Crafting networks to achieve, or not achieve, chaotic states

Sarah De Nigris*
Department of Mathematics and Namur Center for Complex Systems–naXys, University of Namur, 8 rempart de la Vierge 5000 Namur, Belgium

Xavier Leoncini†
Aix Marseille Université, Université de Toulon, CNRS, CPT UMR 7332, 13288 Marseille, France
(Received 27 May 2014; revised manuscript received 12 September 2014; published 27 April 2015)

The influence of networks topology on collective properties of dynamical systems defined upon it is studied in the thermodynamic limit. A network model construction scheme is proposed where the number of links and the average eccentricity are controlled. This is done by rewiring links of a regular one-dimensional chain according to a probability p within a specific range r that can depend on the number of vertices N. We compute the thermodynamical behavior of a system defined on the network, the XY-rotors model, and monitor how it is affected by the topological changes. We identify the network effective dimension d as a crucial parameter: topologies with $d < 2$ exhibit no phase transitions, while topologies with $d > 2$ display a second-order phase transition. Topologies with $d = 2$ exhibit states characterized by infinite susceptibility and macroscopic chaotic, turbulent dynamical behavior. These features are also captured by d in the finite size context.

DOI: 10.1103/PhysRevE.91.042809 PACS number(s): 89.75.Hc, 05.20.−y, 05.45.−a

I. INTRODUCTION

Networks are ubiquitous in the reality surrounding us, and indeed the network perspective for systems of interacting agents has seen a real paradigm shift in various realms ranging from physics to biology, sociology, and economics [1–4]. One pivotal feature shared by many existing networks, like the World Wide Web [5–7] or social networks [8], is the so-called "small-world" property: two nodes are separated by a short path consisting in just a few edges thanks to the presence of long-range connections, the *shortcuts*, in the network. Since this property often arises in a self-organized fashion, it could seem natural at first to infer that those shortcuts favor the flow of information and more easily lead to collective states, as if a kind of evolutionary principle is at play. But are those long-range links always beneficial to enhance global coherence? A striking example of this dilemma can be the brain: from one side it displays the small-world property [9], but at the same time, there are evidences of *chaotic* response in living neural systems [10]. In contrast, small-world topologies can be a fertile substrate to enhance transport phenomena as navigation [11] and, more recently, it has also been shown that the overall conductance of a network is advantaged by the introduction of long-range links [12]. It hence appears highly nontrivial, when dealing with interacting agents upon a network, to ask oneself what kind of collective behavior they can possibly display, since a chaotic response can arise along with a coherent response due to the presence of long-range links.

This work is given within the following framework. We provide here a means to construct a class of networks in which the addition of long-range links can give rise to a whole range of dynamical and statistical behaviors and, in particular, it also entails a chaotic state of infinite susceptibility, similar to that encountered in Refs. [13] and [14]. Moreover, our network model is crafted to embed real networks characteristics but via minimal assumptions so as to ensure a certain form of simplicity. As we will display in the following, we related the different behaviors to the network *dimension*, which changes according to the injection of long-range links.

II. MODEL CONSTRUCTION

In our model, networks are built starting from a k-regular network with periodic boundary conditions and degree $k \propto N^{\gamma-1}(1 \leqslant \gamma \leqslant 2)$ (where N is the number of vertices), which constitutes the backbone. Practically, the nodes are laid on a one-dimensional ring and each of them interacts with its k closest neighbors (see Fig. 1). Therefore our starting configuration is completely symmetrical and invariant under rotations. In this work, we set γ close to 1, $\gamma = 1.2$, in order to have a few links per vertex (for instance, we get $k = 12$ for $N = 2^{14}$). This choice for the present work is meant to recover sparseness [4], which is a common feature in many real-world networks; nevertheless, the influence of the k parameter by itself was explored in [13]. We then proceed to a construction similar to the Watts-Strogatz construction for small world networks [15]: we rewire each link with probability p but, differently from [15], we impose to rewire it within a range r (Fig. 1). Therefore with our parameters (γ,p,r) we put three ingredients meant to mimic concrete systems: first the condition of *sparseness* through γ, i.e., a very low vertex degree compared to the system size [4]; second, we introduced the concept of *interaction range* constraining the links to be at most of a fixed length r; and last we inject *randomness* in the structure so to have a nonuniform degree. Hence, from one side, the range parameter r mimics the fact that in many natural and artificial systems interactions can occur only within a certain neighborhood and on the other side the probability p ensures the presence of randomness in the link distribution, so that all the length scales occur. The range concept is reminiscent of the Kleinberg model [16], but in our case, the choice of r entails a sharp cutoff in the distribution of the

*denigris.sarah@gmail.com
†Xavier.Leoncini@cpt.univ-mrs.fr

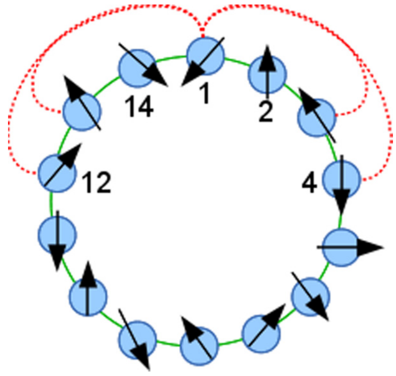

FIG. 1. (Color online) Practical network construction for $N = 14$, $\gamma = 1.2$ thus $k \propto \lfloor N^{0.2} \rfloor = 2$ and $r = \lfloor \sqrt{N} \rfloor = 3$. The starting configuration is the solid (green) line, since we have just two links per spin and the dotted (red) links are the possible rewiring.

accessible link lengths and, moreover, the probability p to rewire a link within the range r is uniform. Before proceeding we would like to stress that the key parameter of interest for the present work is the range: indeed, in two previous works [13,14] we investigated, respectively, the impact of the k parameter on k-regular networks and the interaction between k and p for small-world networks. Now, with the range constraint, we practically enforce a control on the dimension.

III. DEFINITION OF DIMENSION

Heuristically we can forecast that if we choose, for instance, $r \propto \sqrt{N}$, the more links are rewired (i.e., for high p), the more the network will be shaped like a bidimensional object, because we have in some sense crafted from the initial ring a $\sqrt{N} \times \sqrt{N}$ lattice. To give a more quantitative counterpart to this view, we define the dimension d similarly to the dimension on Euclidean lattices. For the latter, it holds a power law relation between the volume and a characteristic length $V \propto r^d$, the exponent d being the dimension. Then in our context of networks, we have to consider a specific length scale. Here we settled for the average of the vertices eccentricity $ec(i)$, i.e., the longest path $\ell_{i,j}$ $i \neq j$ attached to each vertex i. Thus we define our characteristic length ℓ as

$$\ell = \frac{1}{N} \sum_i ec(i). \tag{1}$$

Hence if we consider its scaling with the network volume (size) N, we obtain the following definition of dimension:

$$d = \frac{\log N}{\log \ell}. \tag{2}$$

The definition in Eq. (2) recovers in the $N \to \infty$ limit the one already proposed in [17–19] in which they consider the power law scaling of the average path length ℓ_{av} with the network size N, while we take into account in Eq. (1) the average vertex eccentricity ℓ_{ec}. These two quantities are indeed related since $\ell_{ec} \sim 2\ell_{av}$, and this assumption was also numerically tested. It is hence evident that the difference between the two dimension definitions is a term vanishing logarithmically with the size N, thus proving their equivalence in the $N \to \infty$ limit.

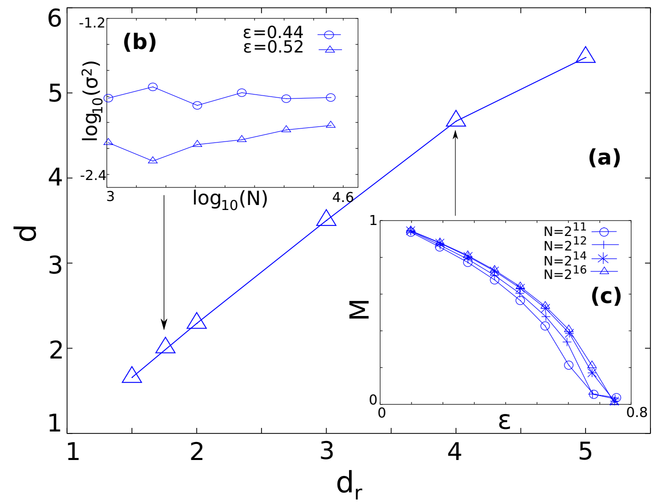

FIG. 2. (Color online) (a) Dimension of a completely rewired network ($p = 1$) with $N = 2^{14}$ and $r \propto N^{\delta}$. The horizontal axis is the parametrization in Eq. (3), which gives, with our choice of r, $d_r = 1/(1 - \delta)$. (b) Scaling of the magnetization variance $\sigma^2(N)$ with the system size N for $d = 2$. (c) Phase transition of the magnetization $M(\varepsilon)$, $\varepsilon = E/N$ for $d > 4$. The error bars are within the dot size.

However, in the range of system sizes used in our simulations, the definition in Eq. (1) was the more suitable choice to grasp the dimension, since the aforementioned difference is still important enough to introduce a small shift in the dimension value. In Eq. (2), it is straightforward to see that the dimension of the completely rewired ($p = 1$) configuration is intrinsically related to the range r: indeed, for $p = 1$, we have that $\ell \sim N/r$, since each node very probably possesses a link rewired at a distance r. Therefore, if $\ell \sim N/r$, we have that Eq. (2) becomes

$$d_r = \frac{\log N}{\log N - \log r}. \tag{3}$$

In what follows we shall use the dimension d_r given by Eq. (3) as our control parameter: in practice d_r corresponds to a reparametrization of the range which we will consider to be of the type $r \sim N^{\delta}$ with $N \gg 1$ and $\delta > \gamma - 1$. If we take our previous example of $r = \sqrt{N}$, we obtain that the corresponding network with $p = 1$ has indeed $d_r = 2$ and, in Fig. 2(a), we display how the measured network dimension for $p = 1$ follows Eq. (3) so that, fixing the range $r(N)$, we can control the resulting dimension once we have rewired all the links, independently from the size.

IV. THE XY– MODEL

Having thus an operative and general way to set and quantify the dimension, we used our network model to investigate the thermodynamic response of a dynamical system defined upon these networks and test the influence of the dimension d in Eq. (2). With this goal in mind, we consider N XY-rotors [20,21], whose dynamics is described by an angle $\theta_i(t)$, and its canonically associated momentum $p_i(t)$. We shall show that the rewiring of a few links, beyond altering the network structure significantly, can also entail different collective behaviors. In particular, we shall investigate if, like on regular lattices, we have a spontaneous symmetry breaking for $d > 2$, which is absent when $d < 2$. This brings some analogies to the extension of the Mermin-Wagner theorem on inhomogeneous structures [22,23], in which the critical parameter to discriminate between different regimes is the spectral dimension [24–26], therefore opening an interesting thread of research. Moreover, we shall focus on $d = 2$, or $r \sim \sqrt{N}$, to see if a chaotic state emerges, displaying some similarities with the one observed in the regular structure discussed in [13]. Returning back to the XY rotors, each rotor i is located on a network vertex and its interactions are provided by the set V_i of vertices attached to it via the links. The Hamiltonian of the system reads

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{J}{2\langle k\rangle} \sum_{i\in V_i} [1 - \cos(\theta_i - \theta_j)], \tag{4}$$

where $J > 0$, $\langle k\rangle$ is the average degree, and $V_i = \{j \neq i | \exists e_{i,j} \in E\}$, E being the ensemble of edges. The dynamics of the network is given by the two Hamilton equations:

$$\begin{cases} \dot{\theta}_i &= p_i \\ \dot{p}_i &= -\frac{J}{\langle k\rangle} \sum_{j\in V_i} \sin(\theta_i - \theta_j). \end{cases} \tag{5}$$

V. SIMULATION PROCEDURE

We run molecular dynamics (MD) simulations of the isolated system in Eq. (5), starting with Gaussian initial conditions for $\{\theta_i, p_i\}$. The simulations are performed by integrating the dynamic equations in Eq. (5) with the fifth-order optimal symplectic integrator, described in [27], with a time step of $\Delta t = 0.05$. Such an integrating scheme allows us to check the correctness of the numerical integration, since we verified at each time step that the conserved quantities of the system, the energy $E = H$ and the total momentum $P = \sum_i p_i/N$, are effectively constant. The total momentum P is set at 0 as the initial condition without loss of generality. In order to grasp the amount of coherence in the system, we define a magnetization $M = |\mathbf{M}|$ as the order parameter

$$\mathbf{M} = \frac{1}{N} \sum_i (\cos\theta_i, \sin\theta_i), \tag{6}$$

and once the system has reached a stationary state, we measure $\overline{M}$, where the bar stands for the temporal mean. Thus, in the stationary state, if $\overline{M} \sim 1$, all the rotors point in the same direction, whereas if $\overline{M} \sim 0$ there is not a preferred direction. Practically, once the network topology and the size N are fixed, we monitor the average magnetization $\overline{M(\varepsilon,N)}$ for each energy $\varepsilon = E/N$ in the physical range. We perform the temporal mean on the second half of the simulation, after checking that the magnetization has reached a stationary state, when it is reached (i.e., not in the case of the chaotic state). The simulation time is typically of order $Tf = 10^4$–10^5.

VI. THERMODYNAMIC BEHAVIOR

In the insets of Fig. 2 we display the dynamical response of the XY model to different dimensions: we chose r so as to have $d = 2$ for $r \propto \sqrt{N}$ and $d > 4$ for $r \propto N^{3/4}$. For the latter, in Fig. 2(c) the magnetization displays a second-order phase transition, seeming to occur at $\varepsilon = E/N \sim 0.75$, in the same fashion of the Hamiltonian mean field (HMF) model [28]. It is noteworthy that, for the XY model, the dimension 4 is the one at which mean field theory starts to apply and, indeed, the phase transition displayed in Fig. 2(c) for $d > 4$ seems to confirm this picture. For the case with $d = 2$ we observed a state similar to the one described in [13,14]; the magnetization, for low energy densities $\varepsilon = E/N$, is affected by important fluctuations, such as if the order parameter was oscillating between the mean field value and zero. Moreover, this regime does not reach the equilibrium on the time scales considered. Its persistence was checked for simulation times $Tf \sim 10^6$, i.e., 10 times longer than in previous cases and nevertheless, it was not possible to observe its relaxing. To give further insight on this chaotic state arising in the network with $d = 2$, we looked at the magnetization variance $\sigma^2 = \overline{(M - \overline{M})^2}$, where the bar stands again for the temporal mean, in order to give a quantitative measure of this regime. As shown in Fig. 2(b), the variance is unaffected by the size. This flat profile is in striking contrast with the variance's canonical scaling $\sigma^2 \propto 1/N$, leading to vanishing fluctuations in the thermodynamic limit. On the contrary, if we take into account the definition of the magnetic susceptibility,

$$\chi \sim \lim_{N\to\infty} N\sigma^2, \tag{7}$$

we have that this regime shall be characterized by an infinite susceptibility in the thermodynamic limit. The peculiar nature of this regime is also highlighted by its persistence in an energy range. Indeed, in the usual XY Kosterlitz-Thouless transition the divergence of the susceptibility occurs at the phase transition point [29], while these "turbulent" states exist in whole interval energies up to the critical state. In fact, these states are somewhat reminiscent of the observed *quasistationary states* (QSSs) occurring in the HMF model or more generally in systems with long-range interactions [30–34]. Nevertheless, as mentioned, we do not observe any relaxation, in contrast with the QSSs' behavior.

VII. FINITE SIZE CASE

Our model brings interesting perspectives for finite size systems as well. As a first observation, note that our construction procedure, like the Watts-Strogatz algorithm for small world networks [15], induces on average $N_R = Nkp \propto N^\gamma p$ rewired links. Hence the fraction of long-range connections increases with the size (in the present study very slightly because of our choice $\gamma = 1.2$). We thus argue the existence of a nontrivial interplay between p, r, and N, so that it is possible, like for small world networks, to tune p in order to change the measured dimension for a given size N. In some sense, d can turn out to be, for a finite size system, a measure of an *effective dimension* produced by the fraction of rewired links. To test our hypothesis, we consider $N = 2^{14}$ and $N = 2^{16}$, and in Figs. 3(a)–3(d) we show how the progressive introduction of long-range links in the network drags the dimension to $d = 2$ for $r = \sqrt{N}$ [Fig. 3(a)] and to $d = 4$ for $r = N^{3/4}$ [Fig. 3(d)]. Indeed, the shift between the two dimension curves mirrors the effect of the two sizes, and it is more pronounced

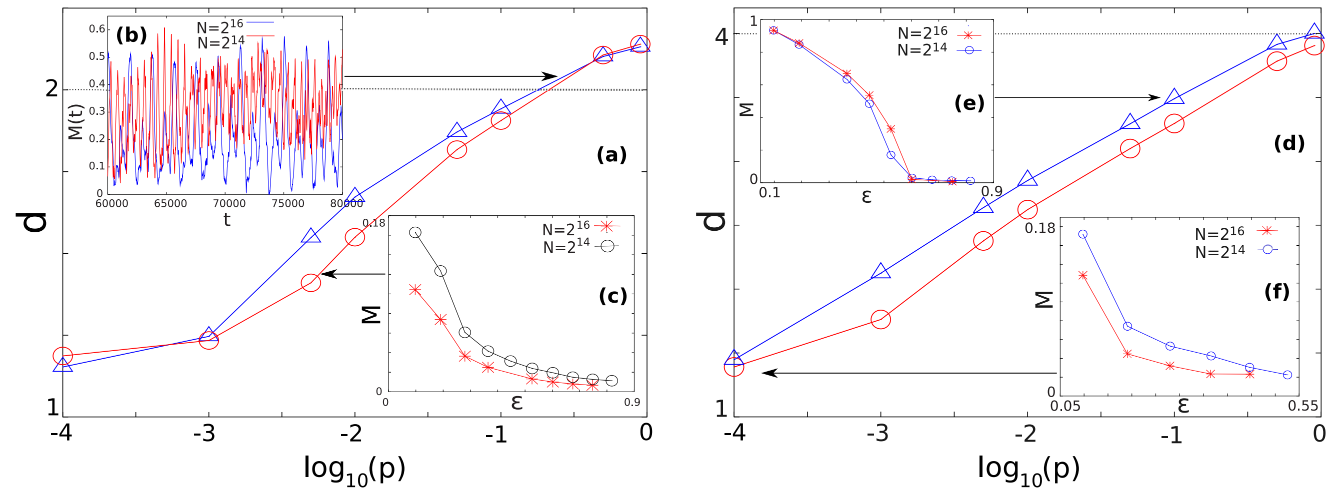

FIG. 3. (Color online) Dimension and its influence on global coherence. The relation between the dimension d and the fraction of links rewired, given by p, for (a) $r = \sqrt{N}$ and (d) $r = N^{3/4}$ for two network sizes, $N = 2^{14}$ (dots) and $N = 2^{16}$ (triangles). In (a) the dimension shifts from 1 to 2, whereas in (d) the increased r drags the dimension up to 4. In the insets we display the corresponding thermodynamical response. In (b) for a network with $d = 2$ the magnetization shows a chaotic behavior at $\varepsilon = 0.350(1)$, while in (c) and (f) the quasi-unidimensional network does not sustain any long-range order, entailing the vanishing of the magnetization for every energy. Finally, for $d \simeq 3$, (e) shows a second-order phase transition at $\varepsilon_c = 0.6$. For the magnetization equilibrium values (c,e,f) and the dimension (a,d), the error bars are within the dot size.

for the largest range $r = N^{3/4}$. Therefore a natural question arises: Does the dynamical behavior relate to this "finite size" dimension? Similarly to what we did in Fig. 2, we analyzed the dynamical response of the XY model, and in Fig. 3 we display our results for $r \sim \sqrt{N}$ and $r \sim N^{3/4}$. To guide our investigation, we can use Figs. 3(a)–3(d) as a map to locate the parameter zones characterized by different dimensions. Focusing first on $r \sim \sqrt{N}$, we chose the probabilities so as to have either a network with $d = 2$, $p = 0.1$, and $p = 0.3$, or a quasi one-dimensional network, $p = 0.005$. In Fig. 3(b) we show that indeed these networks generate a chaotic state similar to the state of Fig. 2(a) and described in Refs. [13] and [14]. The heavy oscillations of the magnetization do not relax even for long time simulations, and their amplitude (i.e., the variance) is unaffected by the size increase. This peculiar state, appearing for low energy densities, seems again intrinsically related to the dimensionality, since the two aforementioned probability values entail $d \sim 2$, as displayed in Fig. 3(a). Moreover, we considered several sizes to investigate the impact of the size increase and, again, there is no significant difference between, for instance, $N = 2^{14}$ and $N = 2^{16}$ in the fluctuation amplitude. On the other hand, it is noteworthy to observe a signature of the different sizes in the oscillation period, which is significantly slower in the $N = 2^{16}$ case. This effect, entangling system size and time scales, can be reminiscent with the lifetime of QSSs [30–34] of the HMF model, which is the mean field version of the XY model. Moreover, the collective oscillation itself recalls a very similar oscillating behavior observed in the HMF case [35] or in the $\alpha -$ HMF case [33] for QSSs. In this latter case of QSSs, this feature was used to perform "Poincaré sections" [33,36]. Nevertheless, we would like to stress that, both in [13] and in the present case, the root of the oscillating state is a *topological* condition on the network and not a dynamical one, as the choice of a particular initial condition. Furthermore, as another point of difference, we were not able to observe the eventual relaxation of those states so far. Anyway, those analogies, like the aforementioned analogy on the phase transition, and those differences both point to very interesting research perspectives to shed light on the connection between these two systems. Continuing in our analysis, for $p = 0.005$, which gives $d \lesssim 1.7$ [Fig. 3(a)], the magnetization vanishes for all the energies, so as to confirm the crucial role played by the crossover to the two-dimensional configuration. Now, taking into account the case $r \sim N^{3/4}$, we show in Fig. 3(e) that the system undergoes a second-order phase transition, as it happens in Fig. 2(c) when $d > 2$. In Fig. 3(e) the probability is set at $p = 0.1$, which entails $d \sim 3$ for the sizes considered. On the other hand, the short-range regime is at play for lower probabilities in Fig. 3(f), where we display the vanishing of the order parameter for $d \leqslant 1.5$.

VIII. CONCLUSION

In conclusion, we have provided a way to construct a class of networks whose dimension d is controllable via the range parameter r. We have shown how this dimension, in the thermodynamic limit, is related to different collective states of the XY model upon those networks. For $d > 2$ the system displays a second-order phase transition that becomes very similar to that of the HMF model for $d > 4$, while for $d = 2$ a regime characterized by an infinite susceptibility is at play. Beyond the analysis in the thermodynamic limit, we also interpreted the dimension d in the case of finite size systems. In this framework d is a function of (N,r,p) so that we can "adjust" the probability of rewiring p to obtain the desired *effective dimension*.

Considering the evidences we have displayed, we may argue that, for general networks, the considered dimension can be a key topological characteristic that in the end governs the final collective behavior of large coupled systems. Moreover, we believe that the peculiar case of networks with $d = 2$, for which the chaotic collective state emerges, could lead to many interesting applications. For instance, the infinite susceptibility could be used to amplify signals, or a better understanding of the dynamics could prove useful in the context of modeling and studying turbulent behaviors in an isolated system. On a closing note, the condition $d > 2$ to have a collective behavior, which is entangled with having a range of interaction $r > O(\sqrt{N})$, bears a strong resemblance to the necessary condition for synchronization of Kuramoto oscillators, as shown in Ref. [37], and this latter analogy could point to the intrinsic importance of this topological feature over the details of the dynamic imposed on the network.

ACKNOWLEDGMENTS

The authors are grateful to a referee for having pointed out Refs. [22–25]. X.L. is partially supported by the FET project Multiplex 317532.

[1] S. N. Dorogotsev, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, UK, 2003).
[2] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, Oxford, UK, 2010).
[3] S. Havlin and R. Cohen, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, Cambridge, UK, 2010).
[4] A. Barrat, M. Barthelemy, and A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, Cambridge, MA, 2008).
[5] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, and J. Wiener, Comput. Networks **33**, 309 (2000).
[6] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[7] A.-L. Barabási, R. Albert, and H. Jeong, Phys. A (Amsterdam, Neth.) **281**, 69 (2000).
[8] J. Travers and S. Milgram, Sociometry **32**, 425 (1969).
[9] E. Bullmore and O. Sporns, Nat. Rev. Neurosci. **10**, 186 (2009).
[10] W. J. Freeman, Int. J. Bifurcation Chaos Appl. Sci. Eng. **02**, 451 (1992).
[11] J. Kleinberg, Nature (London) **406**, 845 (2000).
[12] C. L. N. Oliveira, P. A. Morais, A. A. Moreira, and J. S. Andrade, Phys. Rev. Lett. **112**, 148701 (2014).
[13] S. De Nigris and X. Leoncini, Europhys. Lett. **101**, 10002 (2013).
[14] S. De Nigris and X. Leoncini, Phys. Rev. E **88**, 012131 (2013).
[15] D. J. Watts and S. H. Strogatz, Nature (Nature) **393**, 440 (1998).
[16] J. M. Kleinberg, in *Proceedings of the 32nd Annual ACM Symposium on Theory of Computing*, STOC '00 (ACM, New York, NY, 2000), pp. 163–170.
[17] S. Havlin and A. Bunde, *Fractals and Disordered Systems*, (Springer, Berlin, 1991).
[18] G. Baglietto, E. V. Albano, and J. Candia, J. Stat. Phys. **153**, 270 (2013).
[19] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[20] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *The Theory of Critical Phenomena: An Introduction to the Renormalization Group* (Oxford University Press, Oxford, UK, 1992).
[21] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, UK, 1995).
[22] D. Cassi, Phys. Rev. Lett. **68**, 3631 (1992).
[23] R. Burioni and D. Cassi, Phys. Rev. Lett. **76**, 1091 (1996).
[24] R. Burioni, D. Cassi, and A. Vezzani, Phys. Rev. E **60**, 1500 (1999).
[25] R. Burioni, D. Cassi, and C. Destri, Phys. Rev. Lett. **85**, 1496 (2000).
[26] R. Burioni and D. Cassi, J. Phys. A: Math. Gen. **38**, R45 (2005).
[27] R. I. McLachlan and P. Atela, Nonlinearity **5**, 541 (1992).
[28] A. Campa, T. Dauxois, and S. Ruffo, Phys. Rep. **480**, 57 (2009).
[29] J. M. Kosterlitz and J. D. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973).
[30] W. Ettoumi and M.-C. Firpo, J. Phys. A: Math. Theor. **44**, 175002 (2011).
[31] W. Ettoumi and M.-C. Firpo, Phys. Rev. E **87**, 030102(R) (2013).
[32] P.-H. Chavanis, G. De Ninno, D. Fanelli, and S. Ruffo, Out-of-equilibrium phase transitions in mean-field Hamiltonian dynamics, in *Chaos, Complexity and Transport: Theory and Applications*, edited by C. Chandre, X. Leoncini, and G. M. Zaslavsky (World Scientific, Singapore, 2008), pp. 3–26.
[33] T. L. Van Den Berg, D. Fanelli, and X. Leoncini, Europhys. Lett. **89**, 50010 (2010).
[34] Y. Levin, R. Pakter, F. B. Rizzato, T. N. Teles, and F. P. Benetti, Phys. Rep. **535**, 1 (2014).
[35] H. Morita and K. Kaneko, Phys. Rev. Lett. **96**, 050602 (2006).
[36] R. Bachelard, C. Chandre, D. Fanelli, X. Leoncini, and S. Ruffo, Phys. Rev. Lett. **101**, 260603 (2008).
[37] F. Mori, Phys. Rev. Lett. **104**, 108701 (2010).