt{-g}\frac{d^3 p_a}{p_{a0}}. \tag{74}$$

We substitute the Grad distribution function (37) into the above expression and linearize in the fluxes $\mathsf{J}_a^\mu, \mathsf{q}^\mu, \varpi, \mathsf{p}^{\langle\mu\nu\rangle}$. After integration the entropy flux takes the form

$$\begin{aligned} \Phi^\mu &= \frac{1}{T}\sum_{a=1}^{r}\mathsf{q}_a^\mu + \sum_{a=1}^{r}\mathsf{s}_a\mathsf{J}_a^\mu = \frac{\mathsf{q}^\mu}{T} - \sum_{a=1}^{r}\frac{\mu_a}{T}\mathsf{J}_a^\mu \\ &= \frac{\mathsf{q}^\mu}{T} - \sum_{a=1}^{r-1}\frac{\mu_a - \mu_r}{T}\mathsf{J}_a^\mu. \end{aligned} \tag{75}$$

The first equality above shows that the entropy flux of the mixture is a sum of two terms: One refers to the sum of all partial heat fluxes divided by the temperature and the other is a sum of the transport due to diffusion of the partial entropies per particle. The second equality is well known from nonrelativistic linear irreversible thermodynamics [19] and is connected with the transport of the chemical potentials driven by diffusion.

We can also express the entropy flux of the mixture in terms of the thermal and diffusion generalized forces by substituting the representations (67) and (68) into (75), yielding

$$\Phi^\mu = -\frac{\mathcal{L}}{T^2}\nabla^\mu\mathcal{T} - \sum_{a=1}^{r-1}\frac{\mathcal{L}_a}{T}\mathsf{d}_a^\mu. \tag{76}$$

Here the scalar coefficients $\mathcal{L}$ and $\mathcal{L}_a$ read

$$\mathcal{L} = \widetilde{\lambda} - \sum_{a=1}^{r-1}(\mu_a - \mu_r)\widetilde{\mathcal{D}}_a, \tag{77}$$

$$\mathcal{L}_a = \widetilde{\mathcal{D}}'_a - \sum_{b=1}^{r-1}(\mu_b - \mu_r)\widetilde{\mathcal{D}}_{ba}. \tag{78}$$

It is clear from the definitions of the thermal (51) and diffusive (64) forces that the entropy flux of the mixture (76) depends on the temperature, concentration, and pressure gradients as well as on the acceleration and gravitational potential gradient.

VI. NAVIER-STOKES LAW

In this section we will calculate the constitutive equations for a relativistic Newtonian fluid, in other words, the Navier-Stokes law. This law is usually separated into two equations. The first one is for the nonequilibrium pressures and it is associated with the bulk viscosity. The second one is for the pressure deviator tensor and it is associated with the shear viscosity.

Let us start with the constitutive equation for the partial nonequilibrium pressures ϖ_a of species a; it is obtained as follows. We multiply (42) by $\Delta_{\sigma\tau}p_a^\sigma p_a^\tau$ and integrate over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$. For this purpose, we use the integrals from the appendix. We also eliminate the derivative projections $U^\mu\partial_\mu$ with the help of the partial particle number density and energy per particle balance equations. Such balance equations correspond to a Eulerian fluid, where nonequilibrium quantities $\mathsf{J}_a^\mu, \mathsf{q}_a^\mu \varpi_a, \mathsf{p}_a^{\langle\mu\nu\rangle}$ vanish, that is,

$$U^\mu\partial_\mu \mathsf{n}_a + \mathsf{n}_a\nabla^\mu U_\mu = 0, \tag{79}$$

$$\mathsf{n}_a \mathsf{c}_v^a U^\mu\partial_\mu T + \mathsf{p}_a\nabla^\mu U_\mu = 0. \tag{80}$$

The result of this process becomes a system of equations for ϖ_b and it reads

$$-\left[\frac{\mathsf{p}_a kT}{c^3}\frac{\partial\ln\zeta_a}{\partial\ln\mathsf{c}_v^a}\right]\nabla_\mu U^\mu = \sum_{b=1}^r \mathcal{R}_{ab}\varpi_b. \tag{81}$$

Here we have introduced the matrix $\mathcal{R}_{ab}$, which is defined for different indices $\{a,b\}$ as:

$$\mathcal{R}_{ab} = \frac{U_\mu U_\nu U_\sigma}{c^2\mathsf{p}_b}\int p_a^\mu p_a^\nu \mathcal{I}_{ab}\left[\frac{\partial\ln\zeta_b}{\partial\ln\mathsf{c}_v^b}\left(\frac{U_\tau p_b^\tau}{kT} - \frac{3(\mathsf{c}_p^b+\mathsf{h}_b/T)}{\mathsf{c}_v^b}\right)\frac{p_b^\sigma}{kT}\right]\sqrt{-g}\frac{d^3p_a}{p_{a0}}, \quad a\neq b. \tag{82}$$

Similarly as in previous sections, we write this matrix for equal indices $\{a, b=a\}$ as

$$\mathcal{R}_{aa} = \frac{U_\mu U_\nu U_\sigma}{c^2\mathsf{p}_a}\Bigg\{\sum_{b=1}^r\int p_a^\mu p_a^\nu\mathcal{I}_{ab}\left[\frac{\partial\ln\zeta_a}{\partial\ln\mathsf{c}_v^a}\left(\frac{U_\tau p_a^\tau}{kT} - \frac{3(\mathsf{c}_p^a+\mathsf{h}_a/T)}{\mathsf{c}_v^a}\right)\frac{p_a^\sigma}{kT}\right] + \int p_a^\mu p_a^\nu\mathcal{I}_{aa}\left[\frac{\partial\ln\zeta_a}{\partial\ln\mathsf{c}_v^a}\left(\frac{U_\tau p_a^\tau}{kT} - \frac{3(\mathsf{c}_p^a+\mathsf{h}_a/T)}{\mathsf{c}_v^a}\right)\frac{p_a^\sigma}{kT}\right]\Bigg\}\sqrt{-g}\frac{d^3p_a}{p_{a0}}. \tag{83}$$

The solution of the linear system of Eq. (81) for the partial nonequilibrium pressures ϖ_a is given by

$$\varpi_a = -\left[\sum_{b=1}^r(\mathcal{R}^{-1})_{ab}\frac{\mathsf{p}_b kT}{c^3}\frac{\partial\ln\zeta_b}{\partial\ln\mathsf{c}_v^b}\right]\nabla_\mu U^\mu, \tag{84}$$

where $(\mathcal{R}^{-1})_{ab}$ denotes the inverse of the matrix $\mathcal{R}_{ab}$. The constitutive equation for the nonequilibrium pressure of the mixture is obtained from the sum of (84) over all constituents according to third part of (23). Hence it follows that

$$\varpi = -\eta\nabla_\mu U^\mu, \tag{85}$$

where the bulk viscosity coefficient of the mixture reads

$$\eta = \sum_{a,b=1}^r(\mathcal{R}^{-1})_{ab}\frac{\mathsf{p}_b kT}{c^3}\frac{\partial\ln\zeta_b}{\partial\ln\mathsf{c}_v^b}. \tag{86}$$

For the second equation that conforms the Navier-Stokes law, which is the pressure deviator constitutive one, we proceed in an analogous manner. We take the product of (42) with $[\Delta_\sigma^{(\mu}\Delta_\tau^{\nu)} - \Delta_{\sigma\tau}\Delta^{\mu\nu}/3]p_a^\sigma p_a^\tau$ and integrate over $\sqrt{-g}\frac{d^3p_a}{p_{a0}}$. This process leads to the following linear system of equations for the partial pressure deviator tensors $\mathsf{p}_b^{\langle\mu\nu\rangle}$:

$$2\nabla^{\langle\mu}U^{\nu\rangle} = \sum_{b=1}^r\mathcal{K}_{ab}\mathsf{p}_b^{\langle\mu\nu\rangle}. \tag{87}$$

In this last equation we have introduced the following abbreviation for the symmetric and traceless four-velocity gradient:

$$\nabla^{\langle\mu}U^{\nu\rangle} = \left(\frac{\Delta_\sigma^\mu\Delta_\tau^\nu + \Delta_\sigma^\nu\Delta_\tau^\mu}{2} - \frac{\Delta^{\mu\nu}\Delta_{\sigma\tau}}{3}\right)\partial^\sigma U^\tau. \tag{88}$$

Equation (87) also includes the definition of the matrix $\mathcal{K}_{ab}$ which reads

$$\mathcal{K}_{ab} = -\frac{c^3\Delta_{\mu\langle\sigma}\Delta_{\tau\rangle\nu}}{10\mathsf{p}_a\mathsf{h}_a\mathsf{p}_b}\int p_a^\sigma p_a^\tau\mathcal{I}_{ab}\left[\frac{\zeta_b}{m_b\mathsf{h}_b}p_b^\mu p_b^\nu\right]\times\sqrt{-g}\frac{d^3p_a}{p_{a0}}, \quad a\neq b \tag{89}$$

$$\mathcal{K}_{aa} = -\frac{c^3\Delta_{\mu\langle\sigma}\Delta_{\tau\rangle\nu}}{10\mathsf{p}_a\mathsf{h}_a\mathsf{p}_a}\Bigg\{\sum_{b=1}^r\int p_a^\sigma p_a^\tau\mathcal{I}_{ab}\left[\frac{\zeta_a}{m_a\mathsf{h}_a}p_a^\mu p_a^\nu\right] + \int p_a^\sigma p_a^\tau\mathcal{I}_{aa}\left[\frac{\zeta_a}{m_a\mathsf{h}_a}p_a^\mu p_a^\nu\right]\Bigg\}\sqrt{-g}\frac{d^3p_a}{p_{a0}}. \tag{90}$$

From the solution of the linear system of Eq. (87) for $\mathsf{p}_b^{\langle\mu\nu\rangle}$ and from the relationship $\mathsf{p}^{\langle\mu\nu\rangle} = \sum_{b=1}^r\mathsf{p}_b^{\langle\mu\nu\rangle}$, the constitutive equation for the pressure deviator tensor of the mixture is as follows:

$$\mathsf{p}^{\langle\mu\nu\rangle} = 2\mu\nabla^{\langle\mu}U^{\nu\rangle}. \tag{91}$$

Here the shear viscosity coefficient of the mixture is given by

$$\mu = \sum_{a,b=1}^r(\mathcal{K}^{-1})_{ab}. \tag{92}$$

Equations (85) and (91) are the constitutive equations of a relativistic Newtonian fluid, also known as the Navier-Stokes constitutive equations.

VII. CONCLUSIONS

In this work we have studied a mixture of r species of relativistic gases in the presence of gravitational fields. The curvature of the space-time was introduced by incorporating the Christoffel symbols to the Boltzmann equation. We used the Schwarzschild metric written in isotropic coordinates. A linearized Boltzmann equation was obtained by following a methodology which combines the features of the Chapman-Enskog and Grad methods.

By applying the Chapman-Enskog–Grad combined method to the Boltzmann equation we obtained a linear expression [Eq. (42)] which was used for the determination of the thermodynamic fluxes as functions of the thermodynamic forces. The Navier-Stokes law was derived as well as the generalized of Fourier and Fick laws.

The proof of the validity of the Onsager reciprocity relations was at last possible because of the symmetries of the collisional term of the Boltzmann equation. These symmetries are those associated with the H-theorem and the microscopic reversibility principle. This reinforces the idea that the Onsager reciprocity relations are the macroscopic manifestation of the microscopic symmetries of the trajectories of the particles that conform the gas.

We have introduced the thermal force

$$\nabla^\mu \mathcal{T} = \nabla^\mu T - \frac{T}{c^2}\Delta^{\mu i}\left[U^\nu \frac{\partial U_i}{\partial x^\nu} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial \Phi}{\partial x^i}\right], \tag{93}$$

which is a very eloquent result. Indeed, Eq. (93) turns to be just the gradient of the temperature in the nonrelativistic limit, i.e., $\nabla^\mu T$ because the factor T/c^2 of the second term is of relativistic order. The inclusion of the acceleration term into the thermal force was proposed by Eckart [12] while the one relating the gravitational potential gradient was proposed by Tolman [13,14]. Here these terms appear as a natural consequence of the solution of the relativistic Boltzmann equation in gravitational fields.

On the other hand, we have identified the generalized diffusion force with

$$\mathsf{d}_a^\mu = \nabla^\mu \mathsf{x}_a + (\mathsf{x}_a - 1)\,\nabla^\mu \ln \mathsf{p} - \frac{\mathsf{n}_a \mathsf{h}_a - \mathsf{nh}}{\mathsf{p}c^2}\Delta^{\mu j}\left[U^\tau \frac{\partial U_j}{\partial x^\tau} - \frac{1}{1-\Phi^2/4c^4}\frac{\partial \Phi}{\partial x^j}\right]. \tag{94}$$

This is a new and interesting result because the third term—which is related with the four-acceleration and the gradient of the gravitational potential—does not go to zero in the nonrelativistic limiting case as the thermal force. As was pointed out in Ref. [7], the diffusion force that came out from a nonrelativistic kinetic theory [20,23] has a similar expression to (94). It depends on the concentration and pressure gradients, but it has a term depending on the forces which act on the particle of different species, and such a term vanishes when only gravitational forces are acting on the particles.

Another result obtained is the entropy flux of the relativistic mixture through the use of Grad's distribution function, which has a similar expression as the one of nonrelativistic linear irreversible thermodynamics [19]. Its constitutive equation was written in terms of the generalized thermal and diffusion forces, so it depends also on the acceleration and on the gravitational potential gradient.

Here is the place to discuss two additional issues. The first one is the validity of the Onsager reciprocity relations for the case of a relativistic quantum gas. In such a case, the local equilibrium distribution [which in this work is given by Eq. (6)] would take a form of the Fermi-Dirac and Bose-Einstein distributions for a fermionic and bosonic gas, respectively. Quantum relativistic gases can be described by the relativistic Uehling-Uhlenbeck equation (see, e.g., Ref. [3]). As we have pointed out, the validity of Onsager's reciprocity relations are deeply associated with the symmetries that belong to the H-theorem. In the present work, those symmetries are implied in Eq. (41). Then, to show the validity of the Onsager reciprocity relations for a quantum system, we need the validity of the H-theorem, which has been presented in Refs. [24,25], raising the possibility of exploring that issue. The second topic is related with Tolman's law, which has been derived in Ref. [13] and is valid for all static spherical symmetrical line elements. In the present work we have used the Schwarzschild metric, which, according to Birkoff's theorem, is the most general spherically symmetrical nonrotating and uncharged source of the gravitational field.

As a final comment we call attention to the fact that for the determination of all the transport coefficients, we have to specify the interaction potential of the relativistic particles and evaluate the matrices $\{\mathcal{R}_{ab},\mathcal{K}_{ab},\mathcal{A}_{ab},\mathcal{F}_{ab},\mathcal{H}_{ab}\}$. This represents work in progress and will be published in the future.

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APPENDIX

1. Table of integrals

For the purposes of this work, it is convenient to do the following unique decomposition of the integral operators:

$$X_{ab}^{\mu\nu} = \int p_a^\mu \mathcal{I}_{ab}\left[p_b^\nu\right]\sqrt{-g}\frac{d^3 p_a}{p_{a0}} = \frac{1}{3c^2}\left(4I_{ab}^1 - I_{ab}^2\right)U^\mu U^\nu + \frac{1}{3}\left(I_{ab}^2 - I_{ab}^1\right)g^{\mu\nu}, \tag{A1}$$

$$X_{ab}^{\mu\nu\sigma} = \int p_a^\mu p_a^\nu \mathcal{I}_{ab}\left[p_b^\sigma\right]\sqrt{-g}\frac{d^3 p_a}{p_{a0}} = \frac{2}{3c^3}\left(3I_{ab}^3 - I_{ab}^4\right)U^\mu U^\nu U^\sigma - \frac{1}{3c}I_{ab}^3 g^{\mu\nu}U^\sigma + \frac{1}{3c}\left(I_{ab}^4 - I_{ab}^3\right)\left(g^{\mu\sigma}U^\nu + g^{\nu\sigma}U^\mu\right), \tag{A2}$$

$$X_{ab}^{\mu\nu\sigma\tau}=\int p_a^\mu p_a^\nu \mathcal{I}_{ab}\big[p_b^\sigma p_b^\tau\big]\sqrt{-g}\frac{d^3p_a}{p_{a0}}=\frac{2}{15c^4}\big(24I_{ab}^5-12I_{ab}^6+I_{ab}^7\big)U^\mu U^\nu U^\sigma U^\tau+\frac{1}{15c^2}\big(I_{ab}^7-2I_{ab}^6-6I_{ab}^5\big)$$
$$\times(g^{\mu\nu}U^\sigma U^\tau+g^{\sigma\tau}U^\mu U^\nu)+\frac{1}{30c^2}\big(16I_{ab}^6-3I_{ab}^7-12I_{ab}^5\big)(g^{\mu\sigma}U^\nu U^\tau+g^{\mu\tau}U^\nu U^\sigma+g^{\nu\sigma}U^\mu U^\tau+g^{\nu\tau}U^\mu U^\sigma)$$
$$+\frac{1}{30}\big(3I_{ab}^7-6I_{ab}^6+2I_{ab}^5\big)(g^{\mu\sigma}g^{\nu\tau}+g^{\mu\tau}g^{\nu\sigma})+\frac{1}{15}\big(I_{ab}^5-I_{ab}^7+2I_{ab}^6\big)g^{\mu\nu}g^{\sigma\tau}, \tag{A3}$$

where $I_{ab}^1\dots I_{ab}^7$ are given by

$$I_{ab}^1=\frac{U_\mu U_\nu}{c^2}X_{ab}^{\mu\nu},\quad I_{ab}^2=g_{\mu\nu}X_{ab}^{\mu\nu},\quad I_{ab}^3=\frac{U_\mu U_\nu U_\sigma}{c^3}X_{ab}^{\mu\nu\sigma},\quad I_{ab}^4=\frac{g_{\mu\sigma}U_\nu}{c}X_{ab}^{\mu\nu\sigma}, \tag{A4}$$

$$I_{ab}^5=\frac{U_\mu U_\nu U_\sigma U_\tau}{c^4}X_{ab}^{\mu\nu\sigma\tau},\quad I_{ab}^6=\frac{g_{\nu\tau}U_\mu U_\sigma}{c^2}X_{ab}^{\mu\nu\sigma\tau},\quad I_{ab}^7=g_{\mu\sigma}g_{\nu\tau}X_{ab}^{\mu\nu\sigma\tau}. \tag{A5}$$

Here we list a table of integrals that are used in the previous sections.

$$\int e^{-\frac{1}{kT}U^\lambda p_\lambda}\frac{d^3p}{p_0}=4\pi mkT\,K_1(\zeta),\qquad \int p^\mu e^{-\frac{1}{kT}U_\lambda p^\lambda}\frac{d^3p}{p_0}=4\pi m^2kT\,K_2(\zeta)U^\mu, \tag{A6}$$

$$\int p^\mu p^\nu e^{-\frac{1}{kT}U_\lambda p^\lambda}\frac{d^3p}{p_0}=-4\pi(mkT)^2\left[K_2(\zeta)g^{\mu\nu}-\zeta K_3(\zeta)\frac{U^\mu U^\nu}{c^2}\right], \tag{A7}$$

$$\int p^\mu p^\nu p^\sigma e^{-\frac{1}{kT}U_\lambda p^\lambda}\frac{d^3p}{p_0}=-4\pi m^3(kT)^2\left[\frac{K_3(\zeta)}{3}g^{(\mu\nu}U^{\sigma)}-\zeta K_4(\zeta)\frac{U^\mu U^\nu U^\sigma}{c^2}\right], \tag{A8}$$

$$\int p^\mu p^\nu p^\sigma p^\tau e^{-\frac{1}{kT}U_\lambda p^\lambda}\frac{d^3p}{p_0}=4\pi(mkT)^3\left[\frac{K_3(\zeta)}{3}g^{(\mu\nu}g^{\sigma\tau)}-\zeta K_4(\zeta)\frac{g^{(\mu\nu}U^\sigma U^{\tau)}}{6c^2}+\zeta^2K_5(\zeta)\frac{U^\mu U^\nu U^\sigma U^\tau}{c^4}\right], \tag{A9}$$

$$\int p^\mu p^\nu p^\sigma p^\tau p^\epsilon e^{-\frac{1}{kT}U_\lambda p^\lambda}\frac{d^3p}{p_0}=4\pi m^4(kT)^3\left[\frac{K_4(\zeta)}{15}U^{(\epsilon}g^{\mu\nu}g^{\sigma\tau)}-\zeta K_5(\zeta)\frac{g^{(\mu\nu}U^\sigma U^\tau U^{\epsilon)}}{10c^2}+\zeta^2K_6(\zeta)\frac{U^\mu U^\nu U^\sigma U^\tau U^\epsilon}{c^4}\right], \tag{A10}$$

$$\int\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\tau p_\tau}\frac{d^3p}{p_0}=4\pi m[K_1(\zeta)-\mathrm{Ki}_1(\zeta)],\qquad \int p^\mu\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\tau p_\tau}\frac{d^3p}{p_0}=4\pi m^2\frac{K_1(\zeta)}{\zeta}U^\mu, \tag{A11}$$

$$\int p^\mu p^\nu\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\tau p_\tau}\frac{d^3p}{p_0}=-\frac{4\pi m^2kT}{3}\left\{[K_2(\zeta)-\zeta(K_1(\zeta)-\mathrm{Ki}_1(\zeta))]g^{\mu\nu}-\frac{1}{c^2}[4K_2(\zeta)-\zeta(K_1(\zeta)-\mathrm{Ki}_1(\zeta))]U^\mu U^\nu\right\}, \tag{A12}$$

$$\int p^\mu p^\nu p^\sigma\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\tau p_\tau}\frac{d^3p}{p_0}=-\frac{4\pi m^2k^2T^2}{c^2}\left\{\frac{K_2(\zeta)}{3}g^{(\mu\nu}U^{\sigma)}-[\zeta K_3(\zeta)+2K_2(\zeta)]\frac{U^\mu U^\nu U^\sigma}{c^2}\right\}, \tag{A13}$$

$$\int p^\mu p^\nu p^\sigma p^\tau\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\theta p_\theta}\frac{d^3p}{p_0}=\frac{4\pi m^3k^2T^2}{15}\left\{\frac{3K_3(\zeta)-\zeta K_2(\zeta)+\zeta^2[K_1(\zeta)-\mathrm{Ki}_1(\zeta)]}{3}g^{(\mu\nu}g^{\sigma\tau)}\right.$$
$$-\frac{1}{6c^2}[18K_3(\zeta)-\zeta K_2(\zeta)+\zeta^2(K_1(\zeta)-\mathrm{Ki}_1(\zeta))]g^{(\mu\nu}U^\sigma U^{\tau)}$$
$$\left.+\frac{3}{c^4}[48K_3(\zeta)+4\zeta K_2(\zeta)+\zeta^2(K_1(\zeta)-\mathrm{Ki}_1(\zeta))]U^\mu U^\nu U^\sigma U^\tau\right\}, \tag{A14}$$

$$\int p^\mu p^\nu p^\sigma p^\tau p^\epsilon\frac{e^{-\frac{1}{kT}U^\lambda p_\lambda}}{U^\theta p_\theta}\frac{d^3p}{p_0}=\frac{4\pi m^6c^4}{\zeta^3}\left\{\frac{K_3(\zeta)}{15}U^{(\mu}g^{\nu\tau}g^{\sigma\epsilon)}-\frac{1}{10c^2}[8K_3(\zeta)+\zeta K_2(\zeta)]g^{(\mu\epsilon}U^\nu U^\sigma U^{\tau)}\right.$$
$$\left.+\frac{1}{c^4}[\zeta^2K_3(\zeta)+12\zeta K_2(\zeta)+80K_3(\zeta)]U^\mu U^\nu U^\sigma U^\tau U^\epsilon\right\}. \tag{A15}$$

Above the parenthesis around N indexes indicate a sum over all permutations of these indexes divided by $N!$. Furthermore, $\mathrm{Ki}_n(\zeta)$ denotes the integral

$$\mathrm{Ki}_n(\zeta) = \int_0^\infty \frac{e^{-\zeta \cosh t}}{\cosh^n t} dt. \tag{A16}$$

2. Cristoffel symbols for the Schwarzschild isotropic metric

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{ij} = 0, \quad \Gamma^k_{ij} = 0 \quad (i \neq j \neq k), \quad \Gamma^i_{0j} = 0, \quad \Gamma^{\underline{i}}_{\underline{i}\,j} = \frac{1}{2g_1(r)}\frac{dg_1(r)}{dr}\delta_{jk}\frac{x^k}{r}, \tag{A17}$$

$$\Gamma^0_{0i} = \frac{1}{2g_0(r)}\frac{dg_0(r)}{dr}\delta_{ij}\frac{x^j}{r}, \quad \Gamma^i_{00} = \frac{1}{2g_1(r)}\frac{dg_0(r)}{dr}\frac{x^i}{r}, \quad \Gamma^j_{\underline{i}\,\underline{i}} = -\frac{1}{2g_1(r)}\frac{dg_1(r)}{dr}\frac{x^j}{r} \quad (i \neq j). \tag{A18}$$

The underlined indices above are not summed and

$$\frac{dg_0(r)}{dr} = \frac{2GM}{c^2r^2}\frac{\left(1 - \frac{GM}{2c^2r}\right)}{\left(1 + \frac{GM}{2c^2r}\right)^3}, \quad \frac{dg_1(r)}{dr} = -\frac{2GM}{c^2r^2}\left(1 + \frac{GM}{2c^2r}\right)^3. \tag{A19}$$

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