Bright, dark, and mixed vector soliton solutions of the general coupled nonlinear Schrödinger equations

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The reduction procedure for the general coupled nonlinear Schrödinger (GCNLS) equations with four-wave mixing terms is proposed. It is shown that the GCNLS system is equivalent to the well known integrable families of the Manakov and Makhankov $U(n,m)$-vector models. This equivalence allows us to construct bright-bright and dark-dark solitons and a quasibreather-dark solution with unconventional dynamics: the density of the first component oscillates in space and time, whereas the density of the second component does not. The collision properties of solitons are also studied.

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I. INTRODUCTION

The proof of integrability of the nonlinear Schrödinger equation (NLSE) by Zakharov and Shabat [1] via the inverse scattering transform (IST) method (i.e., solitonic "Big Bang") stimulated the search for important integrable generalizations of the NLSE. There has been great interest in natural vector generalization of the NLSE, namely, N-coupled NLSE (N-NLSE). This is due to the fact that the coupled NLSE appears in numerous branches of physics, such as nonlinear optics [2], Bose-Einstein condensates [3], biophysics [4], plasma physics [5], metamaterial technologies [6], and so on. The 2-NLSE is also integrable when the nonlinear coefficients have the same magnitudes [7–9]. The integrability of the focusing 2-NLSE has been discovered by Manakov, who first solved the 3×3 spectral problem for the vector bright soliton by IST [7]. Afterwards, it was revealed by Makhankov *et al.* that in the mixed nonlinearity case (i.e., the nonlinear coefficients have opposite signs), the system is still integrable [8,9]. Subsequent studies of the soliton solutions of these systems have led to three "traditional" vector solitons: bright-bright (BB) [7,10], bright-dark (BD) [11–13], and dark-dark (DD) [10,14]. One of the distinctive features of vector solitons is that they exhibit certain novel inelastic collision properties, which has wide potential applications in optical computers, multistate logic systems, and so on [15–17]. Based on Ref. [7] this phenomenon was first revealed in Ref. [18] for BB solitons of the focusing 2-NLSE, and it was extended to focusing N-NLSE [19,20]. Further, the nontrivial class of exact mixed type solutions (which consist of s bright and m dark solitons, $s+m=N$) of the general N-NLSE has been presented in Ref. [12], where it is shown that in the $N \geqslant 3$ case mixed solitons undergo nontrivial (with energy sharing) collision, and in the $N=2$ case undergo only standard elastic collision [11–13,20].

Since the focusing Manakov $U(2,0)$-vector model, the Makhankov $U(1,1)$-pseudovector model, and the defocusing Manakov $U(0,2)$-vector model (hereinafter referred to as *basic models*) have important theoretical and practical applications, the broad spectrum of research is aimed at constructing an integrability generalization of these models, and to retrieve their solutions. In this direction, the $(1+1)$-dimensional general coupled nonlinear Schrödinger (GCNLS) equations [21]

$$i\frac{\partial q_j}{\partial t} + \frac{\partial^2 q_j}{\partial x^2} + 2Q(q_1,q_2)q_j = 0, \quad j = 1,2, \qquad (1)$$

where the potential $Q(q_1,q_2) = a|q_1|^2 + c|q_2|^2 + bq_1q_2^* + b^*q_1^*q_2$ is the real-valued function of q_j (the asterisk is used to denote the complex conjugate), have received considerable interest recently. In fiber optic applications, in the above equation, $q_j(x,t)$ represent slowly varying pulse envelopes, and a and c simultaneously account for the self-phase modulation (SPM) and cross-phase modulation (XPM) strengths. The additional phase-dependent terms $(bq_1q_2^* + b^*q_1^*q_2)q_j$ with complex parameter b describe the four-wave mixing (FWM) effect which arises in multichannel communications systems [22]. We remark that coupled NLS equations similar to system (1), but with different phase-dependent nonlinear terms $q_j^* \sum_{k=1,2} \gamma_{jk} q_k^2$ or $\sum_{k=1,2} \gamma_{jk} q_k^* q_{3-k}^2$ in the jth equation, and their integrable cases have been studied previously (see, for example Refs. [23,24] and references therein). The GCNLS system (1) is also completely integrable. Its Painleve-integrability is examined in Ref. [25]. The Lax pair and N-soliton BB solution are derived in Ref. [21]. The BB one- and two- soliton solutions have also been obtained in Ref. [26] through Hirota's bilinear method (HBM). The DD soliton solution in $a<0, c<0$ case and a wide range of rational solutions of the GCNLS equations (1) such as the Kuznetsov-Ma soliton, the Akhmediev breather, and roque wave solution are obtained in Ref. [27] by HBM. In the same paper, the impact of the FWM parameter was also examined. Very recently, in Refs. [28,29] the rogue wave solutions of Eqs. (1) have been obtained via constructing a generalized Darboux transformation. Some comments on Eqs. (1) can be found in Ref. [30].

Evidently, when $b=0$ the system (1) is reduced to the basic models by $q_1 \to \tilde{q}_1/\sqrt{2|a|}$, $q_2 \to \tilde{q}_2/\sqrt{2|c|}$. The main purpose of our paper is to show that the Eqs. (1) can be reduced to the basic models in arbitrary a,b,c parametric cases. It is then possible to use this fact to construct previously known BB, DD solitons in a rather easy and straightforward way. At the same time, we construct a mixed quasibreather-dark (QBD) solution with unusual space- and time-density oscillations in

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TABLE I. The type of the resulting model for various domains of the parameters a, b, c.

Case	a	Constraint on b and c	Symmetry of resulting model
(a)	$a > 0$	$\|b\|^2 < ac$	$U(2,0)$
(b)	$a > 0$	$\|b\|^2 > ac$	$U(1,1)$
(c)	$a < 0$	$\|b\|^2 > ac$	$U(1,1)$
(d)	$a < 0$	$\|b\|^2 < ac$	$U(0,2)$

this way. We find that even though XPM and SPM coefficients are negative (positive), Eqs. (1) would nevertheless admit BB (DD) solitons, unlike the basic models. We also briefly discuss the collision properties, the role of FWM terms, and the existence conditions of these solutions.

The paper is organized as follows. In Sec. II, the reduction procedure is presented. The HBM is illustrated in Sec. III. The BB and DD soliton solutions are obtained in Secs. IV and V, respectively. Section VI is devoted to the QBD solution. Finally, our conclusion is given in Sec. VII.

II. REDUCTION TO THE BASIC MODELS

First, let us consider the function $Q(q_1,q_2)$ in system (1) as a Hermitian form,

$$Q(\mathbf{q}) = \mathbf{q}^\dagger B\, \mathbf{q}$$

with $\mathbf{q} = \mathrm{col}(q_1,q_2)$ and

$$B = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}, \quad \det B = ac - |b|^2 \tag{2}$$

($\dagger$ denotes the conjugate transpose). Suppose $a \neq 0$ and c, b are arbitrary; then the self-adjoint matrix B is transformed into diagonal matrix C of the same size according to the congruent transformation $C = S^\dagger B S$, where

$$S = \begin{pmatrix} 1 & -b^* \\ 0 & a \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & 0 \\ 0 & a^2c - a|b|^2 \end{pmatrix}.$$

Now we set functions ψ_1, ψ_2 from transformation matrix S as

$$q_1 = \psi_1 - b^*\psi_2, \quad q_2 = a\psi_2, \tag{3}$$

By using transformation (3) from Eqs. (1), after some algebra we arrive at

$$i\frac{\partial \psi_1}{\partial t} + \frac{\partial^2 \psi_1}{\partial x^2} + 2a(|\psi_1|^2 + \sigma|\psi_2|^2)\psi_1 = 0, \tag{4a}$$

$$i\frac{\partial \psi_2}{\partial t} + \frac{\partial^2 \psi_2}{\partial x^2} + 2a(|\psi_1|^2 + \sigma|\psi_2|^2)\psi_2 = 0, \tag{4b}$$

where FWM terms are effectively absent and $\sigma = ac - |b|^2$. Continuing this line of reasoning, we see that the basic models can be obtained by using the scaling transformation $\psi_1 \to \tilde{\psi}_1/\sqrt{2|a|}$, $\psi_2 \to \tilde{\psi}_2/\sqrt{2|a\sigma|}$, but we retain all parameters here to conveniently define the existing regions of the solutions. The resulting system's type depends on values of SPM, XPM, and FWM parameters, as shown in Table I. We must admit that the transformation (3) is incomplete. By putting $a = 0$ and $c \neq 0$ we can see that it does not achieve the intended goal. In this case in an analogous manner we find $q_1 = c\psi_2$, $q_2 = \psi_1 - b\psi_2$, which leads to the $U(1,1)$-vector model: $i\partial\psi_j/\partial t + \partial^2\psi_j/\partial x^2 + 2\tilde{Q}(\psi_1,\psi_2)\psi_j = 0$, where $\tilde{Q} = c(|\psi_1|^2 - |b|^2|\psi_2|^2)$. We note that the parametric choice $a = 0, c = 0$ is not trivial [21,31]. In this situation the resultant reducing system will be of the same form as the above equations, with the only difference in $\tilde{Q} = (b + b^*)(|\psi_1|^2 - |\psi_2|^2)/2$, and the corresponding transformation is given by $q_1 = \psi_1/\sqrt{2} + \psi_2/\sqrt{2}$, $q_2 = bb^{*-1}\psi_1/\sqrt{2} - \psi_2/\sqrt{2}$. We also note that the problem is greatly simplified if the Hermitian form Q is degenerate ($|b|^2 = ac$). Thus we cover all possible cases of parameters, and in this paper we will consider the more general case ($a \neq 0$; b, c are arbitrary) and omit here the special cases.

Taking into account the above mentioned considerations, from now on we will focus our attention on constructing solutions of Eqs. (4), and solutions of Eqs. (1) will be recovered via the transformation (3).

III. HIROTA'S BILINEARIZATION METHOD

In this section we briefly present the standard HBM [32] for the Eqs. (4).

To obtain the bilinear form of system (4), we introduce the transformation $\psi_j = G_j/F$ where G_j are complex functions, while F is a real function. Then the bilinear form of Eqs. (4) can be presented as

$$\left(iD_t + D_x^2 - \lambda\right)G_j \cdot F = 0, \tag{5a}$$

$$\left(D_x^2 - \lambda\right)F \cdot F = 2a[|G_1|^2 + \sigma|G_2|^2], \tag{5b}$$

where Hirota's bilinear operators D_t and D_x^2 are defined by $D_\zeta^m U \cdot V = (\partial/\partial\zeta - \partial/\partial\zeta')^m U(\zeta)V(\zeta')|_{\zeta'=\zeta}$. The real constant λ is determined from the boundary conditions.

IV. BRIGHT-BRIGHT SOLITON SOLUTIONS

In order to obtain the BB n-soliton solution of system (4) we expand G_j and F with respect to the formal expansion parameter ε as follows:

$$G_j = \sum_{m=1}^{n} \varepsilon^{2m-1} g_{j,2m-1}, \quad F = 1 + \sum_{m=1}^{n} \varepsilon^{2m} f_{2m}. \tag{6}$$

From the boundary condition of the BB soliton solution $\lim_{|x|\to\infty} \psi_j = 0$ it follows that λ is zero.

One-soliton solution. We substitute the above expansions into the bilinear equations (5), then equate powers of the arbitrary parameter ε. After solving the resulting set of partial differential equations for $n = 1$ recursively, we obtain the explicit BB one-soliton solution of Eqs. (4) as $\psi_j = g_{j,1}/(1 + f_2)$, where $g_{j,1} = \alpha_j e^{\theta}$, $f_2 = re^{\theta+\theta^*}$, $r = [a|\alpha_1|^2 + a\sigma|\alpha_2|^2]/(k + k^*)^2$, $\theta = kx + ik^2t$, and k, α_j are arbitrary complex parameters. Combining this solution and transformation (3), after some simplification the BB one-soliton solution of Eqs. (1) can be written as

$$q_j = \tfrac{1}{2}A_j e^{i[k_I x + (k_R^2 - k_I^2)t]}\,\mathrm{sech}[k_R(x - 2k_I t) + d], \tag{7}$$

where $d = \ln\sqrt{r}$ represents the localization position of the soliton and the complex amplitudes A_j are defined as follows: $A_1 = [|\alpha_1|\exp(i\phi_1) - b^*|\alpha_2|\exp(i\phi_2)]/\sqrt{r}$,

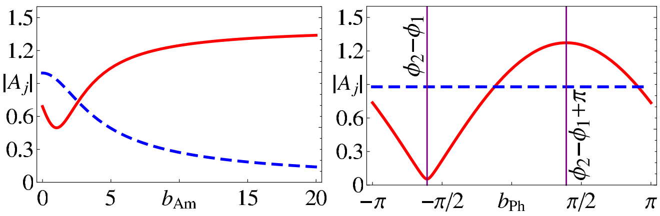

FIG. 1. (Color online) The dependence of amplitudes of solution (7) on b_{Am} (left) and b_{Ph} (right). The red (solid) and blue (dashed) curves correspond to $|A_1|$ and $|A_2|$, respectively. In the left plot b_{Ph} is fixed to $-\pi/3$ and in the right plot b_{Am} is fixed to 1.51. The other parameters are chosen as in the text.

$A_2 = a|\alpha_2|\exp(i\phi_2)/\sqrt{r}$ with $\phi_j = \arg\alpha_j$. The soliton's velocity and width in each mode are determined by k_I and k_R, respectively. Here and elsewhere, suffixes R and I indicate the real and imaginary parts. The solution (7) is valid only when the nonsingularity condition $r > 0$ is satisfied. This condition cannot be reached only in case (d) of Table I. Suppose SPM and XPM parameters are negative and $|b|^2 > ac$. Then, from case (c) it can be seen that, with the choices of parameters α_j under the nonsingularity condition, the GCNLS system (1) admits vector BB solitons, even in the $a < 0, c < 0$ case, in contrast to the defocusing Manakov model.

It should be noted that if we accept $\tilde{\alpha}_1 = \alpha_1 - b^*\alpha_2$, $\tilde{\alpha}_2 = a\alpha_2$ as the new independent parameters and rewrite r in terms of $\tilde{\alpha}_j$, then solution (7) exhibits exactly the same form as the soliton solution reported in Ref. [26].

Now, it is fascinating to analyze the role of the FWM parameter in this solution. It is clear from (7) that b only affects the amplitudes. For definiteness, we set $a = -2, c = 5$, $\alpha_1 = 4 - 3i$, $\alpha_2 = -3 - 2i$, $k = 1 - i$ and take $b = b_{\mathrm{Am}}e^{ib_{\mathrm{Ph}}}$, where $b_{\mathrm{Am}} > 0$ and $b_{\mathrm{Ph}} \in [-\pi,\pi]$. In this parametric choice, for any values of b it follows that $r > 0$. We illustrate the effect of b_{Am} and b_{Ph} in Fig. 1.

Two-soliton solution. Likewise, if we take $n = 2$ in the power series expansions (6) by using Eqs. (5) and (3) we get the BB two-soliton solution as

$$q_1 = \frac{g_{1,1} + g_{1,3} - b^*(g_{2,1} + g_{2,3})}{1 + f_2 + f_4}, \tag{8a}$$

$$q_2 = \frac{a(g_{2,1} + g_{2,3})}{1 + f_2 + f_4}, \tag{8b}$$

where functions $g_{i,j}$, f_j are defined as

$$g_{j,1} = \alpha_{j,1}e^{\theta_1} + \alpha_{j,2}e^{\theta_2}, \quad \theta_j = k_j x + ik_j^2 t,$$

$$g_{j,3} = \beta_{j,1}e^{\theta_1+\theta_1^*+\theta_2} + \beta_{j,2}e^{\theta_2+\theta_2^*+\theta_1},$$

$$f_2 = r_{1,1}e^{\theta_1+\theta_1^*} + r_{1,2}e^{\theta_1+\theta_2^*} + r_{2,1}e^{\theta_1^*+\theta_2} + r_{2,2}e^{\theta_2+\theta_2^*},$$

$$f_4 = \mu e^{\theta_1+\theta_1^*+\theta_2+\theta_2^*},$$

$$r_{i,j} = \frac{a\alpha_{1,i}\alpha_{1,j}^* + a\sigma\alpha_{2,i}\alpha_{2,j}^*}{(k_i + k_j^*)^2},$$

$$\beta_{i,j} = (k_j - k_{3-j})\left(\frac{r_{3-j,j}\alpha_{i,j}}{k_j + k_j^*} - \frac{r_{j,j}\alpha_{i,3-j}}{k_{3-j} + k_j^*}\right),$$

$$\mu = \frac{(k_2^* - k_1^*)[r_{1,2}\beta_{1,1}(k_1^* + k_2) - r_{1,1}\beta_{1,2}(k_2 + k_2^*)]}{\alpha_{1,1}(k_2 + k_2^*)(k_1^* + k_2)},$$

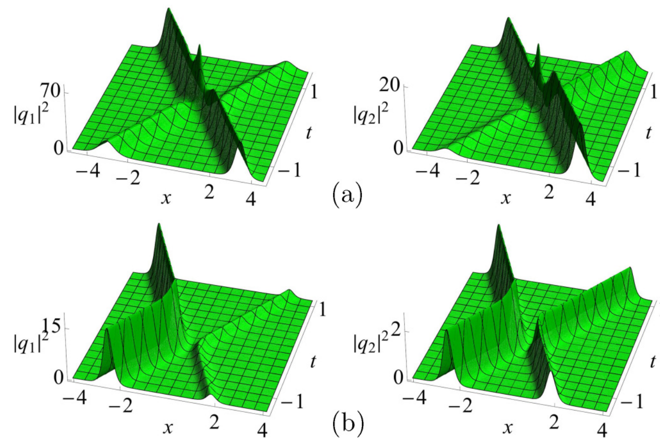

FIG. 2. (Color online) The collisional dynamics of the BB two-soliton solution (8) on the (x,t) plane. (a) Transmissional scenario of soliton collision. Parameters: $a = 1$, $c = 4$, $b = -1.9 - 0.1i$, $k_1 = -2 + i$, $k_2 = 3 - i$, $\alpha_{1,1} = 2 - i$, $\alpha_{1,2} = 1$, $\alpha_{2,1} = 1.25 + i$, $\alpha_{2,2} = -1 + i$. (b) Reflectional scenario of soliton collision. Parameters: $a = -2$, $c = 10.5$, $b = 1.8 - i$, $k_1 = 4 - 0.8i$, $k_2 = -4 + i$, $\alpha_{1,1} = 2.2 + i$, $\alpha_{1,2} = -5.93 + 0.82i$, $\alpha_{2,1} = -3 + 2i$, $\alpha_{2,2} = 1.5 - i$.

and α_j, k_j are arbitrary complex parameters. Figure 2 shows the interaction of two solitons which are well separated before and after collision. As pointed out in Ref. [21] the BB two-soliton solution possesses the intersecting behavior [*soliton transmission*, see Fig. 2(a)] as well as the repulsive behavior [*soliton reflection*, see Fig. 2(b)]. Soliton reflection is the special case of shape-changing collision when left and right solitons in each components visually do not pass through but bounce off each other when they approach. It is known that the transmission and reflection scenario of collision is an inherent property of $U(2,0)$- and $U(1,1)$-vector models, respectively [33,34]. On this basis and taking into account that q_2 is proportional to ψ_2, we have concluded that soliton transmission appears in case (a) from Table I. Accordingly, cases (b) and (c) lead to soliton reflection. Furthermore, for the ideal reflection it is necessary that the profile of the left soliton before collision and the profile of the right soliton (and vice-verse) after collision must be equal to each other. We analyzed the asymptotic profile of the solution (8) and obtained the constraints

$$k_{1R} = -k_{2R},$$

$$\frac{k_1 + k_1^*}{k_1 + k_2^*} = \frac{(|\alpha_{1,1}|^2 + \sigma|\alpha_{2,1}|^2)(b^*\alpha_{1,1}^*\alpha_{2,2} - b\alpha_{1,2}\alpha_{2,1}^*)}{(\alpha_{1,1}^*\alpha_{1,2} + \sigma\alpha_{2,1}^*\alpha_{2,2})(b\alpha_{1,1}\alpha_{2,1}^* - b^*\alpha_{1,1}^*\alpha_{2,1})}$$

for the ideal reflectional scenario of collision.

V. DARK-DARK SOLITON SOLUTION

To construct the DD soliton solutions of Eqs. (1) we will consider the solution of basic models (4) with the nonvanishing large-x asymptotics as $\lim_{x\to\pm\infty}|\psi_j|^2 = |g_{j,0}|^2$, where $g_{j,0} = \rho_j\exp[i(s_j x - p_j t + \xi_j^{0\pm})]$. Similar to Sec. III we expand G_j and F as

$$G_j = g_{j,0}\left(1 + \sum_{m=1}^{n}\varepsilon^m g_{j,m}\right), \quad F = 1 + \sum_{m=1}^{n}\varepsilon^m f_m. \tag{9}$$

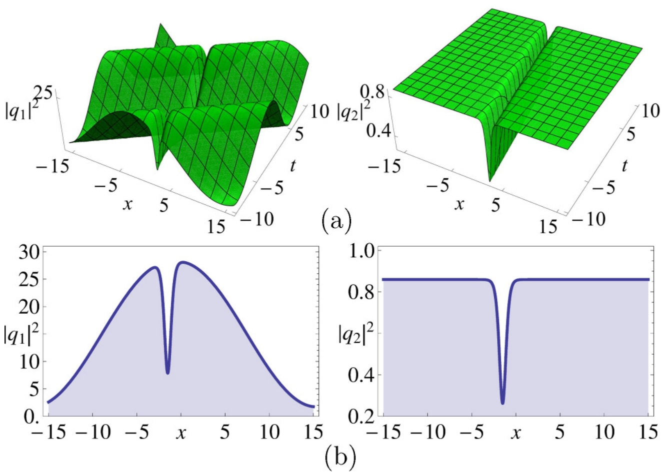

FIG. 3. (Color online) (a) The DD soliton (10) on the (x,t) plane. (b) Snapshots of the density profile for $t=-7$. Parameters: $a=1$, $c=1, k=4, s_1=1, s_2=1.2, b=3.54-0.5i, \omega=1, \rho_1=2, \rho_2=0.927, \xi_{0,1}=4, \xi_{0,2}=3.1, \theta_0=1$.

By substitution these series into Eqs. (5), and then collecting the coefficients of ε^0, it is easy to get $\lambda=-2a\rho_1^2-2a\sigma\rho_2^2$.

One-soliton solution. If we take the first-order case, $n=1$ in Eq. (9), then by solving Eqs. (5) and using (3) we get the following expressions for q_j:

$$q_1=\frac{1}{1+f_1}[g_{1,0}(1+g_{1,1})-b^*g_{2,0}(1+g_{2,1})], \quad (10a)$$

$$q_2=\frac{a}{1+f_1}[g_{2,0}(1+g_{2,1})], \quad (10b)$$

where

$$f_1=e^{\theta}, \quad \theta=kx+\omega t+\theta_0,$$
$$g_{j,0}=\rho_j e^{i\xi_j}, \quad \xi_j=s_jx-(s_j^2+\lambda)t+\xi_{j,0},$$
$$g_{j,1}=B_je^{\theta}, \quad B_j=(-k^2+2iks_j+i\omega)/(k^2+2iks_j+i\omega).$$

All parameters k, ω, ρ_j, s_j, $\xi_{j,0}$, θ_0 are real, and must satisfy the following condition:

$$\frac{\rho_1^2}{k^4+(\omega+2ks_1)^2}+\frac{\sigma\rho_2^2}{k^4+(\omega+2ks_2)^2}=-\frac{1}{4ak^2}. \quad (11)$$

This equality can be achieved in cases (b), (c), (d) of Table I. In Fig. 3 we illustrate this solution.

Reference [30] raised the question of the existence of the DD solution with background asymptotics different from the monochromatic plane wave. Without loss of generality we assume that $k>0$; then it is easy to find large distance $(x\to\pm\infty)$ asymptotics of above solution: $\lim_{x\to+\infty}q_1=\rho_1B_1e^{i\xi_1}-b^*\rho_2B_2e^{i\xi_2}$, $\lim_{x\to+\infty}q_2=a\rho_2B_2e^{i\xi_2}$, $\lim_{x\to-\infty}q_1=\rho_1e^{i\xi_1}-b^*\rho_2e^{i\xi_2}$, $\lim_{x\to-\infty}q_2=a\rho_2e^{i\xi_2}$. We see that the component q_1 resides on the superposition of two monochromatic plane waves: $\sim e^{i\xi_1}$ and $\sim e^{i\xi_2}$ (it can be clearly seen from Fig. 3).

Now let us prove that the traditional DD soliton on a monochromatic plane-wave background is a special case of the solution (10). The component q_1 is built on top of one plane-wave background iff $s_1=s_2=s$, which leads to $B_1=B_2$. Using this constraint the solution (10) can be rewritten in

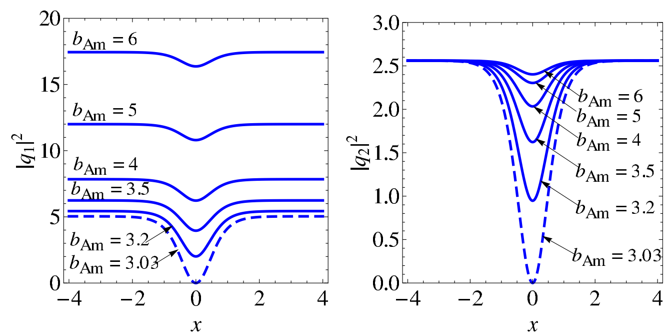

FIG. 4. (Color online) The dependence of the grayness of solution (12) on b_{Am}. The left and right plots correspond to the density profile of component q_1 and q_2, respectively. The dashed curves correspond to the black soliton. The parameter b_{Ph} is fixed to $\pi/3$. Other parameters are chosen as in the text.

terms of hyperbolic and trigonometric functions,

$$q_j=A_j\{i\sin\beta+\cos\beta\tanh[(kx+\omega t+\theta_0)/2]\}$$
$$\times e^{isx-i(s^2+\lambda)t+i\tilde{\xi}_0}, \quad (12)$$

where $\beta=-\arctan\{(\omega+2ks)/k^2\}$, $A_1=\rho_1-b^*\rho_2$, $A_2=a\rho_2$, and $\tilde{\xi}_0$ is a certain real parameter. Now the connecting identity (11) can be rewritten as

$$4\rho_1^2a\cos^2\beta+4\rho_2^2a\sigma\cos^2\beta=-k^2. \quad (13)$$

From the exact form of the DD soliton solution (12) one can easily note that components q_1 and q_2 are proportional to each other. This proportionality is a necessary condition to obtain the DD soliton which is set on the top of one plane-wave background [30].

Let us now examine the effect of the FWM term on the basic characteristics of solution (12) as a peak power (maximum value of $|q_j|^2$) and a grayness (depth of the pulse relative to background). It is clear that the peak powers of each component q_j are $|A_j|^2$ ($|A_1|^2=\rho_1^2+b_{\mathrm{Am}}^2\rho_2^2-2\rho_1\rho_2b_{\mathrm{Am}}\cos b_{\mathrm{Ph}}$ and $|A_2|^2=a^2\rho_2^2$).

The relationship between the grayness and FWM parameter is less obvious, since many of solution's parameters are closely related to each other by the implicit relationship (13). For definiteness we chose the parameters from solution (12) as $a=2$, $c=0.6$, $\omega=4$, $k=3$, $\rho_1=2$, $\rho_2=0.8$, $\tilde{\xi}_0=\theta_0=0$. For illustrative purposes we plot the density of the profile of q_j in Fig. 4 for the various values of b_{Am}. For the black (or fundamental) soliton, b_{Am} is fixed to $\sqrt{k^2+4a\rho_1^2+4a^2c\rho_2^2}/(2\sqrt{a}\rho_2)\approx 3.03$. It should be mentioned that if the above statement is not real then the black soliton cannot be reached.

Two-soliton solution. Taking into account the next ε^2 terms in expansions (9), the DD two-soliton solution can be obtained analogously. We do not present its explicit form here because of its cumbersome nature.

VI. QUASIBREATHER-DARK SOLUTION

Finally, let us present a mixed-type solution of Eqs. (1). To reach this goal we will consider the BD soliton solution of Eqs. (4). In this situation, the appropriate boundary conditions

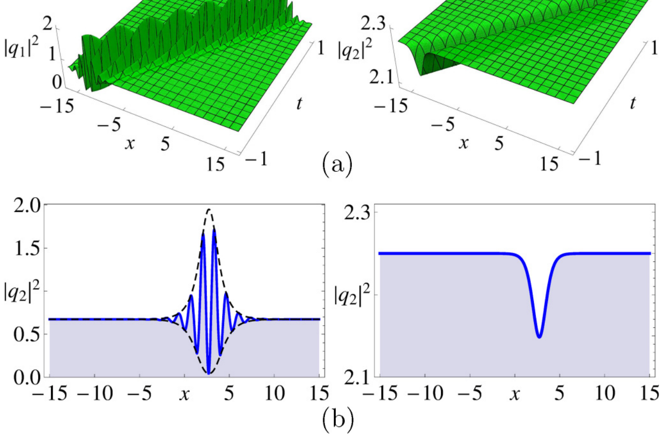

FIG. 5. (Color online) (a) The QBD soliton solution (15) on the (x,t) plane. (b) Snapshots of the density profile for $t = 0.3$. Parameters: $a = 3$, $c = 1.5$, $k = 1 + 6i$, $s = 1.4$, $\rho = 0.5$, $\alpha = 2 - 2i$, $\xi_0 = 1$, $b = 1 - 1.3i$.

are defined as $\lim_{|x|\to\infty} |\psi_j| = \delta_{2,j}\rho$, where ρ is real parameter and $\delta_{i,j}$ is Kronecker delta, and the power series are defined as

$$G_1 = \sum_{m=1}^{n} \varepsilon^{2m-1} g_{1,2m-1},$$
$$G_2 = g_{2,0}\left(1 + \sum_{m=1}^{n} \varepsilon^{2m} g_{2,2m}\right), \tag{14}$$
$$F = 1 + \sum_{m=1}^{n} \varepsilon^{2m} f_{2m}.$$

In this case λ is determined as $\lambda = -2a\sigma\rho^2$.

One-soliton solution. Suppose that $n = 1$ in expansions (14). Then, using the route as in the previous sections, we can get the next five-parametric spatially localized solution

$$q_1 = \frac{1}{1 + f_2}[g_{1,1} - b^* g_{2,0}(1 + g_{2,2})], \tag{15a}$$
$$q_2 = \frac{a}{1 + f_2}[g_{2,0}(1 + g_{2,2})], \tag{15b}$$

where

$$g_{1,1} = \alpha e^{\theta}, \quad \theta = kx + i(k^2 - \lambda)t,$$
$$g_{2,0} = \rho e^{i\xi}, \quad \xi = sx - (s^2 + \lambda)t + \xi_0,$$
$$g_{2,2} = \gamma r e^{\theta+\theta^*}, \quad \gamma = -(k - is)/(k^* + is)$$
$$f_2 = re^{\theta+\theta^*}, \quad r = a|\alpha|^2[(k + k^*)^2 - \lambda(\gamma + \gamma^* - 2)/2]^{-1}.$$

Here α,k are complex and ρ,s, ξ_0 are real parameters. This solution is depicted in Fig. 5. The condition of nonsingularity $r > 0$ can be satisfied in all cases of Table I.

It can be seen that the component q_1 of this solution has propagating and oscillating behavior (breather-like), while the other component q_2 is a dark soliton. We call this solution the *quasibreather-dark* (QBD) soliton. Let us now make some remarks on this solution:

(i) In the limit $b \to 0$ the first component (15a) turns into the bright soliton, since it represents the overlay of dark and bright solitons:

$$q_1 = \frac{\alpha}{2\sqrt{r}} e^{(\theta-\theta^*)/2}\,\text{sech}\,\eta - \frac{b^*}{a} q_2,$$
$$q_2 = a\rho e^{isx - i(s^2+\lambda)t + i\tilde{\xi}_0}\{i \sin\beta + \cos\beta \tanh\eta\},$$

where $\beta = \arctan\{(k_I - s)/k_R\}$, $\eta = (\theta + \theta^* + \ln r)/2$ and $\tilde{\xi}_0$ is a certain real parameter.

(ii) This solution has the following asymptotics:

$$\lim_{x\to+\infty} q_1 = -b^*\rho\gamma e^{i\xi}, \quad \lim_{x\to+\infty} q_2 = a\rho\gamma e^{i\xi},$$
$$\lim_{x\to-\infty} q_1 = -b^*\rho e^{i\xi}, \quad \lim_{x\to-\infty} q_2 = a\rho e^{i\xi}.$$

(iii) The localized space and time oscillations of density in the first component are ensured by its interference structure, while the second component does not possess similar pattern:

$$|q_1|^2 = I_1^2 + I_2^2 - 2I_1 I_2 \cos(\Phi - \Phi_0),$$
$$|q_2|^2 = a^2|b|^{-2} I_2^2,$$

where

$$I_1 = |\alpha|/\sqrt{4r}\,\text{sech}\,\eta,$$
$$I_2 = |b\rho\cos\beta|\sqrt{\tan^2\beta + \tanh^2\eta},$$
$$\Phi = \tilde{k}x + \Omega t, \quad \Phi_0 = \arctan\{\tan\beta/\tanh\eta\} + \phi_0,$$
$$\tilde{k} = k_I - s, \quad \Omega = k_R^2 - k_I^2 + s^2,$$

and ϕ_0 is a certain real constant. The upper and lower envelopes [dashed lines in Fig. 5(b)] of the interference pattern are given by $I_1^2 + I_2^2 \pm 2I_1 I_2$.

(iv) There are two types of the QBD soliton: standing ($k_I = 0$) or traveling ($k_I \neq 0$) ones. In the stationary case the density of component (15a) is periodic in time. Its oscillation period is $T = 2\pi/\Omega$.

It should be noted that similar bound states were found in the basic mixed model [35]. However, in contrast to the solution from Ref. [35], the density of the QBD soliton is space and time oscillated with frequencies $\tilde{k} = k_I - s$ and $\Omega = k_R^2 - k_I^2 + s^2$, correspondingly. To the best of our knowledge, such a solitonic "atom" with oscillating internal structure has not been reported before.

We note that Eqs. (1) admit the quasibreather-bright soliton solution also. It can be obtained by taking a dark-bright soliton solution of Eqs. (4) instead of the BD soliton.

Two-soliton solution. Now to understand the nature of such mixed soliton solutions, and their propagation and collision dynamics, we obtain the solution for the $n = 2$ case in expansions (14). In the same way as before we get

$$q_1 = \frac{g_{1,1} + g_{1,3} - b^* g_{2,0}(1 + g_{2,2} + g_{2,4})}{1 + f_2 + f_4}, \tag{16a}$$
$$q_2 = \frac{a g_{2,0}(1 + g_{2,2} + g_{2,4})}{1 + f_2 + f_4}, \tag{16b}$$

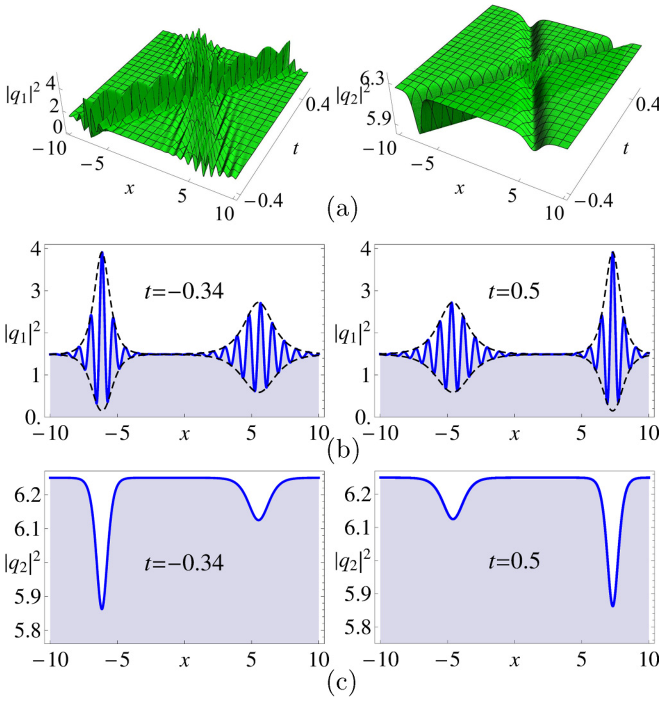

FIG. 6. (Color online) (a) The collisional dynamics of the QBD two-soliton solutions (16) on the (x,t) plane. (b),(c) Snapshots of density profile, before $(t=-0.34)$ and after $(t=0.5)$ collision. Parameters are chosen as in the text.

where

$$g_{1,1}=\alpha_1 e^{\theta_1}+\alpha_2 e^{\theta_2},\quad \theta_j=k_j x+i\left(k_j^2-\lambda\right)t,$$

$$g_{2,0}=\rho e^{i\xi},\quad \xi=sx-(s^2+\lambda)t+\xi_0,$$

$$f_2=r_{1,1}e^{\theta_1+\theta_1^*}+r_{1,2}e^{\theta_1+\theta_2^*}+r_{2,1}e^{\theta_1^*+\theta_2}+r_{2,2}e^{\theta_2+\theta_2^*},$$

$$r_{i,j}=\frac{a\alpha_i\alpha_j(k_i-is)(k_j^*+is)}{[ik_is+s^2-\lambda/2+(k_i-is)k_j^*](k_i+k_j^*)^2},$$

$$g_{2,2}=\beta_{1,1}e^{\theta_1+\theta_1^*}+\beta_{1,2}e^{\theta_1+\theta_2^*}+\beta_{2,1}e^{\theta_1^*+\theta_2}+\beta_{2,2}e^{\theta_2+\theta_2^*},$$

$$\beta_{i,j}=-(k_i-is)/(is+k_j^*),$$

$$g_{1,3}=\sigma_1 e^{\theta_1+\theta_1^*+\theta_2}+\sigma_2 e^{\theta_2+\theta_2^*+\theta_1},$$

$$\sigma_j=\frac{(k_2-k_1)[k_1r_{1,j}\alpha_2-k_2r_{2,j}\alpha_1+(r_{1,j}\alpha_2-r_{2,j}\alpha_1)k_2^*]}{(k_j+k_j^*)(k_{3-j}+k_j^*)},$$

$$f_4=\mu e^{\theta_1+\theta_1^*+\theta_2+\theta_2^*},$$

$$\mu=\frac{(k_2^*-k_1^*)(k_2r_{1,2}\delta_1-k_2r_{1,1}\delta_2)+r_{1,2}\delta_1k_1^*-r_{1,1}\delta_2k_2^*}{\alpha_1(k_2+k_1^*)(k_2+k_2^*)},$$

$$g_{2,4}=\gamma e^{\theta_1+\theta_1^*+\theta_2+\theta_2^*},\quad \gamma=(-ik_2^*+s)\beta_{1,1}^*(s+ik_2)^{-1}/r_{1,1}.$$

In above formulas α_j,k_j are complex and ρ,s,ξ_0 are real parameters. Similarly to one-soliton case, $\tilde{k}_j=k_{jI}-s$ and $\Omega_j=k_{jR}^2-k_{jI}^2+s^2$ correspond to frequencies of the space and time oscillations of the jth soliton in the first component, respectively. One can check that as $t\to\pm\infty$ the QBD two-soliton solution (16) has the asymptotic form of a superposition of the QBD one-soliton solutions with velocities $v_j=2k_{jI}$. The dynamics of two solitons which move in opposite directions and collide is presented in Fig. 6. Parameters are chosen as $a=5$, $c=1$, $k_1=1.8+8i$, $k_2=-1-6i$, $s=1$, $\alpha_1=2.3+5i$, $\alpha_2=3+2i$, $b=2-1.4i$, $\rho=0.5$, $\xi_0=0.2$. Figure 6(c) shows the fact that the collision among dark solitons is shape-preserving, as the profiles of colliding solitons remain the same before and after interaction; Fig. 6(b) shows that the quasibreathers remain confined and propagate without changing their forms and dynamical structures. Therefore the vector soliton (16) exhibits an elastic collision. We connect this elastic behavior with two facts: (i) the transformation (3) is linear and (ii) the BD soliton of $U(2,0)$-, $U(0,2)$-vector Manakov and $U(1,1)$-pseudovector Makhankov models do not exhibit shape-changing collision [12,20].

VII. CONCLUSION

In this paper, we have developed the procedure for reducing the GCNLS system to the family $U(n,m)$-vector models. The resulting models have been classified into three classes $U(0,2)$, $U(2,0)$, and $U(1,1)$ based on the values of XPM, SPM, and FWM parameters. By this reduction we have obtained the BB soliton solution, which had been previously proposed in the literature. Further, we have given the DD soliton with a superposition of plane-wave background, which reduces to the standard DD soliton in a particular case. As one of the interesting results we have presented a mixed quasibreather-dark solution with unconventional properties; namely, the first component's density oscillates while the second one does not. We also have studied their collisional dynamics and the effect of the FWM parameter briefly.

In conclusion, it should also be noted that the proposed method enables us to construct a broad class of exact solutions for Eqs. (1). For instance, it is interesting to consider the cases where the solution of Eqs. (4) is rational [36–38]. The details of such a study will be presented elsewhere. We hope that our results will be useful in the study of soliton theory and its applications.

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[1] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].

[2] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).

[3] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[4] A. C. Scott, Phys. Scr. **29**, 279 (1984).
[5] B. K. Som, M. R. Gupta, and B. Dasgupta, Phys. Lett. A **72**, 111 (1979).
[6] N. Lazarides and G. P. Tsironis, Phys. Rev. E **71**, 036614 (2005).
[7] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].
[8] V. G. Makhankov and O. K. Pashaev, Theor. Math. Phys. **53**, 979 (1982).
[9] V. E. Zakharov and E. I. Schulman, Physica D **4**, 270 (1982).
[10] R. Radhakrishnan and M. Lakshmanan, J. Phys. A **28**, 2683 (1995).
[11] A. P. Sheppard and Y. S. Kivshar, Phys. Rev. E **55**, 4773 (1997).
[12] A. M. Agalarov and R. M. Magomedmirzaev, JETP Lett. **76**, 414 (2002).
[13] M. Vijayajayanthi, T. Kanna, and M. Lakshmanan, Phys. Rev. A **77**, 013820 (2008).
[14] B. Prinari, M. J. Ablowitz, and G. Biondiny, J. Math. Phys. **47**, 063508 (2006).
[15] M. H. Jakubowski, K. Steiglitz, and R. Squier, Phys. Rev. E **58**, 6752 (1998).
[16] K. Steiglitz, Phys. Rev. E **63**, 016608 (2000).
[17] J. Dove, C. Chudzicki, and J. H. Shapiro, Phys. Rev. A **90**, 062314 (2014).
[18] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Phys. Rev. E **56**, 2213 (1997).
[19] T. Kanna and M. Lakshmanan, Phys. Rev. Lett. **86**, 5043 (2001).
[20] Vijayajayanthi, T. Kanna, and M. Lakshmanan, Eur. Phys. J. Spec. Top. **173**, 57 (2009).
[21] D. S. Wang, D. J. Zhang, and J. Yang, J. Math. Phys. **51**, 023510 (2010).
[22] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, London, 1995).
[23] K. Sakkaravarthi and T. Kanna, J. Math. Phys. **54**, 013701 (2013).
[24] Q. H. Park and H. J. Shin, Phys. Rev. E **59**, 2373 (1999).
[25] X. Lü and M. Peng, Nonlin. Dynam. **73**, 405 (2013).
[26] M. Wang, W. R. Shan, X. Lü, Y. S. Xue, Z. Q. Lin, and B. Tian, Appl. Math. Comput. **219**, 11258 (2013).
[27] N. V. Priya, M. Senthilvelan, and M. Lakshmanan, Phys. Rev. E **89**, 062901 (2014).
[28] N. V. Priya and M. Senthilvelan, Commun. Nonlin. Sci. Numer. Simul. **20**, 401 (2015).
[29] N. V. Priya and M. Senthilvelan, Wave Motion **54**, 125 (2015).
[30] Y. Ohta, D. S. Wang, and J. K. Yang, Stud. Appl. Math. **127**, 345 (2011).
[31] V. G. Makhankov, Y. P. Rybakov, and V. I. Sanyuk, Phys. Usp. **37**, 113 (1994).
[32] R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
[33] T. Kanna, M. Lakshmanan, P. T. Dinda, and N. Akhmediev, Phys. Rev. E **73**, 026604 (2006).
[34] J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).
[35] V. G. Makhankov, N. V. Makhaldiani, and O. K. Pashaev, Phys. Lett. A **81**, 161 (1981).
[36] V. E. Zakharov and A. A. Gelash, Phys. Rev. Lett. **111**, 054101 (2013).
[37] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, Phys. Rev. Lett. **109**, 044102 (2012).
[38] C. Kalla, J. Phys. A: Math. Theor. **44**, 335210 (2011).