Symmetries and exact solutions of a class of nonlocal nonlinear Schrödinger equations with self-induced parity-time-symmetric potential

Debdeep Sinha* and Pijush K. Ghosh†

Department of Physics, Siksha-Bhavana, Visva-Bharati University, Santiniketan 731 235, India

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A class of nonlocal nonlinear Schrödinger equations (NLSEs) is considered in an external potential with a space-time modulated coefficient of the nonlinear interaction term as well as confining and/or loss-gain terms. This is a generalization of a recently introduced integrable nonlocal NLSE with self-induced potential that is parity-time-symmetric in the corresponding stationary problem. Exact soliton solutions are obtained for the inhomogeneous and/or nonautonomous nonlocal NLSE by using similarity transformation, and the method is illustrated with a few examples. It is found that only those transformations are allowed for which the transformed spatial coordinate is odd under the parity transformation of the original one. It is shown that the nonlocal NLSE without the external potential and a $(d+1)$-dimensional generalization of it admits all the symmetries of the $(d+1)$-dimensional Schrödinger group. The conserved Noether charges associated with the time translation, dilatation, and special conformal transformation are shown to be real-valued in spite of being non-Hermitian. Finally, the dynamics of different moments are studied with an exact description of the time evolution of the "pseudowidth" of the wave packet for the special case in which the system admits a $O(2,1)$ conformal symmetry.

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I. INTRODUCTION

Ever since it was realized that parity-time-symmetric ($\mathcal{PT}$-symmetric) non-Hermitian systems may exhibit real spectra [1], a great deal of investigation has been carried out in this field [2–8]. As theoretical understanding proceeds, attempts have been made to realize non-Hermitian $\mathcal{PT}$-symmetric systems experimentally. Since the paraxial equation of diffraction is similar in structure to the Schrödinger equation, it was believed that optics may be a testing ground for $\mathcal{PT}$-symmetric systems [9]. In fact, the phase transition between broken and unbroken phases of a non-Hermitian system has been observed experimentally [10], stimulating a great deal of research [11–13] in optical systems with $\mathcal{PT}$ symmetry.

The nonlinear Schrödinger equation (NLSE) admits soliton solutions and finds applications in many diverse branches of modern science, such as Bose-Einstein condensation (BEC) [14], plasma physics [15], gravity waves [16], α-helix protein dynamics [17], etc., and especially optics where it describes wave propagation in nonlinear media [18]. The study of solitons in the NLSE was mainly confined to homogeneous and autonomous systems during the earlier years of its development, where time merely played the role of a parameter in the nonlinear evolution equation. However, it became apparent that integrability of the NLSE may be preserved [19] if different coefficients appearing in it are given specific space-time dependences. This led to the concept of nonautonomous solitons [20]. A great deal of research [21–23] work has been carried out recently on the inhomogeneous and/or nonautonomous NLSE in an external potential due to its physical and experimental relevance. Such systems appear in the study of Bose-Einstein condensation, soliton lasers, ultrafast soliton switches, and logic gates [24]. The time dependence of different coefficients may arise due to time-dependent external forces, whereas inhomogeneity may be introduced through optical control of Feshback resonances [25]. One may use the method of similarity transformation [26,27] to find exact solutions of such inhomogeneous and nonautonomous NLSEs, and there are many such exactly solvable systems.

An integrable nonlocal NLSE was introduced in Ref. [28] for which exact solutions were obtained through the inverse scattering method. In contrast to the standard formulation of the NLSE, the Schrödinger field and its parity ($\mathcal{P}$)-transformed complex conjugate are treated as two independent fields. The self-induced potential in the corresponding stationary problem is non-Hermitian but $\mathcal{PT}$-symmetric. It was shown later [29] that this nonlocal NLSE admits both dark and bright solitons for the case of attractive interaction. Several periodic soliton solutions of this equation have been obtained analytically [30]. A two-component generalization of the nonlocal NLSE is considered in Ref. [30], while a nonlocal NLSE on a one-dimensional lattice is introduced in Refs. [29,30]. Nonlocal nonlinearity arises whenever the nonlinear effect at a particular point depends on the influences from other points [31]. This kind of nonlinearity manifests itself in many natural events; for example, Bose-Einstein condensation in a system having long-range interaction is reported in Refs. [32,33]. This kind of nonlinearity is also observed in transport processes associated with heat conduction in media having a thermal influence, and during diffusion of charge carriers, atoms, or molecules in atomic vapors [34,35]. The propagation of highly nonlocal solitons in nematic liquid crystals is mentioned in Ref. [36].

The purpose of this paper is to introduce and study an inhomogeneous and nonautonomous version of the integrable nonlocal NLSE of Ref. [28]. In particular, we consider a class of nonlocal NLSEs in an external potential with a space-time modulated coefficient of the nonlinear interaction term as well as confining and/or loss-gain terms. We find exact soliton solutions for this generalized class of nonlocal NLSEs by using a similarity transformation. We find that only those transformations are allowed for which the transformed

*debdeepsinha.rs@visva-bharati.ac.in

†pijushkanti.ghosh@visva-bharati.ac.in

spatial coordinate is odd under the parity transformation of the original one. This is in contrast to the findings of similar studies for the local NLSE, for which no such restriction is necessary. Although such a condition puts restrictions on the possible types of external potentials, loss-gain terms, space-time modulated coefficients, etc., the choices are still infinitely many, including most of the physically interesting cases. We consider a few examples with the explicit expressions for the external potential and space-time modulated coefficient of the nonlinear interaction term. It is worth mentioning here that integrability of the nonlocal NLSE with spatiotemporally varying coefficients of the dispersion as well as the nonlinear term has been considered recently by using the Lax-pair formulation [37]. However, the integrability condition in Ref. [37] restricts the spatial dependence of the coefficients to a specific form, and it cannot reproduce the class of nonlocal NLSE considered in this paper.

We introduce a $(d+1)$-dimensional homogeneous and autonomous nonlocal NLSE without any external potential and study its Schrödinger invariance. The system is invariant under all the symmetry transformations associated with the $(d+1)$-dimensional Schrödinger group. We find that the formal expressions for the corresponding conserved Noether charges are non-Hermitian. However, the conserved charges associated with the time translation, dilatation, and special conformal transformation are shown to be real-valued only. On the other hand, the total momentum as well as the boost are complex in any spatial dimensions. Consequently, the angular momentum turns out to be real in odd spatial dimensions and is complex in even spatial dimensions. The conserved charges are shown to satisfy the $(d+1)$-dimensional Schrödinger algebra.

Finally, we consider an inhomogeneous and nonautonomous version of this higher-dimensional nonlocal NLSE. We introduce different moments and study their dynamics. Although the formal expressions for these moments are non-Hermitian, they are shown to be real-valued. We find an exact description of the time evolution of the "pseudowidth" of the wave packet for the special case in which the system admits a $O(2,1)$ conformal symmetry.

II. EXACT SOLUTION OF THE NONLOCAL NLSE

An integrable nonlocal NLSE in 1+1 dimensions was introduced in [28]:

$$i\psi_t(x,t) = -\frac{1}{2}\psi_{xx}(x,t) + G\,\psi^*(-x,t)\psi(x,t)\psi(x,t),\ G \in R. \tag{1}$$

The self-induced potential in the corresponding stationary problem has the form $V(x) = \psi^*(-x)\psi(x)$, which is $\mathcal{PT}$-symmetric, i.e., $V^*(-x) = V(x)$. The equation is nonlocal in the sense that the value of the potential $V(x)$ at x requires the information on ψ at x as well as at $-x$. It has been shown in Ref. [28] that this equation possesses a Lax pair and an infinite number of conserved quantities, and therefore it is integrable. In contrast to the usual local NLSE, Eq. (1) admits dark as well as bright soliton solutions for $g < 0$ [29]. Several periodic soliton solutions of this equation have also been found [30]. It is interesting to note that Eq. (1) admits a solution with a special shift in coordinate x, but not with an arbitrary shift [29,30]. The occurrence of nonlocal nonlinearity is not rare in nature. For example, it appears in the case of diffusion of charge carriers, atoms, or molecules in atomic vapors [34,35]. In the study of BEC with a long-range interaction, the BEC with magnetic dipole-dipole forces was considered in Ref. [32] and BEC of chromium was investigated in Ref. [33]. The optical spatial soliton in a highly nonlocal medium was observed in Ref. [36]. The nonlinearity occurring in Eq. (1) has the possibility to be realized in the case of a coupled waveguide or in an infinite array of waveguide system [29].

In this section, we investigate the possible exact solutions of the following nonautonomous NLSE:

$$\begin{aligned} i\psi_t &= -\tfrac{1}{2}\psi_{xx} + [V(x,t) + iW(x,t)]\psi + g(x,t)\psi^{*p} \\ &\quad \times (-x,t)\psi^p(x,t)\psi(x,t),\ p \in N, \end{aligned} \tag{2}$$

where $g(x,t)$ is the space-time-dependent strength of the nonlinear interaction. There are many applications of the corresponding local nonautonomous NLSE of this kind with space-time modulated coefficients, e.g., in the study of Bose-Einstein condensation, soliton lasers, ultrafast soliton switches, logic gates [24], etc. It may be noted that the nonlinear interaction term is nonlocal as well as $\mathcal{PT}$-symmetric. The external potential $v(x,t) = V(x,t) + iW(x,t)$ is chosen to be complex, with $V(x,t)$ and $W(x,t)$ being the real and imaginary parts, respectively. The effect of $V(x,t)$ is to confine the particle, whereas $W(x,t)$ is considered to be a gain-loss coefficient. The external potential $v(x,t)$ becomes $\mathcal{PT}$-symmetric for $V(x,t) = V(-x,-t)$ and $W(x,t) = -W(-x,-t)$. It is worth mentioning here that the $\mathcal{PT}$-symmetric periodic potential was investigated in Refs. [9,12] in the realm of optics with the possibility of double refraction, power oscillation, and secondary emissions. In the corresponding local analysis, the situation $p > 1$ arises in the realization of BEC in lower dimensions, in which case the Gross-Pitaevskii equation needs to be modified by taking $p > 1$ [38,39].

The above equation reduces to a homogeneous nonlocal NLSE,

$$i\psi_t = -\tfrac{1}{2}\psi_{xx} + G\psi^{*p}(-x,t)\psi^p(x,t)\psi(x,t) \tag{3}$$

for $V(x,t) = W(x,t) = 0$, $g(x,t) = G$. A further choice of $p = 1$ reproduces the nonlocal NLSE in Eq. (1), which is exactly solvable. We find an exact solution of Eq. (3) for arbitrary p,

$$\psi(x,t) = \Phi_0\, e^{i\frac{A^2}{2p^2}t}\operatorname{sech}^{\frac{1}{p}}(Ax), \tag{4}$$

where

$$G = -\frac{A^2(1+p)}{2p^2\Phi_0^{2p}} \tag{5}$$

is necessarily negative. It may be noted that unlike the soliton solutions of the corresponding local NLSE, an arbitrary constant shift of the transverse coordinate in $\psi(x)$ does not produce an exact solution of (3). The known bright soliton solutions of the nonlocal NLSE with cubic nonlinearity may be reproduced by setting $p = 1$ in Eqs. (4) and (5).

We use the similarity transformation [27]

$$\psi(x,t) = \rho(x,t)e^{i\phi(x,t)}\Phi(X), \quad X \equiv X(x,t) \tag{6}$$

to map Eq. (2) to the following equation:

$$\mu\Phi(X) = -\tfrac{1}{2}\Phi_{XX}(X) + G\Phi^{*p}(-X)\Phi^{p}(X)\Phi(X). \quad (7)$$

Consequently, the known exact solution of Eq. (3) may be used to construct a large class of exactly solvable nonautonomous nonlocal NLSEs of the type of Eq. (2). We find that Eq. (2) reduces to the stationary nonlocal NLSE (7) only when $X(x,t)$ is an odd function of x, i.e.,

$$X(-x,t) = -X(x,t), \quad (8)$$

and the following additional consistency conditions hold simultaneously:

$$2\rho\rho_t + (\rho^2\phi_x)_x = 2\rho^2 W(x,t), \quad (9)$$

$$(\rho^2 X_x)_x = 0, \quad (10)$$

$$X_t + \phi_x X_x = 0, \quad (11)$$

$$V(x,t) = \frac{\rho_{xx}}{2\rho} - \phi_t - \frac{\phi_x^2}{2} - \mu X_x^2, \quad (12)$$

$$g(x,t) = \frac{G}{\rho^p(-x,t)\rho^p(x,t)e^{ip[\phi(x,t)-\phi(-x,t)]}} X_x^2. \quad (13)$$

The above conditions are obtained by exploiting the facts that ψ and Φ satisfy Eqs. (2) and (7), respectively, and they are related by transformation in Eq. (6). It may be noted that the oddness of $X(x,t)$ in x, as in Eq. (8), is not necessary for the similarity transformation from the local NLSE to its inhomogeneous counterpart. Condition (8) arises solely due to the nonlocal nature of the nonlinear interaction and forbids any purely time-dependent shift in the choice of X in terms of x and t. This is consistent with the fact that the solutions of the nonlocal NLSE are not invariant under any shift of the transverse coordinate x [29]. All the consistency conditions in Eqs. (9)–(13), except for the expression of $g(x,t)$, are identical with the corresponding expressions [27] obtained for the mapping of the local NLSE to its inhomogeneous counterpart. Further, it is evident that $g(x,t)$ becomes a complex function if $\phi(x,t)$ is not an even function of x, while the consistency conditions stated above are based on the assumption of real $g(x,t)$. This apparent contradiction is removed by the use of Eq. (8), which reduces $g(x,t)$ to be real. Toward that end, we solve Eqs. (10) and (11) to obtain ρ and ϕ:

$$\rho(x,t) = \sqrt{\frac{\delta(t)}{X_x}}, \quad \phi(x,t) = -\int dx \frac{X_t}{X_x} + \phi_0(t), \quad (14)$$

where $\delta(t)$ and $\phi_0(t)$ are two integration constants. It immediately follows that both ρ and $\phi(x,t)$ are even in x, which allows us to rewrite $g(x,t)$ in Eq. (13) as

$$g(x,t) = \frac{G\delta^2(t)}{\rho^{2(p+2)}}. \quad (15)$$

A choice of X will determine ρ and ϕ through Eq. (14) up to two integration constants that may be fixed by using appropriate conditions on $V(x,t)$. The expressions of X, ρ, and ϕ may be used to determine $W(x,t)$, $V(x,t)$, and $g(x,t)$ from Eqs. (9), (12), and (15), respectively.

A. Inhomogeneous autonomous nonlocal NLSE

Consider a special class of similarity transformation by considering

$$\rho(x,t) \equiv \rho(x), \quad \phi(x,t) \equiv -Et, \quad X \equiv X(x) \quad (16)$$

in Eq. (6). In this case, Eq. (11) is satisfied automatically, and the consistency condition of Eq. (9) determines $W(x,t) = 0$, which implies that no gain-loss term can be generated under this similarity transformation. From Eqs. (10), (12), and (15), $X(x)$, $g(x)$, and $V(x)$ can be determined as

$$X(x) = \int_o^x \frac{ds}{\rho^2(s)}, \quad (17)$$

$$g(x) = \frac{G}{\rho^{2(p+2)}}, \quad (18)$$

$$V(x) = \frac{\rho_{xx}}{2\rho} + E - \frac{\mu}{\rho^4}, \quad (19)$$

Equation (17) implies that ρ must have a definite parity as X is an odd function of x. It immediately follows from Eqs. (18) and (19) that both $g(x)$ and $V(x)$ must be an even function of x. In particular,

$$\rho(-x) = \pm\rho(x), \quad g(-x) = g(x), \quad V(-x) = V(x). \quad (20)$$

The reality of $\rho(x)$, $g(x)$, and $V(x)$ ensures that these functions are also $\mathcal{PT}$-symmetric. It may be noted that for the similarity transformation of the local NLSE to its inhomogeneous counterpart [26], no conditions such as those in Eqs. (8) and (20) are necessary. Thus, we have the important result that the similarity transformation technique [26] is applicable to the nonlocal NLSE only when both the confining potential $V(x)$ and the space-modulated nonlinear interaction term $g(x)$ are even in x.

The expressions for $X(x)$ and $g(x)$ can be obtained once an explicit form of $\rho(x)$ is known. We use Eq. (19) to find $\rho(x)$ for a given $V(x)$. We rewrite Eq. (19) as

$$\frac{1}{2}\rho_{xx} + [E - V(x)]\rho = \frac{\mu}{\rho^3}, \quad (21)$$

which is the Ermakov-Pinney equation [26]. The solution of this equation may be written as

$$\rho = \left[a\phi_1^2(x) + 2b\phi_1(x)\phi_2(x) + c\phi_2^2(x)\right]^{\frac{1}{2}}, \quad (22)$$

where a, b, and c are constants, and $\phi_1(x)$ and $\phi_2(x)$ are the two linearly independent solutions of the equation

$$-\tfrac{1}{2}\phi_{xx} + V(x)\phi(x) = E\phi(x). \quad (23)$$

The constant μ is determined as $\mu = (ac - b^2)[\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x)]^2$. The confining potential $V(x)$ has even parity. Thus, $\phi_{1,2}(x)$ can always be chosen to be either even or odd. The requirement of a definite parity for $\rho(x)$ can always be ensured by suitably choosing the constants a,b,c for a given set of linearly independent solutions ϕ_1 and ϕ_2.

We consider a few specific examples, as follows.

1. Vanishing external potential

The first example deals with the case of no external potential, i.e., $V(x) = 0$. There are two cases, depending on whether $E > 0$ or $E < 0$, which are treated separately. For $E > 0$, Eqs. (17)–(23) can be solved consistently, leading to the following expressions for the function $\rho(x)$ and the space-modulated coefficient $g(x)$:

$$\rho(x) = [1 + \alpha\cos(\omega x)]^{\frac{1}{2}}, \quad g(x,t) = G[1 + \alpha\cos(\omega x)]^{-(p+2)}, \tag{24}$$

where $\omega = 2\sqrt{2|E|}$ and $\mu = (1-\alpha^2)E$. The transformed coordinate $X(x)$ is determined as

$$X_+(x) = \frac{2}{\omega\sqrt{1-\alpha^2}}\tan^{-1}\left[\sqrt{\frac{1-\alpha}{1+\alpha}}\tan\left(\frac{\omega x}{2}\right)\right] \quad \text{for} \quad |\alpha| < 1,$$
$$X_-(x) = \frac{1}{\omega\sqrt{\alpha^2-1}}\ln\left[\frac{\tan\left(\frac{\omega x}{2}\right) + \sqrt{\frac{\alpha+1}{\alpha-1}}}{\tan\left(\frac{\omega x}{2}\right) - \sqrt{\frac{\alpha+1}{\alpha-1}}}\right] \quad \text{for} \quad |\alpha| > 1, \tag{25}$$

where the subscripts refer to the fact that μ is positive for the solution $X_+(x)$, whereas it is negative for $X_-(x)$. A solution of Eq. (2) for $G < 0$, $V = W = 0$, and $g(x,t)$ given by Eq. (24) reads

$$\psi(x,t) = e^{-iEt}\left(\frac{E(\alpha^2-1)(p+1)}{|G|}\right)^{\frac{1}{2p}}[1+\alpha\cos(\omega x)]^{\frac{1}{2}} \times \operatorname{sech}^{\frac{1}{p}}(p\sqrt{2E(\alpha^2-1)}X_-(x)). \tag{26}$$

For $p = 1$, under the same conditions as stated above, Eq. (2) also admits the following solution:

$$\psi(x,t) = e^{-iEt}\left(\frac{E(1-\alpha^2)}{|G|}[1+\alpha\cos(\omega x)]\right)^{\frac{1}{2}} \times \tanh(\sqrt{(1-\alpha^2)E}X_+(x)). \tag{27}$$

It may be noted that Eq. (27) is also a solution of the corresponding local NLSE, but for $G > 0$.

For $E < 0$, Eqs. (17)–(23) can be solved consistently with the following expressions for the function $\rho(x)$, the space-modulated coefficient $g(x)$, and the transformed coordinate $X(x)$:

$$\rho(x) = \cosh^{\frac{1}{2}}(\omega x), \quad g(x) = G\cosh^{-(p+2)}(\omega x), \quad X(x) = -\frac{1}{\omega}\cos^{-1}[\tanh(\omega x)], \tag{28}$$

where μ is determined as $\mu = 2|E|$, which is positive-definite. Unlike x, which is defined on the whole line, X is bounded within the range $0 \leqslant X \leqslant \frac{\pi}{\omega}$, and any solution of Eq. (7) must vanish at the end points. There are many exact periodic solutions [30] of Eq. (7) for $p = 1$ in terms of Jacobi elliptic functions. The type-V and type-VIII solutions of Ref. [30] are of particular interest to the present problem. In particular,

$$\psi_{\rm V} = e^{-iEt}\left(\frac{2m\mu}{|G|(1+m)}\cosh(\omega x)\right)^{\frac{1}{2}}\operatorname{sn}\left(\sqrt{\frac{2\mu}{1+m}}X,m\right) \tag{29}$$

is an exact solution of Eq. (2) with $p = 1$ and $G < 0$, where $\frac{1}{2} < \mu \leqslant 1$. The value of m within the range $0 < m \leqslant 1$ is determined from the condition

$$4nK(m)\sqrt{a_m} = \pi, \quad K(m) \equiv \int_0^{\frac{\pi}{2}}(1 - m\sin^2\theta)^{-\frac{1}{2}}d\theta, \tag{30}$$

where n is any positive integer and $a_m = m + 1$. The above equation determining the allowed values of m arises from the condition that $\operatorname{sn}(\sqrt{\frac{2\mu\pi}{(1+m)\omega}},m) = 0$, and for every n it has a unique solution [26]. The boundary condition at $X = 0$ is automatically satisfied by the elliptic function $\operatorname{sn}(\sqrt{\frac{2\mu}{1+m}}X,m)$. A second solution of Eq. (2) with $p = 1$ and $G > 0$ is

$$\psi_{\rm VIII} = e^{-iEt}\left(\frac{2m\mu(1-m)}{|G|(2m-1)}\cosh(\omega x)\right)^{\frac{1}{2}}\frac{\operatorname{sn}\left(\sqrt{\frac{2\mu}{1-2m}}X,m\right)}{\operatorname{dn}\left(\sqrt{\frac{2\mu}{1-2m}}X,m\right)}, \tag{31}$$

where the values of m within the range $0 < m < \frac{1}{2}$ are again determined from Eq. (30) with $a_m = 1 - 2m$. Both $\psi_{\rm V}$ and $\psi_{\rm VIII}$ describe bound states of multisoliton states. The inhomogeneous local NLSE corresponding to Eq. (2) also admits these novel states [26], but for $G < 0$.

2. *Harmonic confinement*

We choose $V(x) = \frac{1}{2}x^2$ and $E = 0$, for which (17)–(23) can be solved consistently with the following solutions:

$$\rho(x) = e^{\frac{x^2}{2}}, \quad g(x) = Ge^{-(p+2)x^2}, \quad X(x) = \frac{\sqrt{\pi}}{2}\operatorname{erf}x. \tag{32}$$

Note that $\mu = 0$ and $-\frac{\sqrt{\pi}}{2} \leqslant X \leqslant \frac{\sqrt{\pi}}{2}$. We choose $p = 1$ for which solutions of type-II and type-VIII of Ref. [30] with $m = \frac{1}{2}$ are relevant for the present discussion. In particular,

$$\psi_{\rm II}^n = \frac{2nK\left(\frac{1}{2}\right)}{\sqrt{2\pi|G|}}e^{-iEt}e^{\frac{x^2}{2}}\operatorname{cn}\left(\theta_n,\frac{1}{2}\right), \quad n = 1,3,\ldots,$$
$$\psi_{\rm VIII}^n = \frac{nK\left(\frac{1}{2}\right)}{\sqrt{\pi|G|}}e^{-iEt}e^{\frac{x^2}{2}}\frac{\operatorname{sn}\left(\theta_n,\frac{1}{2}\right)}{\operatorname{dn}\left(\theta_n,\frac{1}{2}\right)}, \quad n = 2,4,\ldots \tag{33}$$

are solutions of Eq. (2) for $G < 0$ and $G > 0$, respectively, where θ_n is defined as

$$\theta_n(x) = nK\left(\frac{1}{2}\right)\operatorname{erf}x, \quad n = 1,2,\ldots. \tag{34}$$

It may be recalled that both $\psi_{\rm II}$ and $\psi_{\rm VIII}$ are solutions of the corresponding local NLSE for $G < 0$ [26]. The difference between the local and the nonlocal cases arises due to the fact that $\operatorname{cn}(X)$ and $\operatorname{dn}(x)$ are even functions of X, while $\operatorname{sn}(X)$ is an odd function of its argument. Both $\psi_{\rm II}$ and $\psi_{\rm VIII}$ are localized in space, and each of them has $n-1$ zeros for a fixed n [26].

3. *Reflectionless potential*

We choose $E = 0$ and the potential

$$V(x) = \tfrac{1}{2}A^2 - \tfrac{1}{2}A(A+1)\operatorname{sech}^2 x, \quad A \in N, \tag{35}$$

for which

$$\rho(x) = (\cosh x)^A, \quad g(x) = G(\text{sech}x)^{2A(p+2)},$$
$$X(x) = \sum_{k=0}^{A-1} \frac{(-1)^k}{2k+1}\, {}^{A-1}C_k (\tanh x)^{2k+1} \tag{36}$$

are consistent with Eqs. (17)–(23) and μ is determined as $\mu = 0$. The range of X is given by $-L \leqslant X \leqslant L$, $L = \sum_{k=0}^{A-1} \frac{(-1)^k}{2k+1}\, {}^{A-1}C_k$. We choose $p = 1$, for which

$$\psi_{\text{II}}^n = \frac{nK\left(\frac{1}{2}\right)}{L\sqrt{2|G|}} e^{-iEt} (\cosh x)^A \, \text{cn}\left(\chi_n, \frac{1}{2}\right), \quad n = 1,3,\ldots,$$
$$\psi_{\text{VIII}}^n = \frac{nK\left(\frac{1}{2}\right)}{L2\sqrt{|G|}} e^{-iEt} (\cosh x)^A \frac{\text{sn}\left(\chi_n, \frac{1}{2}\right)}{\text{dn}\left(\chi_n, \frac{1}{2}\right)}, \quad n = 2,4,\ldots \tag{37}$$

are solutions of Eq, (2) for $G < 0$ and $G > 0$, respectively, where χ_n is defined as

$$\chi_n(x) = \frac{\sqrt{\pi} n K\left(\frac{1}{2}\right)}{2L} X(x), \quad n = 1,2,\ldots. \tag{38}$$

Both of the solutions are localized in space, and each of them has $n-1$ zeros for fixed n.

B. Nonautonomous nonlocal NLSE

Condition (8) can be implemented in several ways. We discuss two different classes of $X(x,t)$ depending on its separability or nonseparability in terms of its arguments x and t. It turns out that for the nonseparable case, the gain-loss coefficient is essentially zero, while it may be chosen to be nonzero for the separable case.

1. Nonseparable $X(x,t)$

One may choose the following ansatz:

$$X(t,x) = F(\xi), \quad \xi(t,x) \equiv \gamma(t)x, \quad F(-\xi) = -F(\xi), \tag{39}$$

where $\gamma(t)$ is an arbitrary function of t. Note that unlike in the case of the local NLSE [27], a purely time-dependent term cannot be added to the ansatz for $X(x,t)$ due to condition (8). Further, the consistency of Eqs. (12)–(14) fixes $W(x,t) = 0$. Thus, the above ansatz is not suitable for systems with a loss-gain term. We obtain the following expressions:

$$\phi(x,t) = -\frac{\gamma_t}{2\gamma} x^2 + \phi_0(t),$$
$$\rho(x,t) = \sqrt{\frac{\gamma}{F'(\xi)}},$$
$$g(x,t) = G\gamma^{2-p}[F'(\xi)]^{p+2},$$
$$V(x,t) = \frac{\gamma^2}{8[F'(\xi)]^2}[3\{F''(\xi)\}^2 - 2F''(\xi)F'''(\xi)$$
$$- 8\mu\{F'(\xi)\}^4] + \frac{1}{2}\omega(t)x^2 - \phi_{0t}, \tag{40}$$

where we have assumed $\delta = \gamma^2$, and for a given $\omega(t)$, $u = \gamma^{-1}$ is determined from the equation

$$u_{tt} + \omega(t)u = 0. \tag{41}$$

The above ansatz leads to harmonic confinement irrespective of the choice of $F(\xi)$. It may happen that the first term in $V(x,t)$ contains a term proportional to ξ^2 for specific choices of $F(\xi)$ for which Eq. (41) gets transformed into the Ermakov-Pinney equation of [27].

The example considered in Ref. [27] for the case of a corresponding local NLSE with $p = 1$ is that of exponentially localized nonlinearity with a combination of harmonic and dipole traps. The motivation behind such a choice is the experimental scenario related to Bose-Einstein condensation. It may be noted that $F(\xi) = \int e^{-\xi^2} d\xi$ is an odd function of its arguments and satisfies the conditions (39). Thus, the results of Ref. [26] are equally valid for the nonlocal NLSE with $p = 1$ also, except for the following differences:

(i) The discussions in Ref. [26] for the local NLSE are for attractive interaction ($G = -1$), whereas the same results are valid for the nonlocal NLSE under consideration for repulsive interaction ($G = 1$) only.

(ii) The nonlocal NLSE admits resonant solitons, breathing solitons, and quasiperiodic solutions. However, moving solitons are not allowed for the nonlocal NLSE due to the condition (8), which forbids the addition of a purely time-dependent term to the ansatz for $X(x,t)$.

2. Separable $X(x,t)$

We choose an expression for $X(x,t)$ that is separable in terms of its arguments, and the spatial part is an odd function of x:

$$X(x,t) \equiv \alpha(t)f(x), \quad f(-x) = -f(x). \tag{42}$$

With this choice of X, Eqs. (12)–(14) take the following form in terms of $\alpha(t)$ and $f(x)$:

$$\rho(x,t) = \sqrt{\frac{\delta(t)}{\alpha(t)f'(x)}},$$
$$\phi(x,t) = -\frac{\alpha_t}{\alpha(t)} \int dx \frac{f(x)}{f'(x)}, \tag{43}$$
$$g(x,t) = \frac{G\alpha^{p+2}}{\delta^p}(f')^{p+2},$$

$$W(x,t) = \frac{1}{2\alpha(t)\delta(t)}(\delta_t\alpha - 2\alpha_t\delta) + \frac{\alpha_t}{\alpha}\left(\frac{f''f}{f'^2}\right), \tag{44}$$

$$V(x,t) = -\left(\frac{2f'''f' - 3f''^2}{8f'^2}\right) + \frac{\alpha_{tt}\alpha - \alpha_t^2}{\alpha^2}\int \frac{f(x)}{f'(x)}dx$$
$$- \frac{\alpha_t^2 f^2}{2\alpha^2 f'^2} - \mu\alpha^2 f'^2, \tag{45}$$

where $f'(x) = \frac{df}{dx}$. We have chosen $\phi_0(t)$ to be zero since its sole effect is to add a purely time-dependent term to $V(x,t)$, which can always be removed through a phase rotation. The following points are in order at this point:

(i) *$\mathcal{PT}$-symmetry*. It may be noted that W is odd and V is even under $\mathcal{PT}$ symmetry whenever both $\alpha(t)$ and $\delta(t)$ have definite parity. Thus, this is also the condition for the external potential $v(x,t)$ to be $\mathcal{PT}$-symmetric. The space-time modulated nonlinear interaction $g(x,t)$ becomes $\mathcal{PT}$-symmetric when additional conditions are imposed. In particular, it becomes $\mathcal{PT}$-symmetric when both $\delta(t)$ and $\alpha(t)$ have the same parity or p is even.

(ii) *Parameter fixing*. A purely time-dependent part $W_0(t) = \frac{1}{2\alpha(t)\delta(t)}(\delta_t\alpha - 2\alpha_t\delta)$ of W can be gauged from Eq. (2) through a redefinition of $g(x,t)$:

$$\psi(x,t) \to \psi(x,t)\, e^{\int_0^t W_0(t')dt'}, \quad g(x,t) \to g(x,t)e^{2p\int_0^t W_0(t')dt'}. \tag{46}$$

Thus, without any loss of generality, we may choose $\delta(t) = \alpha^2(t)$ so that $W_0(t) = 0$. The expression for V and ϕ remains unchanged for this particular choice, while ρ, g, and W read

$$\begin{aligned} \rho(x,t) &= \sqrt{\frac{\alpha}{f'(x)}}, \\ g(x,t) &= G\alpha^{2-p}(f')^{p+2}, \\ W(x,t) &= \frac{\alpha_t}{\alpha}\left(\frac{f''f}{f'^2}\right). \end{aligned} \tag{47}$$

The system described by Eq. (3) has a conformal symmetry for $p = 2$ for which $g(x,t)$ becomes independent of time.

(iii) *Harmonic confinement.* The loss-gain term $W(x,t)$ is purely time-dependent for $f(x)$ satisfying the following equation:

$$f''f = f_0(f')^2, \quad f_0 \in R. \tag{48}$$

The odd solution of the above equation with $f_0 = \frac{2n}{2n+1}$ is $f(x) = x^{2n+1}$, $n \in N_0$. For the special choice of $n = 0$, $V(x,t)$ becomes a purely time-dependent harmonic potential:

$$V(x,t) = \tfrac{1}{2}\omega(t)x^2 - \mu\alpha^2, \tag{49}$$

with $W(x,t) = 0$ and $g(x,t) = G\alpha^{2-p}$. Note that Eq. (41) can be used to determine $\omega(t)$ for a given $u = \alpha^{-1}$ or *vice versa*. The system described by Eq. (3) has a conformal symmetry for $p = 2$ for which $g(x,t)$ becomes space-time-independent.

A particular choice may be the constant $\omega(t) = \omega_0^2$. The general solution of Eq. (41) in this case yields

$$\alpha(t) = [C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]^{-1}, \tag{50}$$

where C_1, C_2 are two arbitrary constants. In this case, $g(x,t)$, $\phi(x,t)$, $\rho(x,t)$, and $X(x,t)$ have the following expressions:

$$\begin{aligned} g(x,t) &= G[C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]^{p-2}, \\ \phi(x,t) &= -\frac{\omega_0[C_1\sin(\omega_0 t) - C_2\cos(\omega_0 t)]}{2[C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]}x^2, \\ \rho &= [C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]^{-\frac{1}{2}}, \\ X(x,t) &= [C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]^{-1}x. \end{aligned} \tag{51}$$

We use the type-V solution of Ref. [30] to obtain a solution of Eq. (2) with $p = 1$ and $G < 0$,

$$\psi_{\rm V} = \left(\frac{2\mu m}{G(1+m)\,[C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)]}\right)^{\frac{1}{2}} \times e^{-i\frac{\omega_0[C_1\sin(\omega_0 t) - C_2\cos(\omega_0 t)]}{2[C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)]}x^2}\,{\rm sn}\left(\sqrt{\frac{2\mu}{1+m}}X, m\right),$$

where $\frac{1}{2} < \mu \leqslant 1$, and the value of m within the range $0 < m \leqslant 1$ is determined from the condition given in (30).

Another solution of Eq. (2) for $G < 0$ and arbitrary p as given by Eqs. (4) and (6) reads

$$\psi = \Phi_0\rho e^{i\phi}{\rm sech}^{\frac{1}{p}}(AX), \tag{52}$$

where ρ, ϕ, and X are given by Eq. (51) and

$$A = -2\mu p^2, \quad \phi_0 = \left(\frac{\mu(1+p)}{|G|}\right)^{\frac{1}{2p}}. \tag{53}$$

(iv) *Nonpolynomial external potential.* A space-time-dependent $W(x,t)$ can be produced with nonpolynomial $f(x)$. We choose $f(x) = \sinh x$, for which $g(x,t)$, $W(x,t)$, and $V(x,t)$ have the following expressions:

$$\begin{aligned} g(x,t) &= G\alpha^{2-p}\cosh^{p+2}x, \quad W(x,t) = \Gamma(t)\tanh^2 x, \\ V(x,t) &= -\frac{1}{4} + \frac{3}{8}\tanh^2 x + \left(\frac{d\Gamma}{dt}\right)\ln(\cosh x) \\ &\quad - \frac{\Gamma^2}{2}\tanh^2 x - \mu\alpha^2\cosh^2 x, \end{aligned} \tag{54}$$

where $\Gamma(t) = \frac{\alpha_t}{\alpha}$. The function $\psi(x,t)$ reads

$$\begin{aligned} \psi(x,t) &= \Phi_0\sqrt{\alpha}\,{\rm sech}^{\frac{1}{2}}x\exp[-i\Gamma(t)\ln(\cosh x)]{\rm sech}^{\frac{1}{p}} \\ &\quad \times [A\alpha(t)\sinh x], \end{aligned} \tag{55}$$

where $A^2 = -2\mu p^2$, $\Phi_0 = (\frac{\mu(1+p)}{|G|})^{\frac{1}{2p}}$.

III. SCHRÖDINGER INVARIANCE OF NONLOCAL NLSE

A $(d+1)$-dimensional generalization of (1) may be written as

$$i\psi_t(\mathbf{x},t) = -\tfrac{1}{2}\nabla^2\psi(\mathbf{x},t) + g\{\psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t)\}^p\psi(\mathbf{x},t). \tag{56}$$

The potential in the corresponding stationary problem has the form $V(\mathbf{x}) = [\psi^*(\mathcal{P}\mathbf{x})\psi(\mathbf{x})]^p$, which is $\mathcal{PT}$-symmetric in any spatial dimensions. It may be recalled that $\mathbf{x} \to -\mathbf{x}$ describes a rotation in even space dimensions, while it is parity transformation in odd spatial dimensions. Thus, $\psi^*(-x,t)$ is replaced with $\psi^*(\mathcal{P}x,t)$ for the higher-dimensional generalization of (1). The parity transformation in higher dimensions is not unique and may be parametrized in terms of $d-1$ parameters. All such parity transformations are related to each other through rotations in d-dimensional space. One may choose N set of values of these $d-1$ parameters and define the corresponding parity operations as $\mathcal{P}_i$, $i = 1,2,\ldots,N$. The corresponding $\mathcal{PT}$-symmetric potentials,

$$\tilde{V}_i(\mathbf{x}) = \{\psi^*(\mathcal{P}_i\mathbf{x})\psi(\mathbf{x})\}^p, \tag{57}$$

are related to each other through spatial rotation. However, for systems without rotational invariance, $\tilde{V}_i(\mathbf{x})$'s are to be treated as independent of each other. For example, if Eq. (56) is considered in an external potential with a space-modulated coefficient of the nonlinear interaction term that is not invariant under spatial rotation, then each $\tilde{V}_i(\mathbf{x})$ corresponds to different systems. As mentioned earlier, case $p > 1$ becomes important in lower-dimensional analysis of BEC [38,39]. The choice $d > 1$ arises in the study of BEC in more than one dimension [40,41]. It should be mentioned here that the integrability of Eq. (56) with arbitrary d and/or p is still not known.

A Lagrangian formulation of Eq. (56) may be given in terms of the Lagrangian density,

$$\mathcal{L} = i\psi^*(\mathcal{P}\mathbf{x},t)\partial_t\psi(x,t) - \frac{1}{2}\nabla\psi^*(\mathcal{P}\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) - \frac{g}{p+1}\{\psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t)\}^{p+1}, \tag{58}$$

where $\psi(\mathbf{x},t)$ and $\psi^*(\mathcal{P}\mathbf{x},t)$ are treated as two independent fields. The conjugate momentum associated with $\psi(\mathbf{x},t)$ is $\Pi_\psi(\mathbf{x},t) = i\psi^*(\mathcal{P}\mathbf{x},t)$ and the equal-time Poisson bracket between them leads to the relation

$$\{\psi(\mathbf{x},t),\psi^*(\mathcal{P}\mathbf{y},t)\} = -i\delta^d(\mathbf{x}-\mathbf{y}). \tag{59}$$

It may be recalled that in the Lagrangian formulation of the usual local NLSE and other field theoretical models involving a complex scalar field, $\psi(x,t)$ and its complex conjugate $\psi^*(x,t)$ are treated as independent fields. The equal-time Poisson bracket relation between $\psi(x,t)$ and $\psi^*(x,t)$ in the standard formulation is similar to Eq. (59), i.e., $\{\psi(\mathbf{x},t),\psi^*(\mathbf{y},t)\} = -i\delta^d(\mathbf{x}-\mathbf{y})$.

The action $\mathcal{A} = \int \mathcal{L} d^d\mathbf{x} dt$ is invariant under space-time translations, spatial rotation, Galilean transformation, and a global gauge transformation. The action $\mathcal{A}$ is invariant under dilatation and special conformal transformation for the special case $pd = 2$. The symmetries of the action are discussed in the following subsections.

A. Global U(1) invariance

The action $\mathcal{A}$ is invariant under a global U(1) transformation, $\psi(\mathbf{x},t) \to \psi'(\mathbf{x},t) = e^{is}\psi(\mathbf{x},t)$, where s is a real constant. The corresponding conserved charge is the total number N,

$$N = \int \rho(\mathbf{x},t)d^d\mathbf{x}, \quad \rho(\mathbf{x},t) \equiv \psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t). \tag{60}$$

Note that N is neither Hermitian nor a semipositive-definite quantity. Thus, N is identified as quasipower in the literature [28]. We now show that N is real-valued. It is always possible to decompose $\psi(\mathbf{x},t)$ as a sum of parity-even and parity-odd terms:

$$\psi(\mathbf{x},t) = \psi_e(\mathbf{x},t) + \psi_o(\mathbf{x},t), \tag{61}$$

where

$$\psi_e(\mathbf{x},t) = \frac{\psi(\mathbf{x},t)+\psi(\mathcal{P}\mathbf{x},t)}{2}, \quad \psi_o(\mathbf{x},t) = \frac{\psi(\mathbf{x},t)-\psi(\mathcal{P}\mathbf{x},t)}{2}. \tag{62}$$

With this decomposition of $\psi(\mathbf{x},t)$, the density ρ can be expressed in terms of the redefined field variables as sum of a real-valued parity-even term and a parity-odd term that is purely imaginary. In particular,

$$\rho(\mathbf{x},t) = \rho_r(\mathbf{x},t) + \rho_c(\mathbf{x},t) \tag{63}$$

with

$$\rho_r(\mathbf{x},t) = |\psi_e(\mathbf{x},t)|^2 - |\psi_o(\mathbf{x},t)|^2, \quad \rho_c(\mathbf{x},t) = \psi_e^*(\mathbf{x},t)\psi_o(\mathbf{x},t) - \psi_o^*(\mathbf{x},t)\psi_e(\mathbf{x},t). \tag{64}$$

Note the following properties of $\rho_r(\mathbf{x},t)$ and $\rho_c(\mathbf{x},t)$:

$$\rho_r^*(\mathbf{x},t) = \rho_r(\mathbf{x},t), \quad \mathcal{P}\rho_r(\mathbf{x},t) = \rho_r(\mathbf{x},t), \quad \rho_c^*(\mathbf{x},t) = -\rho_c(\mathbf{x},t), \quad \mathcal{P}\rho_c(\mathbf{x},t) = -\rho_c(\mathbf{x},t). \tag{65}$$

The density is a complex-valued function. However, the total number N, as defined by Eq. (60), does not receive any contribution from the parity-odd purely imaginary term $\rho_c(\mathbf{x},t)$ and is real, $N = \int d^d\mathbf{x}\rho_r(\mathbf{x},t)$. This result is valid for any spatial dimensions, and we have illustrated it in Appendix A for one and two spatial dimensions. Note that unlike the local NLSE, N can take positive as well as negative values. Thus, a proper interpretation is required for the total number operator N in the corresponding quantum theory.

The continuity equation for Eq. (56) reads

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = \mathbf{0}, \quad \mathbf{J} = \frac{i}{2}[\psi(\mathbf{x},t)\nabla\psi^*(\mathcal{P}\mathbf{x},t) - \psi^*(\mathcal{P}\mathbf{x},t)\nabla\psi(\mathbf{x},t)], \tag{66}$$

where the current density $\mathbf{J}$ can be rewritten in terms of the fields $\psi_e(\mathbf{x},t)$ and $\psi_0(\mathbf{x},t)$ as a sum of a parity-odd real term and a parity-even purely imaginary term, $\mathbf{J} = \mathbf{J}_r + \mathbf{J}_i$, with

$$\mathbf{J}_r = \frac{i}{2}[\psi_e(\mathbf{x},t)\nabla\psi_e^*(\mathbf{x},t) - \psi_o(\mathbf{x},t)\nabla\psi_o^*(\mathbf{x},t) - \psi_e^*(\mathbf{x},t)\nabla\psi_e(\mathbf{x},t) + \psi_o^*(\mathbf{x},t)\nabla\psi_o(\mathbf{x},t)], \tag{67}$$

$$\mathbf{J}_i = \frac{i}{2}[\psi_o(\mathbf{x},t)\nabla\psi_e^*(\mathbf{x},t) - \psi_e(\mathbf{x},t)\nabla\psi_o^*(\mathbf{x},t) + \psi_o^*(\mathbf{x},t)\nabla\psi_e(\mathbf{x},t) - \psi_e^*(\mathbf{x},t)\nabla\psi_o(\mathbf{x},t)]. \tag{68}$$

The following properties of $\mathbf{J}_r$ and $\mathbf{J}_i$ may be noted:

$$\mathbf{J}_r(\mathbf{x},t)^* = \mathbf{J}_r(\mathbf{x},t),\ \mathcal{P}\mathbf{J}_r(\mathbf{x},t) = -\mathbf{J}_r(\mathbf{x},t), \quad \mathbf{J}_i^*(\mathbf{x},t) = -\mathbf{J}_i(\mathbf{x},t), \quad \mathcal{P}\mathbf{J}_i(\mathbf{x},t) = \mathbf{J}_i(\mathbf{x},t), \tag{69}$$

which will be useful in showing real-valuedness of some of the conserved charges and moments to be defined below.

B. Spatial translation

The action is invariant under the spatial translation $\mathbf{x}' = \mathbf{x} + \delta\mathbf{x}$ with

$$\psi'(\mathbf{x}',t) = \psi(\mathbf{x},t), \quad \psi'^*(-\mathbf{x}',t) = \psi(-\mathbf{x},t), \tag{70}$$

giving rise to the momentum $\mathbf{P} = \int \mathbf{J} d^d\mathbf{x}$ as the conserved charge. The exact solutions of Eq. (56) for $d = 1$ are not invariant under an arbitrary shift of the coordinate. Thus, these solutions explicitly break the translational invariance. Defining the center-of-mass location as

$$\mathbf{X} = \frac{1}{Nd}\int \mathbf{x}\rho(\mathbf{x},t)d^d\mathbf{x}, \tag{71}$$

it is easy to verify by using the continuity equation that

$$N\frac{d\mathbf{X}}{dt} = \mathbf{P}, \tag{72}$$

where $|\frac{d\mathbf{X}}{dt}|$ may be identified as the speed of the center of mass. The total momentum $\mathbf{P}$ is complex-valued in even spatial dimensions and is purely imaginary in odd spatial dimensions. This result is presented in Appendix A for $d = 1,2$. Similarly, one can show that the center of mass $\mathbf{X}$ is purely imaginary in odd spatial dimensions, while it is complex in even spatial dimensions. Thus, neither the total momentum nor the center of mass can be considered physical.

C. Time translation

The invariance of $\mathcal{A}$ under time translation leads to the conserved quantity

$$\mathcal{H} = \int \left[\frac{1}{2}\nabla\psi^*(\mathcal{P}\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) + \frac{g}{p+1}\{\psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t)\}^{p+1}\right]d^d\mathbf{x}, \tag{73}$$

which is identified as the Hamiltonian of the system. Note that H is not semipositive-definite, since semipositivity is not ensured for any of the terms appearing in $\mathcal{H}$. The Hamiltonian is also non-Hermitian with the standard definition of norm. This should be contrasted with the Hamiltonian corresponding to the usual local NLSE, for which H is Hermitian, and for the defocusing case it is semipositive-definite.

We now show that the total Hamiltonian $\mathcal{H}$ is real-valued in spite of it being non-Hermitian. The kinetic-energy term in the Hamiltonian density can be decomposed as a parity-even real term and a parity-odd purely imaginary term:

$$\begin{aligned}\nabla\psi^*(\mathcal{P}\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) &= [|\nabla\psi_e(\mathbf{x},t)|^2 - |\nabla\psi_o(\mathbf{x},t)|^2]\\ &\quad + [\nabla\psi_e^*(\mathbf{x},t)\cdot\nabla\psi_o(\mathbf{x},t) - \nabla\psi_o^*(\mathbf{x},t)\cdot\nabla\psi_e(\mathbf{x},t)].\end{aligned} \tag{74}$$

The first term is real and even under parity transformation, while the second term is purely imaginary and odd under parity transformation. Thus, the second term does not contribute to $\mathcal{H}$ and the contribution of the kinetic term to $\mathcal{H}$ is real. Similarly, the interaction term in $\mathcal{H}$ can be shown to be real-valued. In particular,

$$\begin{aligned}\frac{g}{p+1}\int d^d\mathbf{x}\,\rho^{p+1} &= \frac{g}{p+1}\int d^d\mathbf{x}\sum_{j=0}^{p+1}{}^{p+1}C_j\rho_c^j\rho_r^{p+1-j}\\ &= \frac{g}{p+1}\int d^d\mathbf{x}\sum_{k=0}^{[\frac{p+1}{2}]}(-1)^k\,{}^{p+1}C_{2k}|\rho_c|^{2k}\rho_r^{p+1-2k},\end{aligned} \tag{75}$$

where $[n]$ denotes the integral part of n, and ${}^nC_r = \frac{n!}{r!(n-r)!}$. It may be recalled that ρ_c^j is odd under parity transformation for odd j, while ρ_r^{p+1-j} is a parity-even term for any j. Thus, the summation over odd j terms does not contribute to the interaction term. The Hamiltonian $\mathcal{H}$ can be rewritten as

$$\mathcal{H} = \frac{1}{2}\int d^d\mathbf{x}[|\nabla\psi_e(\mathbf{x},t)|^2 - |\nabla\psi_o(\mathbf{x},t)|^2] + \frac{g}{p+1}\int d^d\mathbf{x}\sum_{k=0}^{[\frac{p+1}{2}]}(-1)^k\,{}^{p+1}C_{2k}|\rho_c|^{2k}\rho_r^{p+1-2k}, \tag{76}$$

which is real-valued and can take positive as well as negative values.

D. Spatial rotation

The action is invariant under rotation and the corresponding conserved charge is the angular momentum whose $d(d-1)/2$ components are given by

$$L_{ij} = \int (x_i J_j - x_j J_i)d^d\mathbf{x}, \quad i,j = 1,2,\ldots,d, \tag{77}$$

where J_i is the ith component of the current density $\mathbf{J}$. It may be verified by using Eq. (69) that L_{ij}'s are real in odd spatial dimensions, while they are complex in even spatial dimensions.

E. Galilean transformation

The action is invariant under the Galilean transformation. In particular, the fields $\psi(\mathbf{x},t)$, $\psi^*(\mathcal{P}\mathbf{x},t)$ transform under the Galilean transformation $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ as

$$\psi'(\mathbf{x}',t) = e^{-i\mathbf{v}\cdot(\mathbf{x}'+\frac{1}{2}\mathbf{v}t)}\psi(\mathbf{x},t), \tag{78}$$

$$\psi'^*(\mathcal{P}\mathbf{x}',t) = e^{i\mathbf{v}\cdot(\mathbf{x}'+\frac{1}{2}\mathbf{v}t)}\psi^*(\mathcal{P}\mathbf{x},t). \tag{79}$$

It may be recalled that the exact solutions of Eq. (56) for $d = 1$ are not invariant under a purely time-dependent shift of the coordinate. Thus, these solutions break the Galilean invariance explicitly. The conserved charge associated with the Galilean symmetry is boost,

$$\mathbf{B} = t\,\mathbf{P} - \mathbf{X}, \tag{80}$$

which is complex-valued in even spatial dimensions and purely imaginary for odd d. The conservation of $\mathbf{B}$ directly follows from Eq. (72).

F. Conformal symmetry for $pd = 2$

Consider the following transformations:

$$\mathbf{x} \to \mathbf{x}_h = \dot\tau^{-\frac{1}{2}}(t)\mathbf{x}, \quad t \to \tau = \tau(t),$$

$$\psi(\mathbf{x},t) \to \psi_h(\mathbf{x}_h,\tau) = \dot\tau^{\frac{d}{4}}\exp\left(-i\frac{\ddot\tau}{4\dot\tau}x_h^2\right)\psi(\mathbf{x},t),$$

$$\psi^*(\mathcal{P}\mathbf{x},t) \to \psi_h^*(\mathcal{P}\mathbf{x}_h,\tau) = \dot\tau^{\frac{d}{4}}\exp\left(i\frac{\ddot\tau}{4\dot\tau}x_h^2\right)\psi^*(\mathcal{P}\mathbf{x},t),$$

where

$$\tau(t) = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \alpha\delta - \beta\gamma = 1. \tag{81}$$

The particular choices $\tau(t) = t + \beta$, $\tau(t) = \alpha^2 t$, and $\tau(t) = \frac{t}{1+\gamma t}$ correspond to time translation, dilatation, and special conformal transformation, respectively. The action $\mathcal{A}$ is invariant under time translation in arbitrary d, and the corresponding conserved quantity is given in Eq. (73). The action $\mathcal{A}$ is invariant under dilatation and special conformal transformations for $pd = 2$. This corresponds to a quintic NLSE in $1 + 1$ dimensions and cubic NLSE in $2 + 1$ dimensions. The conserved charges corresponding to dilatation (D) and special conformation transformation (K) are

$$D = tH - I_2, \tag{82}$$

$$K = -t^2 H + 2tD + I_1, \tag{83}$$

where the moments I_1 and I_2 are defined as

$$I_1(t) = \frac{1}{2}\int d^d\mathbf{x}\, x^2\, \rho(\mathbf{x},t),\qquad I_2(t) = \frac{1}{2}\int d^d\mathbf{x}\, \mathbf{x}\cdot\mathbf{J}, \tag{84}$$

where $x^2 = \mathbf{x}\cdot\mathbf{x}$. I_1 may be considered as the "pseudowidth" of the wave packet, and I_2 represents the growth speed of the system. It may be noted that neither I_1 nor I_2 is Hermitian and semipositive-definite. However, both I_1 and I_2 can be shown to be real-valued. For example, the moment I_1 may be rewritten by using Eqs. (63) and (64) as

$$I_1 = \frac{1}{2}\int d^d\mathbf{x}\, x^2\, \rho_r(\mathbf{x},t) = \frac{1}{2}\int d^d\mathbf{x}\, x^2[|\psi_e(\mathbf{x},t)|^2 - |\psi_o(\mathbf{x},t)|^2]. \tag{85}$$

The moment I_1 can be expressed as the difference of two semipositive-definite moments, $I_1 = I_{1e} - I_{1o}$, where $I_{1e} \equiv \frac{1}{2}\int d^d\mathbf{x}\, x^2\, |\psi_e(\mathbf{x},t)|^2$ and $I_{1o} \equiv \frac{1}{2}\int d^d\mathbf{x}\, x^2\, |\psi_o(\mathbf{x},t)|^2$. Unlike the case of local NLSE, I_1 can be positive as well as negative, which restricts the analysis of its dynamics by using the moment method. The reality of I_2 is explained in Appendix A for $d = 1,2$. The dynamics of I_{1e} and I_{1o} are described in Appendix B. Finally, it is worth mentioning here that both D and K are real, since H, I_1, and I_2 are all real.

Following Refs. [42,43], the time development of $I_1(t)$ can be determined as

$$I_1(t) = \left(\sqrt{I_1(0)} + \frac{\dot{I_1}(0)}{2\sqrt{I_1(0)}}t\right)^2 + \frac{Q}{I_1(0)}t^2,\qquad Q \equiv I_1 H - \left(\frac{1}{2}\frac{dI_1}{dt}\right)^2, \tag{86}$$

where $I_1(0)$ and $\dot{I}_1(0)$ are the values of $I_1(t)$ and $\frac{dI_1}{dt}$ at $t = 0$. The Casimir operator Q of the underlying O(2,1) group is a constant of motion and can take real values only. The moment I_1 vanishes at a finite real time t^* for $Q < 0$ only,

$$t^* = \frac{4I_1(0)}{Q + \{\dot{I}_1(0)\}^2}\left[-\frac{\dot{I}_1(0)}{2} \pm \sqrt{-Q}\right]. \tag{87}$$

Note that t^* can be made positive by appropriately choosing $I_1(0)$, $\dot{I}_1(0)$, and H. Unlike the case of the local NLSE, the vanishing of I_1 at a real finite time does not necessarily imply the collapse of the condensate. The vanishing of I_1 rather signifies a transition from positive I_1 to a negative value or *vice versa*. It is not clear at this point whether this transition has any physical significance or not. The vanishing of I_1 at a finite real time can be achieved when any of the following four conditions is satisfied: (i) $I_1(0) > 0, H < 0$; (ii) $I_1(0) > 0, H > 0, \dot{I}_1(0) \leqslant -2\sqrt{I_1(0)H}$; (iii) $I_1(0) < 0, H > 0$; and (iv) $I_1(0) < 0, H < 0, \dot{I}_1(0) \leqslant -2\sqrt{|I_1(0)H|}$. The first two conditions are applicable to the local NLSE also. However, the last two conditions are valid for the nonlocal NLSE only.

The action is invariant under a duality symmetry. Consider a particular $\tau(t)$,

$$\alpha = \delta = 0, \quad \gamma = -\frac{1}{\beta}, \quad \tau = -\frac{\beta^2}{t}, \tag{88}$$

which may be thought of as a combined operation of translation in time by β, followed by a special conformal transformation and again a time translation by β. The transformation of the spatial coordinate and the fields reads

$$\mathbf{x} \to \mathbf{x}_h = \frac{t}{\beta}\mathbf{x} = -\frac{\beta}{\tau}\mathbf{x},$$
$$\psi(\mathbf{x},t) \to \psi_h(\mathbf{x}_h,\tau) = \left(\frac{\beta}{t}\right)^{\frac{d}{2}} \exp\left(i\frac{tx^2}{2\beta^2}\right)\psi(\mathbf{x},t), \tag{89}$$
$$\psi^*(\mathcal{P}\mathbf{x},t) \to \psi_h^*(\mathcal{P}\mathbf{x}_h,\tau) = \left(\frac{\beta}{t}\right)^{\frac{d}{2}} \exp\left(-i\frac{tx^2}{2\beta^2}\right)\psi^*(\mathcal{P}\mathbf{x},t),$$

which is known as lens transformation [44] for the case of a critical local NLSE. The parameter β being real, the theory at $\tau > 0$ is mapped to a theory at a time $t < 0$ and *vice versa* with $\tau = 0 = t$ separating the two regions. We choose the following convention,

$$\beta > 0, \quad 0 \leqslant t \leqslant \infty, \quad -\infty \leqslant \tau \leqslant 0. \tag{90}$$

Following Ref. [43], we find that the system admits explosion-implosion duality either for (a) $H > 0, I_1(0) > 0$ or (b) $H < 0, I_1(0) < 0$ such that $Q > 0$, i.e.,

$$|H| \geqslant \left(\frac{\dot{I}_1(0)}{2\sqrt{|I_1(0)|}}\right)^2. \tag{91}$$

The pseudowidth explodes in the physical problem and implodes in the dual problem for both cases. The physical problem for the first case describes the growth of I_1 from its initial positive value to ∞ at $t = \infty$. On the other hand, for the second case, the initial negative value of I_1 in the physical problem decreases to $-\infty$ at $t = \infty$. The second case described above is not allowed for the local NLSE, since I_1 is a semipositive-definite quantity.

The Noether charges satisfy the $(d + 1)$-dimensional Schrödinger algebra:

$$\begin{aligned}
&\{H,D\} = H, \quad \{H,K\} = 2D, \quad \{D,K\} = K,\\
&\{\mathbf{P},D\} = \frac{1}{2}\mathbf{P}, \quad \{\mathbf{P},K\} = \mathbf{B},\\
&\{P_i,L_{jk}\} = -(\delta_{ij}P_k - \delta_{ik}P_j),\\
&\{L_{ij},L_{kl}\} = (\delta_{ik}L_{jl} - \delta_{il}L_{jk} - \delta_{jk}L_{il} + \delta_{jl}L_{ik}),\\
&\{H,\mathbf{B}\} = \mathbf{P}, \quad \{D,\mathbf{B}\} = \frac{\mathbf{B}}{2}, \quad \{P_i,B_j\} = \delta_{ij}N,\\
&\{B_i,L_{jk},\} = -(\delta_{ij}B_k - \delta_{ik}B_j).
\end{aligned} \tag{92}$$

All other Poisson brackets vanish identically. It may be recalled that all the conserved charges are non-Hermitian. Only H, D, K are real-valued in any dimensions, and L_{ij} are real only in odd spatial dimensions. Further analysis is required to understand the significance of this algebra in the context of the nonlocal NLSE.

IV. DYNAMICS OF MOMENTS

It is hard to find exact solutions of the higher-dimensional NLSE or its various generalizations in its generic form. The exact solution may be found only for particular cases. The qualitative nature of solutions of such systems may be described in terms of the dynamics of various moments [42,43,45,46]. In particular, the moments satisfy a set of coupled first-order differential equations with time as the independent variable. However, in general, this is not a close system of differential equations, and it involves spatial integrals involving fields. An exact time development of some of the moments may be described analytically for systems with dynamical conformal symmetry [42,43]. Consequently, important information regarding the time development of the field for different initial conditions may be inferred.

Consider the following nonautonomous NLSE in $d+1$ dimensions:

$$\begin{aligned} i\psi_t(\mathbf{x},t) = &-\tfrac{1}{2}\nabla^2\psi(\mathbf{x},t) + V(\mathbf{x},t)\psi(\mathbf{x},t) \\ &+ g(\mathbf{x},t)\psi^{*p}(\mathcal{P}\mathbf{x},t)\psi^p(\mathbf{x},t)\psi(\mathbf{x},t). \end{aligned} \tag{93}$$

This is a generalization of Eq. (56) where the system is considered in an external potential and the constant coefficient of the nonlinear term is allowed to become space-time-dependent. We define a moment H in addition to the moments I_1 and I_2 defined in Eqs. (84):

$$H = \frac{1}{2}\int \nabla\psi^*(\mathcal{P}\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t)d^d\mathbf{x} + \int G(\rho,\mathbf{x},t)\, d^d\mathbf{x}, \tag{94}$$

where $G(\rho,x,t) = \frac{g(x,t)}{1+p}\rho^{p+1}$. Defining $g' = g\rho^p$, $\frac{\partial G}{\partial \rho} = g'$. Following the standard technique [45,46], it is straightforward to show that the moments satisfy the following set of equations:

$$\begin{aligned} \frac{dI_1}{dt} &= 2I_2, \\ \frac{dI_2}{dt} &= -\frac{1}{2}\int \rho(\mathbf{x},t)\,(\mathbf{x}\cdot\nabla V)\,d^d\mathbf{x} + \tilde{H} \\ &\quad -\frac{1}{2}\int \rho(\mathbf{x},t)(\mathbf{x}\cdot\nabla g')d^d\mathbf{x}, \\ \frac{dH}{dt} &= -\int \nabla V\cdot\mathbf{J}d^d\mathbf{x} + \int \frac{\partial G}{\partial t}d^d\mathbf{x}, \end{aligned} \tag{95}$$

where $\tilde{H}$ is the first part of Eq. (94). If we restrict to the quadratic potential of the form $V = \frac{1}{2}\omega^2\mathbf{x}\cdot\mathbf{x}$ and $G = g_0\rho^{1+\frac{2}{d}}$, Eqs. (95) give a close system of equations:

$$\begin{aligned} &\frac{dI_1}{dt} = 2I_2, \quad \frac{dI_2}{dt} = -\omega^2 I_1 + H, \\ &\frac{dH}{dt} = -2\omega^2 I_2, \end{aligned} \tag{96}$$

It may be noted that the condition $pd = 2$ is essential in deriving the above set of equations, which corresponds to conformal symmetry for the system described by H. A decoupled equation for the pseudowidth $\mathcal{X} = \sqrt{I_1}$ satisfies Hill's equation [47]:

$$\frac{d^2\mathcal{X}}{dt^2} + \omega^2\mathcal{X} = \frac{Q}{\mathcal{X}^3}. \tag{97}$$

Equation (97) has the same form of a particle moving in an inverse-square potential plus a time-dependent harmonic trap. The general solution of Eq. (97) may be written as

$$\mathcal{X}^2(t) = u^2 + \frac{Q}{W^2}v^2(t), \quad W(t) \equiv u\dot{v} - v\dot{u}, \tag{98}$$

where $u(t)$ and $v(t)$ are two independent solutions of the following equation:

$$\begin{aligned} &\ddot{x} + \omega(t)x = 0, \\ &\quad u(t_0) = \mathcal{X}(t_0), \quad \dot{u}(t_0) = \dot{\mathcal{X}}(t_0), \\ &\quad \dot{v}(t_0) = 0, \quad v(t_0) \neq 0, \end{aligned} \tag{99}$$

and W is the corresponding Wronskian. We conclude this section with the following comments:

(i) The system admits explosion-implosion duality [43] for the special choice of the time-dependent frequency $\omega(t) = (\frac{\omega_0\beta}{t})^2$, $\omega_0 \in R$, and $Q > 0$.

(ii) The system exhibits parametric instability [45] for periodic $\omega(t)$ with period T when the condition $\delta = |u(T) + \dot{v}(T)| > 2$ is satisfied with the normalization $\mathcal{X}(0) = 0$, $\dot{\mathcal{X}}(0) = 1$, $v(0) = 1$. The system is stable for $\delta < 2$.

V. SUMMARY AND DISCUSSIONS

We have considered a generalization of the recently introduced integrable nonlocal NLSE with self-induced potential that is $\mathcal{PT}$-symmetric in the corresponding stationary problem. In contrast to the standard formulation of complex scalar field theory, the Schrödinger field and its parity-transformed complex conjugate are treated as two independent fields. We have studied a class of nonlocal NLSE in an external potential with a space-time-modulated coefficient of the nonlinear interaction term as well as confining and/or loss-gain terms. We have obtained exact soliton solutions for the inhomogeneous and/or nonautonomous nonlocal NLSE by using similarity transformation, and the method is illustrated with a few specific examples. We have found that only those transformations are allowed for which the transformed spatial coordinate is odd under the parity transformation of the original one. This puts some restrictions on the types of external potentials, loss-gain terms, and space-time-modulated coefficients for which the method is applicable. Nevertheless, the choices are infinitely many and most of the physically relevant examples are included. It is interesting to note that all the solutions of the local NLSE are also solutions of the corresponding nonlocal NLSE with identical space-time-modulated coefficients, external potential, loss-gain terms, nonlinear interaction, etc. The difference is that the range of the coupling constant of the nonlinear interaction term for which the solutions exist is different for an odd solution of local NLSE and the corresponding nonlocal NLSE. However, the ranges are identical for an even solution.

We have studied the invariance of the action of a $(d+1)$-dimensional generalization of the nonlocal NLSE under different symmetry transformations. We have found that the action is invariant under space-time translation, rotation, global U(1) gauge transformation, and Galilean transformation. The system is invariant under dilatation and special conformal transformations when $pd = 2$. It is shown that H, D, K, and $\mathbf{L}$ are real-valued, although the formal expressions of these conserved Noether charges are non-Hermitian. The conserved momentum and the total boost are complex-valued in any spatial dimensions. Further, the conserved charges satisfy the $(d+1)$-dimensional Schrödinger algebra. We have also studied the dynamics of different moments with an exact description of the time evolution of the "pseudowidth" of the wave packet for the special case in which the action admits a O(2,1) conformal symmetry.

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APPENDIX A: REAL-VALUEDNESS OF SOME OF THE NON-HERMITIAN NOETHER CHARGES

Parity is a discrete transformation with the determinant of the transformation matrix equal to -1. Thus, in odd spatial dimensions, a parity transformation can be realized by flipping the signs of all the coordinates. On the other hand, the sign of only an odd number of coordinates can be reversed in the case of even spatial dimensions. Thus, for example, we have $\mathcal{P}\psi(x,t) = \psi(-x,t)$ in one spatial dimension. However, in two spatial dimensions, we have either $\mathcal{P}\psi(x,y,t) = \psi(-x,y,t)$ or $\mathcal{P}\psi(x,y,t) = \psi(x,-y,t)$. We choose the first relation as our convention for illustrating results related to the real-valuedness of some of the conserved Noether charges that are non-Hermitian.

(i) N *in* $d=1$ *dimension.* We use the following properties of $\rho_r(x,t)$ and $\rho_c(x,t)$:

$$\begin{aligned} \rho_r^*(x,t) &= \rho_r(x,t), \quad \mathcal{P}\rho_r(x,t) = \rho_r(x,t), \\ \rho_c^*(x,t) &= -\rho_c(x,t), \quad \mathcal{P}\rho_c(x,t) = -\rho_c(x,t), \end{aligned} \tag{A1}$$

which allows us to write $N = \int_{-\infty}^{\infty} dx\, \rho(x,t) = \int_{-\infty}^{\infty} dx[\rho_r(x,t) + \rho_c(x,t)] = \int_{-\infty}^{\infty} dx \rho_r(x,t)$.

(ii) N *in* $d=2$ *dimensions.* Similarly, in two dimensions we have

$$\begin{aligned} \mathcal{P}\rho_r(x,y,t) &= \rho_r(-x,y,t) = \rho_r(x,y,t), \\ \mathcal{P}\rho_c(x,y,t) &= \rho_c(-x,y,t) = -\rho_c(x,y,t), \end{aligned} \tag{A2}$$

$$\begin{aligned} N &= \int_{-\infty}^{\infty} [\rho_r(x,y,t) + \rho_c(x,y,t)]\, dx\, dy \\ &= \int_{-\infty}^{\infty} \left[\int_0^{\infty} [\rho_r(x,y,t) - \rho_c(x,y,t)]\, dx \right. \\ &\quad \left. + \int_0^{\infty} [\rho_r(x,y,t) + \rho_c(x,y,t)]\, dx \right] dy \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} 2\rho_r(x,y,t) dx\, dy. \end{aligned} \tag{A3}$$

Thus it turns out that N is real.

(iii) $\mathbf{P}$ *in* $d=1$ *dimension.* We shall use the following relations:

$$\begin{aligned} J_r(\mathcal{P}x,t) &= J_r(-x,t) = -J_r(x,t), \\ J_i(\mathcal{P}x,t) &= J_i(-x,t) = J_i(x,t), \end{aligned} \tag{A4}$$

to evaluate the integral

$$P = \int_{-\infty}^{\infty} [J_r(x) + J_i(x)] dx, \tag{A5}$$

which turns out to be

$$\begin{aligned} P &= \int_0^{\infty} [-J_r(x) + J_i(x)]\, dx + \int_0^{\infty} [J_r(x) + J_i(x)]\, dx \\ &= 2\int_0^{\infty} J_i(x) dx. \end{aligned} \tag{A6}$$

(iv) $\mathbf{P}$ *in* $d=2$ *dimensions.* We shall use the following relations:

$$\begin{aligned} \mathcal{P}J_{rx}(x,y,t) &= J_{rx}(-x,y,t) = -J_{rx}(x,y,t), \\ \mathcal{P}J_{ix}(x,y,t) &= J_{ix}(-x,y,t) = +J_{ix}(x,y,t), \\ \mathcal{P}J_{ry}(x,y,t) &= J_{ry}(-x,y,t) = +J_{ry}(x,y,t), \\ \mathcal{P}J_{iy}(x,y,t) &= J_{iy}(-x,y,t) = -J_{iy}(x,y,t), \end{aligned} \tag{A7}$$

and the expression of $\mathbf{P}$ is given as

$$\begin{aligned} \mathbf{P} &= \int_{-\infty}^{\infty} \{\mathbf{J}_r(x,y,t) + \mathbf{J}_i(x,y,t)\} dx\, dy \\ &= \hat{\mathbf{x}} \int_{-\infty}^{\infty} \{J_{rx}(x,y,t) + J_{ix}(x,y,t)\} dx\, dy + \hat{\mathbf{y}} \int_{-\infty}^{\infty} \{(J_{ry}(x,y,t) + J_{iy}(x,y,t)\} dx\, dy \\ &= \hat{\mathbf{x}} \int_{-\infty}^{\infty} \int_0^{\infty} \{J_{rx}(-x,y,t) + J_{ix}(-x,y,t) + J_{rx}(x,y,t) + J_{ix}(x,y,t)\}\, dx\, dy \\ &\quad + \hat{\mathbf{y}} \int_{-\infty}^{\infty} \int_0^{\infty} \{J_{ry}(-x,y,t) + J_{iy}(-x,y,t) + J_{ry}(x,y,t) + J_{iy}(x,y,t)\} dx\, dy \end{aligned}$$

$$= \hat{\mathbf{x}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \{-J_{rx}(x,y,t) + J_{ix}(x,y,t) + J_{rx}(x,y,t) + J_{ix}(x,y,t)\} dx\, dy$$
$$+ \hat{\mathbf{y}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \{J_{ry}(x,y,t) - J_{iy}(x,y,t) + J_{ry}(x,y,t) + J_{iy}(x,y,t)\} dx\, dy$$
$$= 2 \int_{-\infty}^{\infty} \int_{0}^{\infty} [\hat{\mathbf{x}} J_{ix}(x,y,t) + \hat{\mathbf{y}} J_{ry}(x,y,t)]\, dx\, dy. \tag{A8}$$

(v) *I_1 in $d = 1$ dimension.*

$$I_1 = \frac{1}{2} \int_{-\infty}^{\infty} x^2 \rho(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} x^2 [\rho_r(x,t) + \rho_c(x,t)] dx$$
$$= \frac{1}{2} \int_{0}^{\infty} \{x^2 \rho_r(-x,t) + x^2 \rho_r(x,t) + x^2 \rho_c(-x,t) dx + x^2 \rho_c(x,t)\} dx$$
$$= \int_{0}^{\infty} x^2 \rho_r(x,t) dx, \tag{A9}$$

where we have used Eqs. (63) and (A1).

(vi) *I_1 in $d = 2$ dimensions.*

$$I_1 = \frac{1}{2} \int_{-\infty}^{\infty} [x^2 \rho(x,y,t) + y^2 \rho(x,y,t)] dx\, dy = \frac{1}{2} \int_{-\infty}^{\infty} (x^2 + y^2)(\rho_r + \rho_c) dx\, dy$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \{x^2 \rho_r(-x,y,t) + x^2 \rho_r(x,y,t) + x^2 \rho_c(-x,y,t) + x^2 \rho_c(x,y,t)\} dx\, dy$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \{y^2 \rho_r(-x,y,t) + y^2 \rho_r(x,y,t) + y^2 \rho_c(-x,y,t) + y^2 \rho_c(x,y,t)\} dx\, dy$$
$$= \int_{-\infty}^{\infty} dy \left[\int_{0}^{\infty} x^2 \rho_r(x,y,t) + y^2 \rho_r(x,y,t) \right] dx, \tag{A10}$$

where we have used Eq. (A2).

(vii) *I_2 in $d = 1$ dimension.*

$$I_2 = \frac{1}{2} \int_{-\infty}^{\infty} dx\, x J = \int_{-\infty}^{\infty} dx\, x\, [J_r(x,t) + J_i(x,t)]$$
$$= \frac{1}{2} \int_{0}^{\infty} \{(-x) J_r(-x,t) + x J_r(x,t) + (-x) J_i(-x,t) + x J_i(x,t)\}\, dx$$
$$= \int_{0}^{\infty} dx\, x J_r(x,t), \tag{A11}$$

where we have used Eq. (A4).

(viii) *I_2 in $d = 2$ dimensions.*

$$I_2 = \frac{1}{2} \int_{-\infty}^{\infty} dx\, dy (x J_x + y J_y)$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} dx\, dy\, [x \{J_{rx}(x,y,t) + J_{ix}(x,y,t)\} + y \{J_{rx}(x,y,t) + J_{ix}(x,y,t)\}]$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \{-x J_{rx}(-x,y,t) + x J_{rx}(x,y,t) - x J_{ix}(-x,y,t) + x J_{ix}(x,y,t)\}\, dx$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \{y J_{ry}(-x,y,t) + y J_{ry}(x,y,t) + y J_{iy}(-x,y,t) + y J_{iy}(x,y,t)\} dx$$
$$= \int_{-\infty}^{\infty} dy \left[\int_{0}^{\infty} x J_{rx}(x,y,t) + \int_{0}^{\infty} y J_{ry}(x,y,t) \right], \tag{A12}$$

where we have used Eq. (A7).

APPENDIX B: DYNAMICS OF I_{1e}, I_{1o}, I_{2e}, AND I_{2o}

In this appendix, we show that the time derivative of I_1 and I_2 admits a partial splitting in terms of ψ_e and ψ_o. We present our results for $d = 1$. However, they can be easily generalized to higher dimensions. The time developments of I_{1e} and I_{1o} are described by the equations

$$\frac{dI_{1e}}{dt} = 2I_{2e} - \frac{ig}{2}\int_{-\infty}^{\infty} x^2\rho^2(\psi_e^*\psi_o + \psi_e\psi_o^*)dx, \quad \text{(B1)}$$

$$\frac{dI_{1o}}{dt} = 2I_{2o} - \frac{ig}{2}\int_{-\infty}^{\infty} x^2\rho^2(\psi_e^*\psi_o + \psi_e\psi_o^*)dx, \quad \text{(B2)}$$

where the moments I_{2e} and I_{2o} are defined as

$$I_{2e} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, x\, \frac{i}{2}\left(\psi_e\frac{\partial\psi_e^*}{\partial x} - \psi_e^*\frac{\partial\psi_e}{\partial x}\right), \quad \text{(B3)}$$

$$I_{2o} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, x\, \frac{i}{2}\left(\psi_o\frac{\partial\psi_o^*}{\partial x} - \psi_o^*\frac{\partial\psi_o}{\partial x}\right). \quad \text{(B4)}$$

If we subtract Eq. (B2) from Eq. (B1), then the left-hand side gives $\frac{dI_1}{dt}$ and the last terms on the right-hand sides cancel, leading to the equation $\frac{dI_1}{dt} = 2I_2$.

The equations satisfied by I_{2e} and I_{2o} are

$$\frac{dI_{2e}}{dt} = H_{ke} - \frac{g}{4}\int_{-\infty}^{\infty} x\frac{\partial\rho^2}{\partial x}\left(2\psi_e^*\psi_e + \psi_o\psi_e^* - \psi_o^*\psi_e\right)dx + \frac{g}{4}\int_{-\infty}^{\infty} x\rho^2\left(\psi_o\frac{\partial\psi_e^*}{\partial x} - \psi_o^*\frac{\partial\psi_e}{\partial x} + \psi_e\frac{\partial\psi_o^*}{\partial x} - \psi_e^*\frac{\partial\psi_o}{\partial x}\right)dx, \quad \text{(B5)}$$

$$\frac{dI_{2o}}{dt} = H_{ko} - \frac{g}{4}\int_{-\infty}^{\infty} x\frac{\partial\rho^2}{\partial x}(2\psi_o^*\psi_o - \psi_o\psi_e^* + \psi_o^*\psi_e)dx + \frac{g}{4}\int_{-\infty}^{\infty} x\rho^2\left(\psi_o\frac{\partial\psi_e^*}{\partial x} - \psi_o^*\frac{\partial\psi_e}{\partial x} + \psi_e\frac{\partial\psi_o^*}{\partial x} - \psi_e^*\frac{\partial\psi_o}{\partial x}\right)dx, \quad \text{(B6)}$$

where H_{ke} and H_{ko} are given by

$$H_{ke} = \frac{1}{2}\frac{\partial\psi_e^*}{\partial x}\frac{\partial\psi_e}{\partial x}, \quad H_{ko} = \frac{1}{2}\frac{\partial\psi_o^*}{\partial x}\frac{\partial\psi_o}{\partial x}. \quad \text{(B7)}$$

If we subtract Eq. (B6) from Eq. (B5), then the left-hand side gives $\frac{dI_2}{dt}$ while the second terms on the right-hand side generate the potential part of the total Hamiltonian and the last terms cancel out. Thus, we recover the equation $\frac{dI_2}{dt} = H$. Note that none of the moments I_{1e}, I_{1o}, I_{2e}, and I_{2o} satisfies a decoupled equation like I_1 and I_2.

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