Cusps and cuspidal edges at fluid interfaces: Existence and application

R. Krechetnikov
Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada AB T6G 2G1
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One of the intriguing questions in fluid dynamics is on the interrelation between dynamic singularities in the solutions of fluid dynamic equations – unboundedness of the velocity field in an appropriate norm – and the geometric ones – divergence of curvature at fluid interfaces. The present work focuses on two generic interfacial singularities – genuine cusps and cuspidal edges – found here in both two and three dimensions thus establishing a relation between real fluid interfaces and geometric singularity theory. The key finding is the necessary condition for the existence of geometric singularities, which is a variation of surface tension. It is also established here that the dynamic and geometric singularities entail each other only in the case of three-dimensional cusps. Explicit asymptotic solutions for the flow field and interface shape near steady-state singularities at fluid interfaces are developed as well. The practical motivation for the present study comes from the fundamental role interfacial singularities play in sustaining self-driven conversion of chemical into mechanical energy.

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The appearance of singularities in the solutions of nonlinear partial differential equations, that is, when modeled physical quantities become unbounded, is a consequence of the simplification continuum theory gives over the molecular level description and usually indicates that some important unresolved physics takes place [1]. On the other hand, singularities can also exhibit themselves geometrically as that of curves in two dimensions and surfaces in three dimensions, e.g., in the context of fronts [2–4] and fluid interfaces [5–7], when geometric quantities (e.g., curvature) diverge.

Indeed, singularities in the solutions of fluid dynamics equations – the focus of the present work – can emerge not only as a divergence of the velocity field in an appropriate norm, which is a central question in the theory of existence of solutions [8], but also geometrically as a divergence of curvature at fluid interfaces, which is usually avoided in the existence studies of fluid dynamics with interfaces [9,10] (i.e., the interface is usually assumed to be smooth throughout the entire evolution). The interrelation between these two kinds of singularities is the *first key question* addressed in the present study and brings together topological and analytical views of fluid dynamics, which are arguably equally important [11,12]. Often, mathematical singularities occur when viscosity and/or surface tension is neglected [13]. The present study shows that one can get a singularity even if these physical effects are both present. While singular solutions are known in the dynamics of viscous flows, especially in fixed geometries such as the Jeffrey-Hamel flow in a converging channel [14] and on a polygon [15], in problems with free interfaces primarily corner [14,16] and cone [5] type solutions were studied, e.g., in the context of Taylor cones [17,18] and chemical-reaction driven tip streaming [19].

The present work focuses on two generic, according to Whitney's theory [20], types of interfacial singularities [21] – *cusps* and *cuspidal edges* shown in Fig. 1 – constructed here in both two and three dimensions thus establishing a relation between real physical interfaces and singularity (catastrophe) theory [22]. Cusps differ from cone singularities as the angle at their apex vanishes, and are known to play an important role in many other areas of physics, e.g., gravitational lensing [23], cuspy halo in cosmology [24], and day-side cusps in magnetosphere [25], to mention a few.

In fluid dynamics, approximations [26] to cusps in the framework of macroscopic (continuum) theory were studied before in two dimensions with the methods of complex variable theory by Jeong and Moffatt [27] in the case of clean interface and by Antanovskii [6] in the presence of surfactants; however, due to the requirement of analyticity of a conformal mapping, these studies were limited to regular solutions, i.e., when the interfacial curvature remains finite except for the case of surface tension vanishing everywhere [27], not just locally. In contrast, the *second key question* in the present work is on the necessary condition for the existence of the genuine cusp and cuspidal edge singularities, which, as will be shown here, is a variation of surface tension thus bringing Marangoni phenomena [28] – fluid flows resulting from variations of interfacial tension – into the picture. As opposed to the singularities forced externally [29–34], cf. Fig. 2(a), self-driven Marangoni singularities have not been thoroughly studied. Marangoni-driven flows exhibiting interfacial singularities, which motivated the present study, were found experimentally only recently [19,36–39] and are shown in Figs. 2(b) and 2(c).

On the theoretical side, it was recently demonstrated that the existence and topology of the observed interfacial singularities driven by Marangoni effects can be deduced using mean-curvature flow theory extended to account for variations of interfacial tension [40]. This, in turn, suggests that some of the physical mechanisms underlying the formation of these interfacial singularities originate from the surface tension flow, but existence and the actual form of these solutions were not demonstrated from first principles, i.e., as satisfying the Navier-Stokes equations (NSEs) governing fluid motion [14].

Since there can be a multitude of physical problems leading to singularities with the same asymptotic behavior and the task of constructing a general global analytic solution is formidable, the analysis offered here is local in nature, which is sufficient for our purpose of establishing only necessary conditions for the existence of singularities. With the asymptotic analysis to follow we will be able to answer both key questions

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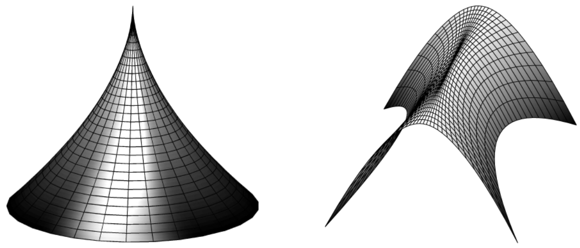


(a) (b)

FIG. 1. Two of the generic singularities [22] in three dimensions relevant to the situation when there is a continuum media (fluid) on one side of the interface: (a) cusp, (b) cuspidal edge (singular part of a cuspidal lip [4] – its transverse cross section is a two-dimensional cusp).

formulated above. In fact, a local approach is very common when analyzing solutions near geometric singularities, e.g., in fluid dynamics [14,41–43], electrodynamics [44], and in a general mathematical context [45]. Such constructed leading order local solutions are determined up to some unknown constants, whose values can be found from the global solution only as is common for elliptic problems [46].

In two dimensions, the analysis is relevant for both studying three-dimensional (3D) cuspidal edges, cf. Fig. 1(b), as well as cusps in two dimensions, which can be considered as a transverse cross section of cuspidal edges. Hence, we will start with incompressible viscous flow formulation, which in two dimensions can be written in terms of the stream function ψ defined such that the x- and y-velocity components are given by $(u,v) = (\psi_y, -\psi_x)$:

$$(\psi_y \partial_x - \psi_x \partial_y)\Delta\psi = \nu \Delta^2 \psi, \tag{1}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian and ν is the kinematic viscosity. The problem is closed with dynamic normal and tangent boundary conditions at the cusp interface $h(x) = c\, x^\alpha$, where $0 < \alpha < 1$ and for negative x one takes $-x$:

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = -\sigma\, \boldsymbol{\nabla} \cdot \mathbf{n}, \tag{2a}$$

$$\mathbf{t} \cdot \mathbf{T} \cdot \mathbf{n} = \mathbf{t} \cdot \boldsymbol{\nabla}_s \sigma, \tag{2b}$$

as well as the kinematic boundary condition:

$$\mathbf{u} \cdot \mathbf{n} = 0, \tag{3}$$

i.e., $-h_x u + v = 0$. Here $T_{ij} = -p\, \delta_{ij} + 2\, \mu\, e_{ij}$ is the stress tensor, e_{ij} is the rate of strain tensor, μ is the dynamic viscosity, p is the pressure, σ is the interfacial tension, $\mathbf{t}$ and $\mathbf{n}$ are the tangential and outer (with respect to the fluid phase) normal vectors, respectively, $\boldsymbol{\nabla}_s$ is the interfacial gradient, s is the arc length parametrizing the interface, and $\boldsymbol{\nabla} \cdot \mathbf{n}$ is the interface curvature, which diverges as $\boldsymbol{\nabla} \cdot \mathbf{n} \simeq h_{xx}/h_x^3 = O(x^{1-2\alpha})$.

Without loss of generality, we will place the system of coordinates at the apex of the singularity, which will simplify notations and the form of solutions. In the case when inertia in (1) can be neglected, which will be verified *a posteriori*, the analysis of (1) reduces to the Stokes approximation

$$\Delta^2 \psi = 0, \tag{4}$$

i.e., the solution is considered at small spatial scales where viscosity dominates. If one represents the stream function as $\psi(x,y) = x^\beta y^\gamma$, Eq. (4) admits any values of the exponents β and γ from the following sets: $(\beta,\gamma) = (\{0,1,2,3\},\{0,1\})$ or $(\{0,1\},\{0,1,2,3\})$, which implies that a solution can be a combination of any admissible values of the exponents β and γ thus giving the following basic elements of the stream function:

$$\psi = \{1; x, x^2, x^3; y, y^2, y^3; x\, y, x\, y^2, x\, y^3; x^2\, y, x^3\, y\}. \tag{5}$$

In addition, in the context of cusp singularities it will be also relevant to consider solutions of the form

$$\begin{aligned} \psi(x,y) &= \psi_0(x) + \psi_1(x,y) + \cdots \\ &= d_0 x^{\beta_0} + d_1 x^{\beta_1} y^{\gamma_1} + \cdots, \end{aligned} \tag{6}$$

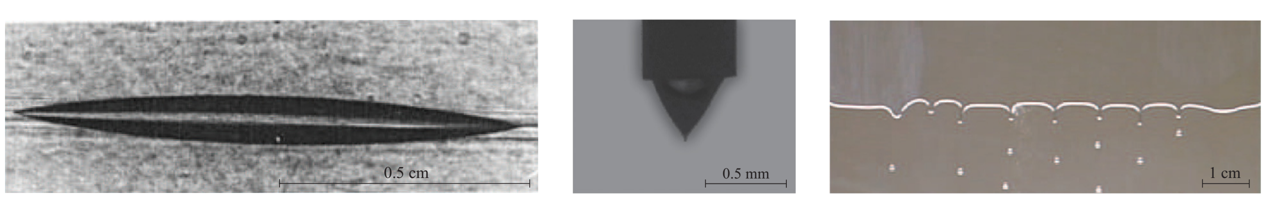
which differs substantially from the works of Lugt [42] and Brøns [41] on nonsingular interfaces in the sense that Taylor series expansions (i.e., with integer powers) are no longer applicable as the solutions are not expected to be regular near cusps – instead, the first term depends on one coordinate only and the second term possesses noninteger powers. Leaving aside the irrelevant solutions corresponding to $\beta_0 = 0$ (which gives the constant leading-order term) and $\gamma_1 = 0$ (which gives $\beta_0 = \beta_1$ and no dependence on y), we arrive at the following possible exponents:

$$(\gamma_1, \beta_1) = (2, \beta_0 - 2) \quad \text{or} \quad (4, \beta_0 - 4);\ \beta_0 = \{1,2,3\}. \tag{7}$$

Let us identify the conditions under which a solution local near a cusp singularity exists in the Stokes approximation. The corresponding stream function (6) satisfies the kinematic boundary condition (3) provided

$$\beta_1 = \beta_0 - \alpha \gamma_1, \tag{8a}$$

$$d_0 + d_1\, c^{\gamma_1} = 0, \tag{8b}$$



(a) (b) (c)

FIG. 2. (Color online) Physical examples of steady-state interfacial singularities: (a) drop deformed due to external forcing of an extensional flow [17,35] – viscous stresses in the surrounding phase deform the drop, which forms pointed ends; (b) chemical reaction-driven tip streaming [19,36,37] – the acid-base chemical reaction at the water-oil interface produces a surfactant, which drives the Marangoni flow along the interface leading to a conical shape of the drop with a singular *cone tip*; (c) surfactant-driven fingering [38,39] – soapy water displaces air in the narrow space between two glass plates (Hele-Shaw cell) and eventually leads to fingering with *cusps* between the fingers.

which allows us to rewrite the formula (6) as

$$\psi(x,y) = d_0 x^{\beta_0} + d_1 x^{\beta_0}\left(\frac{y}{x^{\alpha}}\right)^{\gamma_1} + \cdots, \tag{9}$$

where the ratio $y/x^{\alpha} \geqslant 1$ in the interior of the singularity (fluid phase). Thus, to make the representation (6) meaningful one needs to bring the second term to the next order of approximation compared to the first term, i.e., one must have

$$\gamma_1 < 0. \tag{10}$$

Note that because of (8b), the stream function vanishes at the interface $\psi = 0$ (at the leading order) as it should. The determined conditions (8) and (10) narrow down the class of possible solutions (6) [47].

The dynamic boundary conditions (2) produce at the leading order

$$-\mu\, h_x \psi_{xx} = \sigma_x, \tag{11a}$$

$$-p + \frac{2\,\mu}{h_x}\psi_{xx} = \sigma\frac{h_{xx}}{h_x^3}. \tag{11b}$$

Given the form of the interface $h(x) = c\,x^{\alpha}$ and the leading order term of the stream function $\psi_0 = d_0\,x^{\beta_0}$, one gets the following key scalings for the surface tension and pressure when $\beta_0 = \{2,3\}$:

$$\sigma \sim x^{\alpha+\beta_0-2}, \tag{12a}$$

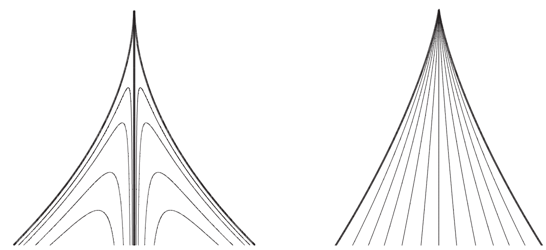
$$p \sim x^{-\alpha+\beta_0-1}, \tag{12b}$$

respectively. Note that in this case ($\beta_0 = \{2,3\}$) both surface tension and pressure are bounded and vanish at the cuspidal point as $x \to 0$; the y component of the velocity field scales as $O(v) = x^{\beta_0-1}$ and thus vanishes as well, for $x \to 0$. Also, the vorticity, $\omega = -\Delta\psi$, scales as $\sim d_0\,x^{\beta_0-2}$, i.e., is bounded near the cusp apex. The surface tension gradient takes the form

$$\sigma_x = -\mu\, c\,\beta_0(\beta_0-1)\,d_0 x^{\alpha+\beta_0-3}, \tag{13}$$

where, as was pointed out before, the unknown constants can be determined only from knowledge of the global solution, i.e., the boundary conditions away from the singularity set the values of these constants. The presence of unknown constants in (13) is also a reflection of the self-similar character of the solution (12) since there is no characteristic length scale in the problem. This also suggests that Marangoni-driven singularities can be physically realized on different length scales (as long as the flow near the singularity is in the Stokes regime). In the context of the cases $\beta_0 = \{2,3\}$, it must be noted that existence of a genuine two-dimensional (2D) cusp with vanishing surface tension at the cuspidal point was predicted by Antanovskii [6] in the particular case of the flow driven externally by two counter-rotating cylinders [27]. Also, the mesoscale theory of Pismen [48], which aimed to resolve the singularity in the same problem with constant nonzero surface tension [27], showed that the cusp is formed due to a decrease of surface tension, caused by mesoscopic physical phenomena, i.e., different from the result established here within the framework of the continuum theory (NSEs).

As for the case $\beta_0 = 1$, it corresponds to ejection of fluid as in the tip-streaming mode ($v \sim$ const; cf. Fig. 4) and gives the variation of surface tension $\sigma \sim x^{\alpha-1}$ at the next order thanks



(a) (b)

FIG. 3. Flow fields – streamline patterns – in (a) 2D cusp and 3D cuspidal edge and (b) 3D cusp – cross section at $y = 0$.

to the contribution of ψ_1, while pressure scales as $p \sim x^{-\alpha}$; thus both surface tension and pressure are singular in this case [49] and the flow (recirculation) direction in Fig. 3(a) is the opposite to that in the cases $\beta_0 = \{2,3\}$.

Concluding the discussion of the 2D case, we can verify that the constructed solutions indeed correspond to the Stokes flow approximation at the leading order despite their singular behavior [50]. Noting that since $|x| \ll 1$ and $0 < \alpha < 1$ in the fluid domain we have $x \ll x^{\alpha} \leqslant y$, so that differentiation with respect to x dominates that with respect to y and hence the leading order part of (1) is

$$-\psi_{0x}\,\Delta\psi_{1y} = \nu\Delta^2\psi_0. \tag{14}$$

Plugging (9) in (14) and evaluating nonlinear inertia and viscous terms and normalizing with respect to the latter leads to

$$x^{\beta_0}\left(\frac{y}{x^{\alpha}}\right)^{\gamma_1}\left(\frac{y}{x}\right)^{-1};\quad x^{\beta_0}\left(\frac{y}{x^{\alpha}}\right)^{\gamma_1}\left(\frac{y}{x}\right)^{-3};\ 1. \tag{15}$$

Since the first two terms representing the order of magnitude of nonlinear inertia are asymptotically small compared to the last (Stokes) term for each of the three possible cases, $\beta_0 = \{1,2,3\}$, one concludes that inertia does not contribute at leading order.

In three dimensions, while it is tempting to perform the analysis of cusps in the Cartesian system of coordinates as was done in the 2D case above, the interface representation $y = h(x,z) = c\,x^{\alpha_x}\,z^{\alpha_z}$ does not correspond to a cusp. Since the goal is to establish existence of cusps in three dimensions, it is sufficient to consider the axisymmetric case, cf. Fig. 1(a),

$$\theta = h(r) = c\,r^{\alpha}, \quad \alpha > 0 \quad \text{s.t.} \quad \theta \to 0 \quad \text{as}\ r \to 0. \tag{16}$$

In the mathematical formulation we choose to work with spherical coordinates centered at the apex of the cusp; cf. Fig. 1(a): $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, where z is directed along the axis of symmetry, $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$. Defining the interface in terms of (r,θ) variables, $\theta = h(r)$, the interfacial curvature is calculated via $\nabla\cdot\mathbf{n} \simeq (r\,h)^{-1}$.

Representing the solution for the axisymmetric stream function $\psi(r,\theta) = r^n\,f(\theta)$ and taking into account that the cusp shape is given by (16) we will look for solutions in the narrow sector of angle $|\theta| \ll 1$, i.e., $f(\theta) = d\,\theta^m$, so that the corresponding axisymmetric biharmonic problem in spherical coordinates

$$E^2\psi = 0, \quad \text{where} \quad E = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right), \tag{17}$$

produces

$$m(m-4)(m-2)^2 + \tfrac{2}{3}m(m-2)[8+3(n-3)n+m]\theta^2 + O(\theta^4) = 0, \tag{18}$$

that is for any n the leading order solution corresponds to $m = \{0,2,4\}$.

It is natural to expect the solution for the stream function $\psi(r,\theta)$ in the form analogous to (6), but the first term cannot be $\psi_0(\theta)$ simply because it would produce $v = 0$ and $u \sim r^{-2}$ so that the kinematic boundary condition (3) would imply that $h = \text{const}$, i.e., a cone-type interface, which contradicts the assumption of a cusp (16). Thus, one needs to assume

$$\psi(r,\theta) = \psi_0(r,\theta) + \psi_1(r,\theta) + \cdots, \tag{19}$$

which, of course, implies that the leading order term gives both r- and θ-velocity components to be nonvanishing $u,v \neq 0$ – this is the key difference from the 2D case. However, the θ component of velocity has lower singularity compared to the r component; namely, if at any order

$$\psi(r,\theta) = d\, r^n\, \theta^m, \tag{20}$$

then

$$u = d\, m\, r^{n-2}\theta^{m-2}, \quad v = -n\, d\, r^{n-2}\theta^{m-1}. \tag{21}$$

Given the allowed values $\{0,2,4\}$ of m, one concludes that $m = 0$ is not relevant, while $m = 4$ lets $u \to 0$ as $\theta \to 0$ provided $n \geqslant 2$.

The kinematic boundary condition (3) at $\theta = h(r)$ gives

$$m\,\alpha = -n, \quad \text{i.e., } \psi(r,\theta) = d\, r^{-m\alpha}\theta^m, \tag{22}$$

and at each order ψ_0, ψ_1, ... are balanced individually as opposed to the 2D case. As it should be, according to (22), the stream function at the interface assumes a constant value $\psi = d\, c^m$. Also, the condition (22) suggests that since $m = 2,4$ and $\alpha > 0$, there should be $n < 0$, which implies that the fluid velocity is unbounded at the cusp apex.

Balancing the capillary, viscous, and Marangoni effects in the dynamic conditions (2) leads to the following scalings for the surface tension and pressure:

$$\sigma \sim r^{n-2+\alpha(m-1)} = r^{-2-\alpha}, \tag{23a}$$

$$p \sim r^{n-3+\alpha(m-3)} = r^{-3-2\alpha}, \tag{23b}$$

where the restriction (22) imposed by the kinematic boundary condition was used. Clearly, both surface tension and pressure are divergent at the cusp apex, which distinguishes the 3D case from the results in the 2D case. As for vorticity, it proves to be singular as well:

$$\omega = -\frac{1}{r\sin\theta}E\psi \sim r^{-m\alpha-3}\theta^{m-3}. \tag{24}$$

Thus, the *key distinction* between the constructed solutions (12) in two dimensions and (23) in three dimensions is that the former admits a singular interface shape with nonsingular surface tension, velocity, and pressure fields. This entails the difference in the flow patterns between the 2D and 3D cases; cf. Figs. 3(a) and 3(b), respectively. From the mathematical point of view, a singularity of the velocity field at the apex of a cusp or cuspidal edge can be understood in two ways: (a) abstractly, as the condition for the existence of a singularity in the fluid dynamics equations, and (b) practically, as an approximation of reality in the same way as, for example, the self-similar solution in the chemical-reaction driven tip streaming [5,19] is singular at the apex of the cone, but real physical effects may limit the existence of (smoothen out) the actual singularity at the submacroscopic level, cf. Fig. 2(b). Resolution of interfacial singularities should be performed at the mesoscopic level as was done, for example, in the context of the vortex dipole flow [27] by Pismen [48] through nanoscale molecular interactions. In general, such an effort would require the use of a microscale theory of interfaces [51] to be generalized onto the case of surface tension varying due to interfacial chemistry (surfactants and/or chemical reactions) and due to curvature [52–54].

The key fact established here is that the existence of cusp or cuspidal edge singularities accompanied by diverging curvature *requires* a variation of surface tension (and thus Marangoni flow) along the interface both in two and three dimensions with either surface tension vanishing or diverging at the singularity apex. In the former case ($\sigma \to 0$), vanishing surface tension occurs, e.g., due to very high concentration of surfactants, which is called ultralow surface tension and achievable in practice [55]. In the latter case ($\sigma \to \infty$), this implies that the interface either becomes clean (free of surfactant) or rigid as surface energy increases significantly compared to that of the liquid and in practice means relatively large values of σ. One must note that there are several physical mechanisms due to which surface tension varies substantially along the interface in nature and applications – this can be due to the presence of surface active substances (soap molecules, i.e., so-called surfactants) [56], temperature gradients along the interface [57], or a nonuniform electric field [58] – and believed to assume even negative values [59,60]. In the case of chemically driven surface tension variations, they range from 0.1 [55] to 600 mN/m for liquid gallium; in the case of water interface, surface tension may vary from 72 mN/m for clean interface down to below 0.1 mN/m when the surfactant is produced by a chemical reaction. Such variations of several orders of magnitude warrant the existence of macroscopic singularities at interfaces both in the case when the mathematical analysis presented in this paper requires either vanishing or diverging surface tension at the point of singularity. The divergence of surface tension is understood in the same asymptotic sense as the singularity itself, i.e., surface tension having much higher value at the singularity compared to that away from the singularity (a similar argument applies to the case of surface tension vanishing at the singular point). Depending on the actual range of surface tension values, the singularity may propagate down to the length scales at which the continuum assumption built into the NSEs is no longer valid.

While the above theoretical arguments suggest existence of both types of generic geometric singularities, i.e., cusps and cuspidal edges, the observability of these singularities depends on a few factors. Gravity, for example, limits the existence of these singularities to small scales when the gravity effects can be neglected [5]. Also, the realistic limited range of surface tension values and dynamic stability of singularities to time-dependent perturbations may affect their observability as well, which, in fact, is suggested by the observations in Fig. 2 – while the singularities appear to be at the macroscopic scale, they are smoothed out at the microscopic level, e.g., by tip streaming which is an unsteady phenomenon. To the author's knowledge, only one study was done on time-dependent stability of singular solutions, namely by Constantin and Kadanoff [7]. However, while these authors [7] established formation of singularities in finite time, using a model problem in a Hele-Shaw cell based on Darcy's law, there was no surface tension variation in their problem and thus, given the results of the present study, formation of such singularities in the framework of the original (not simplified) fluid dynamics description (1) is not possible. Observability of the determined cusp (23) and cuspidal edge (12) solutions can be conjectured based on the existence of the cone solutions shown in Fig. 2(b) – the cusp should be generic too as any topological perturbation of a cone preserving the interfacial singularity either leaves it a cone or deforms it to a cusp [61]. And, last, the exact power-law form of the equation of state $\sigma(x)$ is not crucial for the existence of a self-similar solution as long as (a) the problem defined by (1–3) is well posed in Hadamard's sense [62] and thus not very sensitive to a variation of the coefficients (e.g., in the equation of state) in the governing equations and boundary conditions, and (b) $\sigma(x)$ is close to the power-law form for some range of x's in the sense of intermediate asymptotics [63].

In conclusion, one may ask the question: "Why is it important to study interfacial singularities?" Besides the fundamental reasons which motivated this study, from a practical point of view the interfacial instabilities often lead to singularities at a macroscopic level [19,38], which are crucial for self-sustained motions such as, for example, singularity formation is instrumental in the chemical-reaction driven tip streaming shown in Fig. 4. If one can identify other geometries relevant for the useful conversion of chemical energy into mechanical energy, they can be exploited to perform a number of functions such as pumping, propulsion, and mixing currently accomplished with complex machinery and active control. The nontrivial feature of the direct chemical-to-mechanical energy conversion is its isothermality as opposed to all heat engines, the efficiency of which is limited by the Carnot cycle, and is analogous to how all motors in living organisms, known for their efficiency, operate. On the experimental side, the fact that Marangoni effects can transfer chemical into mechanical energy directly has been known for a long time, e.g., in the context of camphor scrapings [64,65]. Indeed, as known from experimental observations, among the regimes of interfacial mechanical motion are violent and erratic pulsations [66–69], all of which indicate intermittent formation of singularities.

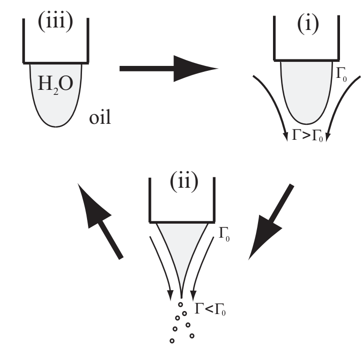


FIG. 4. A motor driven by Marangoni effects at the water-oil interface and the role of singularities: when interfacial surfactant concentration Γ reaches a critical value Γ_c, its magnitude is suddenly reduced by $\Delta\Gamma$, which implies that most of the surfactant is removed in the process of tip streaming. Once triggered, the following sequence of events takes place: (i) sweeping surfactant towards the tip of a new pendant drop, which facilitates the tearing up of the interface; (ii) tip streaming, which removes surfactant from the drop and thus increases interfacial tension, so that the surfactant concentration gradient between the top and the tip of the drop drives Marangoni flow; (iii) drop relaxation back to a round shape due to the increased interfacial tension at its tip.

Since formation of singularities is relevant to Marangoni-driven motors, cf. Fig. 4, understanding of the emergence of such singularities in unsteady solutions as well as their thermodynamics will be a crucial future step. The latter is important since the fluid kinetic energy E_{kin} dissipation rate due to viscosity is infinite in two dimensions for $\beta_0 = 1$ and in three dimensions in general:

$$\dot{E}_{\text{kin}} \sim \begin{cases} x^{2\beta_0+\alpha-3} & \text{in two dimensions;} \\ r^{-4-3\alpha} & \text{in three dimensions,} \end{cases} \tag{25}$$

due to substantial stresses applied to infinitesimally small fluid elements. This means that thermodynamics should come into play to regularize the infinite dissipation rate [70] at the microscopic level. And, in the general context, establishing a connection between singularities of solutions of fluid dynamics equations and that of real fluid interfaces might be relevant in the ongoing research on the existence of solution in three dimensions [71], when realistic boundary conditions are taken into account.

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