Determining the Tsallis parameter via maximum entropy

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The nonextensive entropic measure proposed by Tsallis [C. Tsallis, J. Stat. Phys. **52**, 479 (1988)] introduces a parameter, q, which is not defined but rather must be determined. The value of q is typically determined from a piece of data and then fixed over the range of interest. On the other hand, from a phenomenological viewpoint, there are instances in which q cannot be treated as a constant. We present two distinct approaches for determining q depending on the form of the equations of constraint for the particular system. In the first case the equations of constraint for the operator $\hat{O}$ can be written as $\mathrm{Tr}(F^q\hat{O}) = C$, where C may be an explicit function of the distribution function F. We show that in this case one can solve an equivalent MAXENT problem which yields q as a function of the corresponding Lagrange multiplier. As an illustration the exact solution of the static generalized Fokker-Planck equation (GFPE) is obtained from MAXENT with the Tsallis enropy. As in the case where C is a constant, if q is treated as a variable within the MAXENT framework the entropic measure is maximized trivially for all values of q. Therefore q must be determined from existing data. In the second case an additional equation of constraint exists which cannot be brought into the above form. In this case the additional equation of constraint may be used to determine the fixed value of q.

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Since its introduction, the Tsallis entropy [1–5] has increasingly been utilized as the entropic measure in maximum entropy (MAXENT) calculations [6]. The choice of the Tsallis parameter, q, which is not defined *a priori* [7,8] can yield in the limit $q \to 1$ the Boltzmann-Gibbs (BG) entropy as well as a family of fractal entropies ($q \neq 1$). It has been argued that perhaps this parameter should be constant, if not universally, at least for classes of dynamical systems [9–11], and attempts have been made to set limits on the value of this parameter (see [12] for an extensive list of examples). Within the framework of the MAXENT approach the question arises as to how one should determine q.

For a classical system the stationary state distribution function F can be obtained from the classical Tsallis entropy

$$S_q = \frac{1}{q-1}(F - F^q), \quad (1)$$

via the MAXENT equation [4]

$$\delta_F S_q = 0, \quad (2)$$

along with the equation of constraint for the operator $\hat{O}$,

$$\mathrm{Tr}(F^q\hat{O}) = C. \quad (3)$$

The solution of these equations yields the classical Tsallis distribution given by

$$F(x) = D[1 - \beta(1-q)\hat{O}(x)]^{\frac{1}{1-q}}, \quad (4)$$

where D is a constant.

Here we focus of the equation of constraint. There are two possible cases: the equation of constraint can either be brought into the form of Eq. (3) or it cannot. In the former case, as we shall show in the case of the generalized Fokker-Planck equation (generalized by the presence of F^{2-q} in the diffusion term), C depends on the distribution functions, F. This leads to a dependence of q on the other Lagrange multipliers. Should C only be a function of the Lagrange multipliers we will show that within the framework of the MAXENT formalism q cannot be determined without recourse to a fit to the existing data. In the latter case q may be determined from the equation of constraint.

In the first case all equations of constraint are of the form of Eq. (3). We consider the following nonlinear one-dimensional generalized Fokker-Planck equation (GFPE) [13]:

$$\frac{\partial F(x,t)}{\partial t} = -\frac{\partial}{\partial x}[K(x)F(x,t)] + \frac{1}{2}Q\frac{\partial^2 F^{2-q}(x,t)}{\partial x^2}, \quad (5)$$

where F is the distribution function, Q is the diffusion coefficient, and $K(x)$ is the drift coefficient which determines the potential:

$$V(x) = -\int_{x_0}^{x} K(x)dx. \quad (6)$$

The particular power $q-2$ is chosen in accordance with the quite general discussion of the generalized Bogulubov inequalities [14], which points out that systems which obey Tsallis statistics exhibit abrupt changes at $q = 2$. An exact solution (both static and time dependent) of the GFPE has been found and is shown under certain circumstances to be equivalent to the Tsallis classical distribution functions [4].

In the static case we formulate and solve the equivalent MAXENT problem where C is an explicit function of the distribution function. In the GFPE case one has

$$K(x)F(x) = \frac{Q}{2}\frac{\partial F^{2-q}(x)}{\partial x}, \quad (7)$$

$$\int_{x_0}^{x} K(x)F(x)dx = \frac{Q}{2}[F^{2-q}(x) - F^{2-q}(x_0)]. \quad (8)$$

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Integrating by parts [$K(x) = -\frac{\partial V}{\partial x}$ and $V(x_0) = 0$],

$$V(x)F(x) = \int_{x_0}^{x} V \frac{\partial F(x)}{\partial x} dx - \frac{Q}{2}[F^{2-q}(x) - F^{2-q}(x_0)] \tag{9}$$

or

$$\mathrm{Tr}(VF^q) = \mathrm{Tr}\Bigg[F^{q-1}(x)\bigg(\int_{x_0}^{x} V \frac{\partial F(x)}{\partial x} dx - \frac{Q}{2}(F^{2-q}(x) - F^{2-q}(x_0))\bigg)\Bigg]. \tag{10}$$

Now Eq. (7) is solved by the solution obtained in [13],

$$F(x) = D[1 - \beta(1-q)V(x)]^{\frac{1}{1-q}}. \tag{11}$$

Note in this case the authors have simply shown that a solution of the form given in Eq. (4) is a solution of the GFPE. One can verify by substitution that Eq. (4) is a solution to Eq. (7) provided

$$\beta = \frac{2}{Q}\frac{D^{q-1}}{2-q}. \tag{12}$$

With $D = 1$ the solution is the same as that obtained from the above MAXENT equations [with Eq. (10) as the equation of constraint]. Again the equation of constraint is only satisfied if β is given by Eq. (12) (with $D = 1$). Note in this case there is no solution if $V(x)$ is a constant.

Generally the C in the equation of constraint is a constant rather than an explicit function of the distribution function. In order to simultaneously determine q it has been suggested [5] that an additional equation,

$$\left.\frac{\partial S}{\partial q}\right|_\beta = 0, \tag{13}$$

must be solved. However it should be noted that this equation can be rewritten as

$$\left.\frac{\partial S}{\partial q}\right|_\beta = \left.\frac{\partial S}{\partial F}\right|_\beta \left.\frac{\partial F}{\partial q}\right|_\beta \tag{14}$$

$$= 0. \tag{15}$$

Since Tsallis distribution functions [Eq. (4)] are the solutions of Eqs. (2) and (3) the above equation is trivially satisfied for any value of q. Hence q cannot be determined in this manner and one has no choice but to determine q from the existing data. For the practitioners this has been accepted *de facto* and calculations involving the Tsallis entropy have generally used one piece of data to determine the value of q, which is then fixed over the range of interest [12].

In the second case an additional equation of constraint exists which cannot be brought into the form of Eq. (3). Such is the case in obtaining the distribution functions of the finite temperature BCS equations with the Tsallis single-particle entropic measure. Aside from the self-consistency requirements of the single quasiparticle energies, which are needed, the determination of the distribution functions follows from Eq. (2) along with the appropriate equations of constraint. The existence of a critical point where the gap vanishes yields an additional equation of constraint not of the form given by Eq. (3). As has been shown recently this can be used to determine q [15,16].

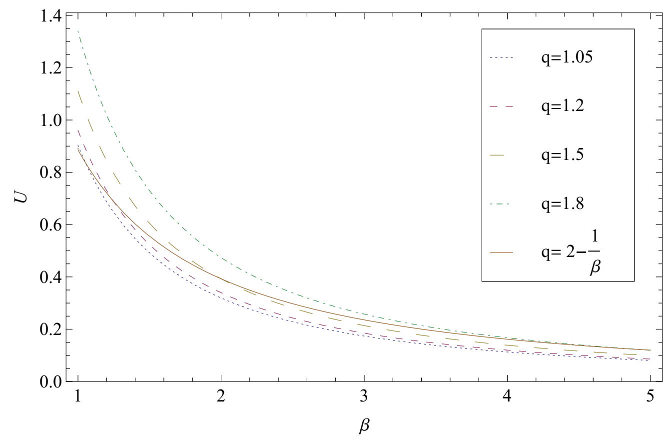

FIG. 1. (Color online) U vs. β for $V(x) = x^2$ and $D = 1$ for fixed values of q and for $q = 2 - \frac{1}{\beta}$, which corresponds to the solution obtained in [13] with $Q = 2$.

The important issue addressed here is how the nonextensive parameter q is to be determined from known data and constraints. We have pointed out that there are two distinct approaches in determining q depending on the specific form of the equations of constraint of the system. If the equations of constraint can be written as $\mathrm{Tr}[F^q\hat{O}] = C(F)$, where C may be an explicit function of the distribution function F, then one can solve an equivalent MAXENT problem which yields q as a function of the corresponding Lagrange multiplier. This is the case, for example, in the GFPE. Exact solutions to the stationary GFPE were obtained via MAXENT, which yielded q as a function of the Lagrange multiplier β. Should C only be a function of the Lagrange multipliers, q cannot be determined without recourse to a fit to the existing data. In the second case an additional equation of constraint exists which cannot be brought into this form. In this case the additional equation of constraint may be used to determine the fixed value of q.

Last we wish to point out that we have been able to obtain the exact solution of the GFPE given in [13] from the Tsallis

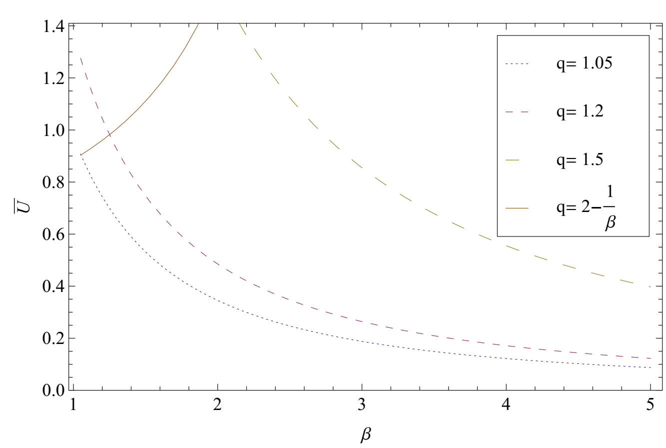

FIG. 2. (Color online) $\bar{U}$ vs. β for $V(x) = x^2$ and $D = 1$ for fixed values of q and for $q = 2 - \frac{1}{\beta}$ with $Q = 2$.

entropy via MAXENT. This is a clear indication that this is the proper choice of entropy for such a system. In the static or equilibrium case the internal energy is given by

$$U_q(\beta) = \mathrm{Tr}[F^q(x)V(x)]. \tag{16}$$

In Fig. 1 a plot of $U(\beta)$ for $V(x) = x^2$ is given for different choices of q. Note that, since the internal energy is not defined as

$$\bar{U}_q(\beta) = \mathrm{Tr}[F(x)V(x)], \tag{17}$$

the H theorem obtained by Shino [17] no longer holds. Figure 2 shows $\bar{U}_q(\beta)$ for different choices of q. It is interesting to note that, in the case of the calculation of the internal energy, $U_q(\beta)$, as a function of β for a fixed value of q for $1.05 \leqslant q \leqslant 1.8$, is in reasonable agreement with the results obtained using the relation $\beta = \frac{1}{2-q}$. This unfortunately is not the case for $\bar{U}_q(\beta)$.

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[2] N. Ito and C. Tsallis, Nuovo Cimento D **52**, 907 (1988).
[3] S. Abe and A. Okamoto, *Nonextensive Statistical Mechanics and Its Applications* (Springer, Heidelberg, 2001).
[4] A. Plastino and A. R. Plastino, Braz. J. Phys. **29**, 50 (1999).
[5] A. R. Plastino, H. G. Miller, and A. Plastino, Continuum Mech. Thermodyn. **16**, 269 (2004).
[6] E. T. Jaynes, *Probability Theory: The Logic of Science* (Cambridge University Press, Cambridge, UK, 2003), http://www.bibsonomy.org/bibtex/2ed3616cca9af65830fb13b9f53e0f19b/josephausterwei
[7] C. Tsallis, Braz. J. Phys. **29**, 79 (1999).
[8] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World* (Springer, New York, 2009).
[9] M. Gell-Mann and C. Tsallis, *Nonextensive Entropy Interdisciplinary Applications* (Oxford University Press, Oxford, 2004).
[10] G. Wilk and Z. Włodarczyk, Phys. Rev. Lett. **84**, 2770 (2000).
[11] U. Tirnakli, C. Beck, and C. Tsallis, Phys. Rev. E **75**, 040106 (2007).
[12] http://tsallis.cat.cbpf.br/biblio.htm
[13] A. R. Plastino and A. Plastino, Physica A (Amsterdam, Neth.) **222**, 347 (1995).
[14] A. Plastino and C. Tsallis, J. Phys. A **175**, L893 (1993).
[15] H. Uys, H. Miller, and F. Khanna, Phys. Lett. A **289**, 264 (2001).
[16] J. M. Conroy and H. G. Miller, Phys. Rev. D **78**, 054010 (2008).
[17] M. Shino, J. Math. Phys. **42**, 2540 (2001).