# Unified trade-off optimization for general heat devices with nonisothermal processes

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(Received 24 July 2014; published 22 April 2015)

An analysis of the efficiency and coefficient of performance (COP) for general heat engines and refrigerators with nonisothermal processes is conducted under the trade-off criterion. The specific heat of the working medium has significant impacts on the optimal configurations of heat devices. For cycles with constant specific heat, the bounds of the efficiency and COP are found to be the same as those obtained through the endoreversible Carnot ones. However, they are independent of the cycle time durations. For cycles with nonconstant specific heat, whose dimensionless contact time approaches infinity, the general alternative upper and lower bounds of the efficiency and COP under the trade-off criteria have been proposed under the asymmetric limits. Furthermore, when the dimensionless contact time approaches zero, the endoreversible Carnot model is recovered. In addition, the efficiency and COP bounds of different kinds of actual heat engines and refrigerators have also been analyzed. This paper may provide practical insight for designing and operating actual heat engines and refrigerators.

DOI: 10.1103/PhysRevE.91.042127 PACS number(s): 05.70.Ln, 05.20.−y

## I. INTRODUCTION

Conditioned on energy saving and fuel depletion, the optimization of real thermodynamic cycles has attracted rising attention. In classical thermodynamics, Carnot efficiency $\eta_C$ and Carnot coefficient of performance (COP) $\varepsilon_C$ have defined the maximum energy conversion rates for heat engines and refrigerators operating between two heat reservoirs [1]. However, the realization of $\eta_C$ and $\varepsilon_C$ leads to vanishing power extracted for heat engines and zero cooling load rate for refrigerators, since they are reached only in reversible cycles where all the processes are quasistatic and the cycle time durations are infinite. The ideal Carnot cycles must be speeded up to meet the actual demand. Finite time thermodynamic analysis has provided a new way for optimizing actual heat devices [2].

By considering finite time durations of the heat transfer processes between the heat reservoirs and working fluid, Curzon and Ahlborn (CA) [3] proposed the concept of an endoreversible Carnot heat engine, and deduced its efficiency at maximum power (MP) output. That is the groundbreaking CA efficiency $\eta_c = 1 - \sqrt{T_c/T_h}$, where $T_h$ and $T_c$ are the temperatures of the hot and cold reservoirs, respectively. Based on the CA model, by allowing for different heat transfer laws between the working medium and the heat reservoirs and the internal dissipations, many revisions have been made to describe the real-life heat engines more accurately, and some good results at the maximum power output criterion have been obtained [4–10]. Furthermore, in view of the entropy generation in isothermal processes, which are treated as the inversed functions of process duration, Esposito *et al.* [11] proposed the low-dissipation model, and obtained the lower and upper bounds of efficiency at MP criterion under asymmetric dissipation limits. Later research has been dedicated to the efficiency and bounds of the low-dissipation heat engines at MP criterion [12–14]. Besides, the efficiency of linear irreversible heat engines described by the Onsager relations and the extended Onsager relations at MP criterion has been also studied [15–17].

However, for refrigerators, the minimum power input is not an appropriate optimization criterion [18], and much research has been focused on selecting figures of merit for optimizing refrigerators. By maximizing the per-unit-time COP, Velasco *et al.* [19] obtained the upper bound of COP, $\varepsilon_{CA} = \sqrt{\varepsilon_C + 1} - 1$, i.e., the CA coefficient of performance, for endoreversible refrigerators with $\varepsilon_C = T_c/(T_h - T_c)$ being the Carnot COP. Furthermore, under the maximum cooling power criterion, Apertet *et al.* [20] studied the endoreversible and exoreversible refrigerators and claimed that the real-life working conditions of the refrigerators do not correspond to a maximum cooling power but rather to a trade-off between cooling power and cooling efficiency. In addition, Yan and Chen [21] conducted the optimization with the objective function $\varepsilon \dot{Q}_c$ where $\dot{Q}_c$ is the cooling load rate of the refrigerators. To go a step further, de Tomas *et al.* [22] introduced the unified optimization criterion $\chi$ both for heat engines and refrigerators. By taking $\chi$ as the objective function, based on the low-dissipation model, Wang *et al.* [18] proposed that the COP at maximum $\chi$ was bounded between 0 and $(\sqrt{9 + 8\varepsilon_C} - 3)/2$. Besides, through the minimally nonlinear irreversible refrigerator model, Izumida *et al.* [23] also obtained the same bounds as those in Ref. [18] under the tight coupling condition. In addition, Allahverdyan *et al.* [24] also investigated quantum refrigerators and obtained some new bounds of COP under the $\chi$ figure of merit. Furthermore, by considering different heat conductance in the heat exchanging processes, the COP under the $\chi$ figure of merit is still the CA coefficient of performance [25].

Actual heat engines or refrigerators may not work at their maximum power output or maximum cooling load rate, but might work under a compromise between energy benefits and losses. Hernández *et al.* [26] proposed a new figure of merit $\Omega$, accounting for both the energy benefits and losses. Based on the $\Omega$ criterion, de Tomas *et al.* [27] and Long *et al.* [28] obtained the COP of refrigerators through the low-dissipation model and the minimally nonlinear irreversible model. The COP of low-dissipation refrigerators with irreversibility in the adiabatic processes was also considered by Hu *et al.*

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[29] under the Ω criterion. In addition, by comparing the bounds and efficiencies of heat engines described by different models under the EMP criterion and Ω criterion, Sánchez-Salas *et al.* [30] showed the maximum Ω regime was more efficient. Furthermore, Apertet *et al.* [20] declared that the real refrigerators do not operate under the maximum cooling power condition but under the trade-off between the cooling power and the COP.

The main merit of the low-dissipation models and the linear irreversible and minimally nonlinear irreversible models is that we do not need to consider the heat transfer law between the working medium and the heat reservoirs. However, in the low-dissipation model, the temperature of the working medium does not change during heat transferring processes. This model is meant for something microscopic (e.g., a quantum dot) and in simultaneous contact with both reservoirs. It is not applicable for a macroscopic system such as the Diesel cycle, Brayton cycle, Otto cycle, and Atkinson cycle. In the linear irreversible model, the temperature difference of the hot and cold reservoirs should be small enough to meet the requirement of the Onsager relations. For actual heat devices, in the heat exchanging processes, the temperatures of the working medium should change continuously to reach the highest or the lowest temperatures. That is to say, the heat exchanging processes should not be treated as isothermal. In this paper, we extend the model proposed in Ref. [31] to describe both the heat engines and refrigerators. This model accounts for the temperature changes of the working substance in heat exchanging processes. Therefore it is more general and realistic. In Sec. II, we first introduce the general mathematical model with nonisothermal processes, and then systemically discuss the efficiency and COP for heat engines and refrigerators under the figure of merit Ω in Secs. III and IV, respectively. The general alternative upper and lower bounds of the efficiency and COP have been proposed. Under the situations where the specific heat stays constant during the cycle, the bounds of the efficiency and COP are found to be the same as those obtained through the endoreversible Carnot ones. However, they are independent of the cycle time durations. In addition, the efficiency and COP bounds of different kinds of heat engines and refrigerators have also been analyzed. Finally some concluding remarks are given.

## II. GENERAL MATHEMATICAL MODEL

For heat devices, a certain amount of heat $Q_i$ is absorbed from the heat reservoir ($T_i$), and then $Q_j$ is evacuated to the heat reservoir ($T_j$) at the end of a cycle. In this paper, the heat transfer law between the heat source and the working medium is assumed to conform to Newton's law of cooling:

$$\frac{dQ}{dt} = cm\frac{dT}{dt} = k(T_s - T), \tag{1}$$

where $c$ is the specific heat, $m$ is the working substance mass, $T$ is the working substance temperature, $T_s$ is heat source temperature, and $k$ is the heat conductance. In the following analysis, we let $i$ and $j$ represent the heat absorbing and releasing processes, respectively. In the heat absorbing process, $T_s = T_i$. While in the heat releasing process, $T_s = T_j$. According to Eq. (1), the working substance temperature ($T_{iw}$) in the heat absorbing process is a function of time $t$:

$$T_{iw}(t) = T_i + (T_{i0} - T_i)e^{-t/\Psi_i}, \tag{2}$$

where $\Psi_i = c_i m/k_i$, which reflects the temperature increase degree of the working medium in the time absorbing process and has the dimension of time. $c_i$ and $k_i$ represent the constant specific heat of the working fluid and the heat conductance between the heat reservoir and the working fluid in the heat absorbing processes, respectively. $T_{i0}$ is the initial temperature of the working fluid in the heat absorbing process. The time duration is denoted as $\tau_i$; thereby the heat absorbed from the hot reservoir can be calculated as

$$Q_i = \int_0^{\tau_i} k_i(T_i - T_{iw})dt = c_i m(T_i - T_{i0})(1 - e^{-\tau_i/\Psi_i}). \tag{3}$$

The relative entropy change of the working substance in the heat absorbing process is given by

$$\Delta s_i = \int_0^{\tau_i} \frac{dQ_i}{T} = c_i m \ln\frac{T_i + (T_{i0} - T_i)e^{-\tau_i/\Psi_i}}{T_{i0}}. \tag{4}$$

Similarly, the heat evacuated and the entropy change during the heat releasing process are given by

$$Q_j = \int_0^{\tau_j} k_j(T_j - T_{jw})dt = c_j m(T_j - T_{j0})(1 - e^{-\tau_j/\Psi_j}), \tag{5}$$

$$\Delta s_j = \int_0^{\tau_j} \frac{dQ_j}{T} = c_j m \ln\frac{T_j - (T_j - T_{j0})e^{-\tau_j/\Psi_j}}{T_{j0}}, \tag{6}$$

where $\Psi_j = c_j m/k_j$, which reflects the temperature increase degree of the working medium in the time releasing process and has the dimension of time. $c_j$ and $k_j$ represent the constant specific heat of the working fluid and the heat conductance between the heat reservoir and the working fluid in the heat releasing processes, respectively. Generally, the values of $c_i$ and $c_j$ are not the same. $T_{j0}$ is the initial temperature of the working fluid in the heat releasing process. $\tau_j$ is the time duration of that process. In this paper, we assume that the compressing and expanding processes are isentropic and the time for completing those processes is zero. After a cycle, the working substance returns to its initial state, and the total entropy change of the working substance should be zero, i.e., $\Delta s_i + \Delta s_j = 0$. According to Eqs. (4) and (6), we have

$$\left[\frac{T_j - (T_j - T_{j0})e^{-\tau_j/\Psi_j}}{T_{j0}}\right]^{c_j}\left[\frac{T_i + (T_{i0} - T_i)e^{-\tau_i/\Psi_i}}{T_{i0}}\right]^{c_i} = 1. \tag{7}$$

According to Eq. (7), we can we can get the relation of $T_{i0}$ and $T_{j0}$, so there is only one unknown parameter, provided the duration of each heat exchanging process is prescribed. As to heat engines, $T_i > T_j$. The system absorbs heat from the hot reservoir, and then releases an amount of heat to the cold reservoir. The ratio of heat capacities of the working medium with higher and lower temperatures is $\gamma = c_i/c_j$. And the efficiency is

$$\eta = 1 + \frac{Q_j}{Q_i}. \tag{8}$$

For refrigerators, $T_i < T_j$. The system absorbs heat from the cold reservoir, and then releases some amount of heat to the hot reservoir. The ratio of heat capacities of the working medium with higher and lower temperatures is $\gamma = c_j/c_i$, and the COP ($\varepsilon$) is

$$\varepsilon = \frac{Q_i}{-Q_i - Q_j}. \tag{9}$$

## III. HEAT ENGINES AND THE Ω CRITERION

For heat engines, the Ω criterion is defined as $\Omega = (2\eta - \eta_C)Q_i$ [26]. It represents a compromise between energy benefits and losses for a specific job. This criterion is easy to implement for any energy converter (either isothermal or nonisothermal), without the requirement of the explicit evaluation of the entropy generation and it is independent of environmental parameters. Then, the objective function $\dot{\Omega} = (2\eta - \eta_C)\dot{Q}_i$ can be expressed as

$$\dot{\Omega} = \frac{(2 - \eta_C)Q_i + 2Q_j}{\tau_i + \tau_j}. \tag{10}$$

In the following analysis of heat engines, to make it more understandable, we replace the subscripts $i$ and $j$ with $h$ and $c$, respectively. In general, the efficiency at the maximum Ω criterion can be derived by using Eqs. (8) and maximizing Eq. (10) with respect to $T_{c0}$. In the following, we will study systematically the efficiency at the maximum trade-off criterion.

### A. Equal heat capacities ($\gamma = 1$)

Under the situations where the heat conductance keeps constant in the cycle, $\gamma = 1$. The maximizing equation (10) with respect to $T_{c0}$ yields

$$\left\{\frac{(1 - e^{-\tau_h/\Psi_h})\varphi}{1 - e^{-\tau_h/\Psi_h}[(1 - e^{-\tau_c/\Psi_c})\varphi + e^{-\tau_c/\Psi_c})]}\right\}^2 = 2\frac{(1 - \eta_C)}{(2 - \eta_C)}, \tag{11}$$

where $\varphi = T_c/T_{c0}$. The efficiency can be rewritten as

$$\eta = 1 - \frac{1 - e^{-\tau_h/\Psi_h}[(1 - e^{-\tau_c/\Psi_c})\varphi + e^{-\tau_c/\Psi_c}]}{(1 - e^{-\tau_h/\Psi_h})\varphi}(1 - \eta_C). \tag{12}$$

The solution of Eq. (11) gives the optimal $\varphi$; then substitute it into Eq. (12), as

$$\eta_\Omega^s = 1 - \sqrt{\frac{(1 - \eta_C)(2 - \eta_C)}{2}} = \frac{3}{4}\eta_C + \frac{1}{32}\eta_C^2 + \frac{3}{128}\eta_C^3 + O(\eta_C^4). \tag{13}$$

It is the same as that obtained through the endoreversible Carnot model [26]. However, they have different physical meanings and the optimization spaces are different. In the endoreversible Carnot model, the efficiency under the Ω criterion is obtained with respect to the time durations of the heat absorbing and releasing processes, while in this model, it is obtained by maximizing the Ω function with respect to the initial temperature of the working medium, and the time durations are treated as constants. Unlike the endoreversible Carnot heat engines, in this model the temperature of the working medium in either heat exchanging process does not keep constant (see Fig. 1 for $\gamma = 1$). Therefore it should be more practical and realistic.

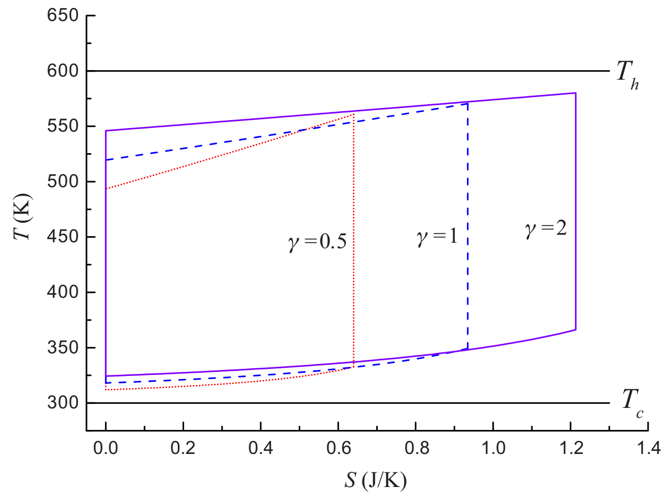


FIG. 1. (Color online) The $T$-$S$ diagram of an optimal heat engine cycle where $\gamma = 0.5, 1, 2$, $T_h = 600$ K, $T_c = 300$ K, $\tau_h/\Psi_h = \tau_c/\Psi_c = 1$, and $c_c m = 10$ J/K.

### B. Nonequal heat capacities ($\gamma \neq 1$)

We define $\tau/\Psi$ ($\tau_h/\Psi_h$ and $\tau_c/\Psi_c$) as the dimensionless contact time, connoting the equilibrium degree of the temperature between the working medium and heat reservoir. Larger $\tau/\Psi$ means the working medium makes contact longer with the heat reservoirs, and will lead to higher final temperature in the heat absorbing process and a lower one in the heat releasing process. Generally, under the conditions where $\gamma \neq 1$, Eq. (7) is transcendental and cannot be solved explicitly. Numerical calculations are conducted to investigate the impacts of the parameters on the optimal efficiencies. The $T$-$S$ diagrams of the heat engine cycles for different $\gamma$ are plotted in Fig. 1, which shows that the value of $\gamma$ has significant impacts on the configuration of heat engine cycles. As depicted in Figs. 2–4, when $\gamma < 1$, the optimal efficiency will increase with increasing $\tau/\Psi$ and will achieve its maximum value when $\tau/\Psi \to \infty$. The lower bound is achieved when $\tau/\Psi \to 0$, and is equal to $\eta_\Omega^s$, and is independent of the specific heat ratio. When $\gamma > 1$, the optimal efficiency will decrease with increasing $\tau/\Psi$, and will obtain its minimum value when $\tau/\Psi \to \infty$. The upper bound is achieved when $\tau/\Psi \to 0$, and is equal to $\eta_\Omega^s$ and is also independent of the specific heat ratio.

Therefore in the heat engine cycles such as the Diesel cycle ($c_h = c_p$, $c_c = c_v$), Brayton cycle ($c_h = c_v = c_p$), and Otto cycle ($c_h = c_c = c_v$),where the specific heat in the heat absorbing process is not less than that in the heat releasing process, the upper bound of the efficiency under the Ω figure of merit is $\eta_\Omega^s$, while in the cycles such as the Atkinson cycle ($c_h = c_v$, $c_c = c_p$), where the specific heat in the heat absorbing process is less than that in the heat releasing process, $\eta_\Omega^s$ is the lower bound.

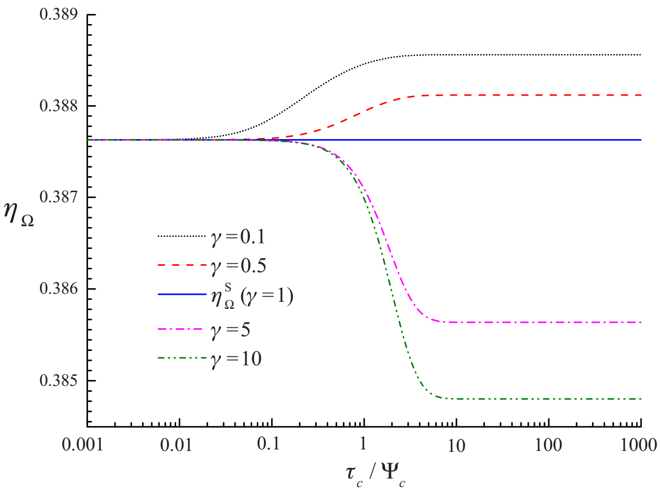


FIG. 2. (Color online) Optimal efficiencies with dimensionless contact times in the heat releasing process under different specific heat ratios ($\gamma = 0.1, 0.5, 1, 5, 10$), where $\eta_C = 0.5$ and $\tau_h/\Psi_h = 1$.

Furthermore, the general upper and lower bounds of the optimal efficiency can be obtained in the situations where $\tau/\Psi \to \infty$, by applying the asymmetric specific heat limits, $\gamma \to 0$ and $\gamma \to \infty$, respectively. In the following, the efficiency under the $\Omega$ figure of merit in asymmetric contact time limits ($\tau/\Psi \to 0$) is studied analytically.

Short dimensionless contact time limit; When $\tau/\Psi \to 0$, the heat exchanging processes are so short that the final temperature of the working substance is almost equal to its initial temperature after either process. Expanding $\exp(-\tau/\Psi)$ to the first order of $\tau/\Psi$ and maximizing Eq. (10) with respect to $T_{c0}$, we have

$$\left(\frac{\varphi}{1 + \frac{(1-\varphi)\tau_c/\Psi_c}{\gamma \tau_h/\Psi_h}}\right)^2 = 2\frac{(1-\eta_C)}{(2-\eta_C)}, \tag{14}$$

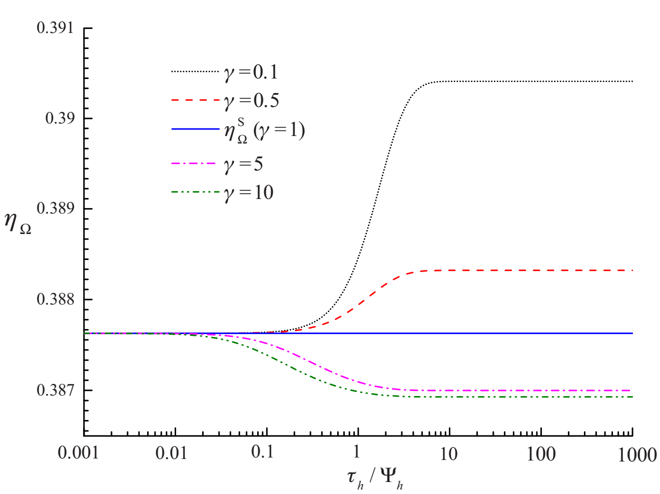


FIG. 3. (Color online) Optimal efficiency with dimensionless contact times in the heat absorbing process under different specific heat ratios ($\gamma = 0.1, 0.5, 1, 5, 10$), where $\eta_C = 0.5$ and $\tau_c/\Psi_c = 1$.

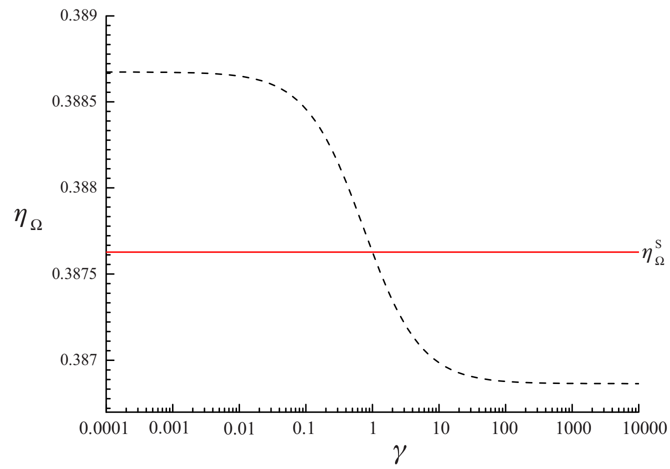


FIG. 4. (Color online) Optimal efficiency with different specific heat ratios where $\eta_C = 0.5$ and $\tau_c/\Psi_c = \tau_h/\Psi_h = 1$.

and the efficiency can be written as

$$\eta = 1 - \frac{1 + \frac{(1-\varphi)\tau_c/\Psi_c}{\gamma \tau_h/\Psi_h}}{\varphi}(1 - \eta_C). \tag{15}$$

Combining Eqs. (14) and (15), we have the same expression as Eq. (13). It is independent of the specific heat ratio, which is in accord with the above analysis. Under the short contact time limit, the heat absorbing and releasing processes are nearly isothermal. The specific heats have no impact on the temperature changes during the heat exchanging processes. Thereby, the endoreversible Carnot model is recovered, as shown in Fig. 5(a).

Long dimensionless contact time limit: Under the conditions where $\tau/\Psi \to \infty$, the contact time is long enough that heat exchange between the working substance and heat reservoirs is sufficient and the final temperature of the working substance is almost equal to that of the heat reservoir [see Fig. 5(b)]. The exponential terms $\exp(-\tau/\Psi)$ can be

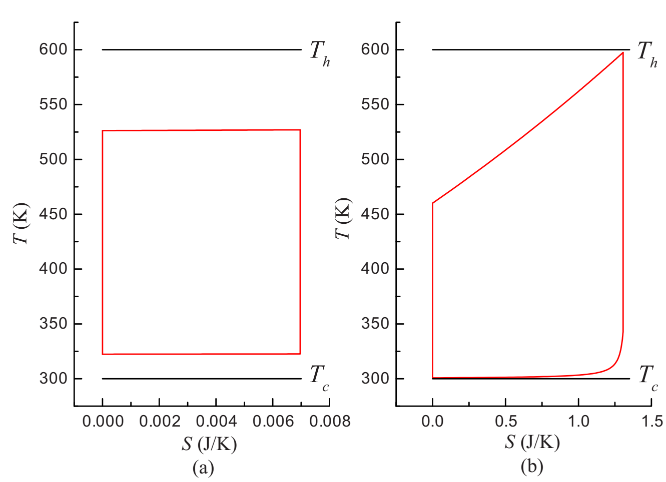


FIG. 5. (Color online) The $T$-$S$ diagrams of two optimal heat engine cycles under asymmetric contact time limits: $\tau_h/\Psi_h = \tau_c/\Psi_c = 0.01$ (a); $\tau_h/\Psi_h = \tau_c/\Psi_c = 4$ (b), where $\gamma = 0.5$, $T_h = 600$ K, $T_c = 300$ K, $\tau_h/\Psi_h = \tau_c/\Psi_c = 1$, and $c_c m = 10$ J/K.

eliminated; therefore maximizing Eq. (10) with respect to $T_{c0}$, we have

$$\varphi^{1/\gamma+1} = 2\frac{(1-\eta_C)}{(2-\eta_C)}, \tag{16}$$

and the efficiency can be written as

$$\eta = 1 - \frac{(1-\varphi)}{\gamma(\varphi - \varphi^{1/\gamma+1})}(1-\eta_C). \tag{17}$$

Combining Eqs. (16) and (17), we have

$$\eta_\Omega = 1 - \frac{1-[2(1-\eta_C)/(2-\eta_C)]^{\frac{\gamma}{\gamma+1}}}{\gamma\left\{[22(1-\eta_C)/(2-\eta_C)]^{\frac{\gamma}{\gamma+1}} - 22(1-\eta_C)/(2-\eta_C)\right\}}(1-\eta_C). \tag{18}$$

As mentioned above, the general upper and lower bounds of the optimal efficiency can be obtained by applying the asymmetric limits $\gamma \to 0$ and $\gamma \to \infty$, so we have

$$\eta_\Omega^+ = 1 + \frac{(2-\eta_C)(1-\eta_C)\ln\left(\frac{2-2\eta_C}{2-\eta_C}\right)}{\eta_C} = \frac{3}{4}\eta_C + \frac{1}{24}\eta_C^2 + \frac{1}{32}\eta_C^3 + \mathrm{O}(\eta_C^4), \tag{19}$$

$$\eta_\Omega^- = 1 + \frac{\eta_C}{2\ln\left(\frac{2-2\eta_C}{2-\eta_C}\right)} = \frac{3}{4}\eta_C + \frac{1}{48}\eta_C^2 + \frac{1}{64}\eta_C^3 + \mathrm{O}(\eta_C^4). \tag{20}$$

According to above equations, $\eta_\Omega^s \approx (\eta_\Omega^+ + \eta_\Omega^-)/2$. In addition, the lower bound obtained through the low-dissipation model [27] is $\eta_\Omega^{LD-} = 3\eta_C/4$, and the upper bound is given by

$$\eta_\Omega^{LD+} = \frac{3-2\eta_C}{4-3\eta_C}\eta_C = \frac{3}{4}\eta_C + \frac{1}{16}\eta_C^2 + \frac{3}{64}\eta_C^3 + \mathrm{O}(\eta_C^4). \tag{21}$$

Besides, the efficiencies under the same criterion for the stochastic heat engine cycle model and the nanothermoelectric engine mode [32] are

$$\eta_\Omega^{SS} = \frac{3}{4}\eta_C + \frac{1}{32}\eta_C^2 + \frac{1}{64}\eta_C^3 + O(\eta_C^4) \tag{22}$$

and

$$\eta_\Omega{}^{ELB} = \frac{3}{4}\eta_C + \frac{1}{32}\eta_C^2 + \frac{19+\mathrm{csch}^2(a_0/2)}{768}\eta_C^3 + O(\eta_C^4), \tag{23}$$

where $a_0$ is the root of the transcendental equation $a_0 = 2\coth(a_0/2)$ at the first order.

It is clear all the efficiencies under the trade-off criterion are coincident in the coefficients 3/4 of the linear term and model-dependent differences appears at second and higher terms. Furthermore, this coincidence also exists in the low-dissipation and minimally nonlinear irreversible models under the symmetric situations [27,28].

## IV. REFRIGERATORS AND THE Ω CRITERION

For refrigerators, the $\Omega$ criterion is defined as $\Omega = (2\varepsilon - \varepsilon_{\max})W$ in Ref. [26]. Then, the objective function $\dot{\Omega} = (2\varepsilon - \varepsilon_{\max})\dot{W}$ can be expressed as

$$\dot{\Omega} = \frac{(2+\varepsilon_C)Q_i + \varepsilon_C Q_j}{\tau_i + \tau_j}. \tag{24}$$

In the following analysis of refrigerators, to make it more understandable, we replace the subscripts $i$ and $j$ with $c$ and $h$, respectively. Generally, the COP under the maximum $\Omega$ criterion can be derived by Eq. (9) and by maximizing Eq. (24) with respect to $T_{c0}$,which will be discussed in the following sections.

### A. Equal heat capacities ($\gamma = 1$)

Under the situations where the heat conductance keeps constant in the cycle, $\gamma = 1$. By maximizing Eq. (24) with respect to $T_{c0}$, we have

$$\frac{[(1-e^{-\tau_h/\Psi_h})\varphi]^2}{\{1-e^{-\tau_h/\Psi_h}[(1-e^{-\tau_c/\Psi_c})\varphi + e^{-\tau_c/\Psi_c}]\}^2} = \frac{\varepsilon_C+2}{\varepsilon_C+1}, \tag{25}$$

and the COP can be written as

$$\varepsilon = \frac{1}{\frac{\varepsilon_C+1}{\varepsilon_C}\frac{(1-e^{-\tau_h/\Psi_h})\varphi}{1-e^{-\tau_h\Psi_h}[(1-e^{-\tau_c/\Psi_c})\varphi+e^{-\tau_c/\Psi_c}]} - 1}. \tag{26}$$

The solution of Eq. (25) gives the optimal $\varphi$; then substitute it into Eq. (26) as

$$\varepsilon_\Omega^s = \frac{\varepsilon_C}{\sqrt{(1+\varepsilon_C)(2+\varepsilon_C)} - \varepsilon_C} = \frac{2}{3}\frac{1}{\frac{1}{\varepsilon_C} - \frac{1}{12\varepsilon_C^2} + \frac{1}{8\varepsilon_C^3} + \mathrm{O}(1/\varepsilon_C^4)}. \tag{27}$$

It is the same as that obtained through the endoreversible Carnot model [26]. Although for those two models the upper bounds of the COP are the same, they have different physical meanings and the optimization spaces are different. In the endoreversible model, the upper bound of the COP is obtained by maximizing $\Omega$ with respect to the time durations of the heat absorbing and releasing processes, respectively, while in this model, the upper bound is obtained by maximizing $\Omega$ with respect to the initial temperature of the working medium, and the time durations are treated as constants. Unlike the Carnot refrigerators, in this model the temperature of the working medium in either heat exchanging process does not need to keep constant (see Fig. 6 for $\gamma = 1$). The model studied in this paper should be more practical and realistic than the endoreversible refrigerator one.

### B. Nonequal heat capacities ($\gamma \neq 1$)

Generally, under the conditions where $\gamma \neq 1$, Eq. (7) is transcendental and cannot be solved explicitly. Numerical calculations are conducted to investigate the impacts of

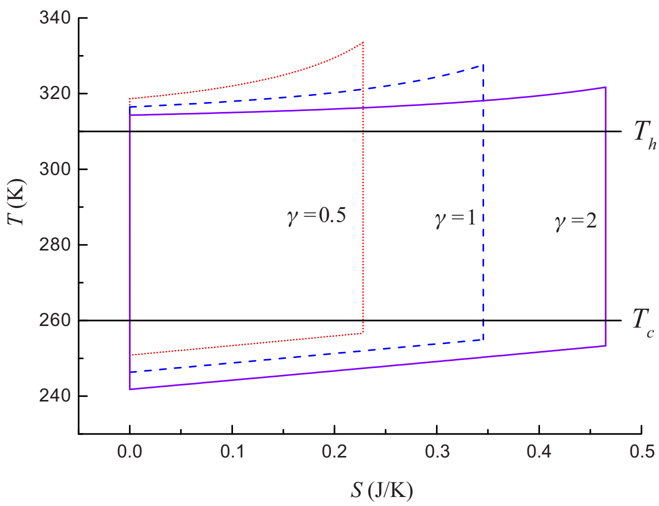


FIG. 6. (Color online) The $T$-$S$ diagram of an optimal refrigerator cycle where $\gamma = 0.5,1,2$, $T_h = 310$ K, $T_c = 260$ K, $\tau_h/\Psi_h = \tau_c/\Psi_c = 1$, and $c_c m = 10$ J/K.

the parameters on the optimal COPs. The $T$-$S$ diagrams of the refrigerator cycles for different $\gamma$ are plotted in Fig. 6, reflecting that the value of $\gamma$ profoundly affects the configuration of refrigerator cycles. As depicted in Figs. 7–9, when $\gamma > 1$, the optimal COP will increase with increasing $\tau/\Psi$ and will achieve its maximum value when $\tau/\Psi \to \infty$. The lower bound is achieved when $\tau/\Psi \to 0$, and is equal to $\varepsilon_\Omega^s$ and is independent of the specific heat ratio. When $\gamma < 1$, the optimal COP will decrease with increasing $\tau/\Psi$, and will obtain its minimum value when $\tau/\Psi \to \infty$. The upper bound is achieved when $\tau/\Psi \to 0$, and is equal to $\varepsilon_\Omega^s$ and is independent of the specific heat ratio.

Therefore in the refrigerator cycles such as the reversed Diesel cycle ($c_c = c_v$, $c_h = c_p$), reversed Brayton cycle ($c_c = c_h = c_p$), and reversed Otto cycle ($c_c = c_h = c_v$), where the specific heat in the heat absorbing process is not less than

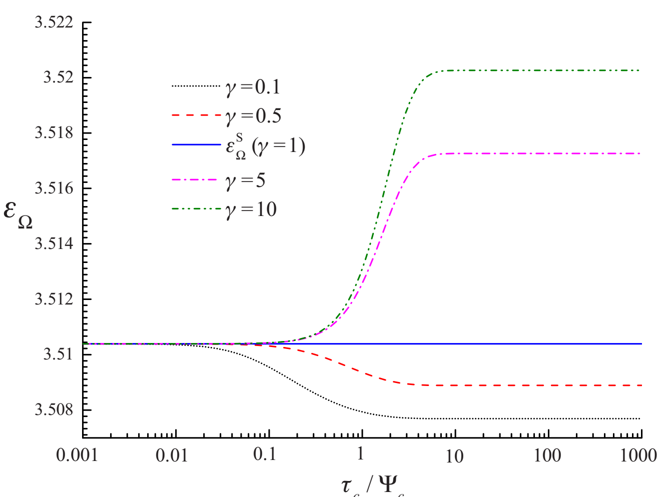


FIG. 7. (Color online) Optimal COPs with dimensionless contact times of the heat releasing process under different specific heat ratios ($\gamma = 0.1,0.5,1,5,10$), where $\varepsilon_C = 5.2$ and $\tau_h/\Psi_h = 1$.

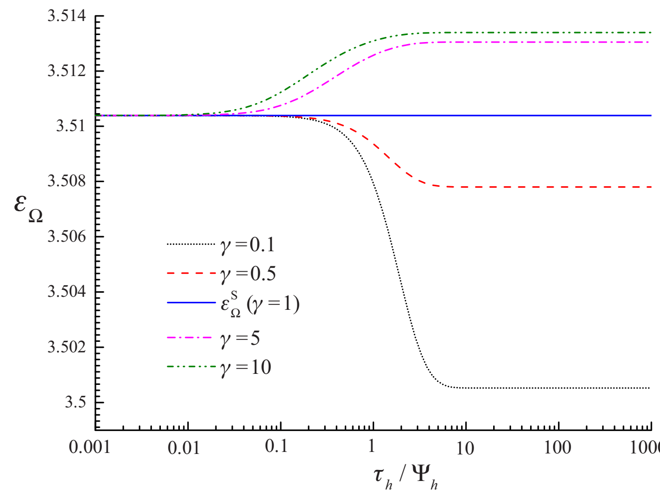


FIG. 8. (Color online) Optimal COPs with dimensionless contact times of the heat absorbing process under different specific heat ratios ($\gamma = 0.1,0.5,1,5,10$), where $\varepsilon_C = 5.2$ and $\tau_c/\Psi_c = 1$.

that in the heat releasing process, the lower bound of the COP under the trade-off criterion is $\varepsilon_\Omega^s$, while in the cycles such as the reversed Atkinson cycle ($c_c = c_p$, $c_h = c_v$) where the specific heat in the heat absorbing process is less than that in the heat releasing process, $\varepsilon_\Omega^s$ is the upper bound. This might be of great guidance for designing and operating actual refrigerators.

Furthermore, the general lower and upper bounds of the optimal COP can be obtained in the situations where $\tau/\Psi \to \infty$, by applying the asymmetric specific heat limits, $\gamma \to 0$ and $\gamma \to \infty$, respectively. In the following, the COP under the $\Omega$ figure of merit in asymmetric contact time limits ($\tau/\Psi \to 0$) is studied analytically.

Short dimensionless contact time limit: Under the conditions similar to heat engines, we expand $\exp(-\tau/\Psi)$ to the first order of $\tau/\Psi$. By maximizing Eq. (24) with respect to

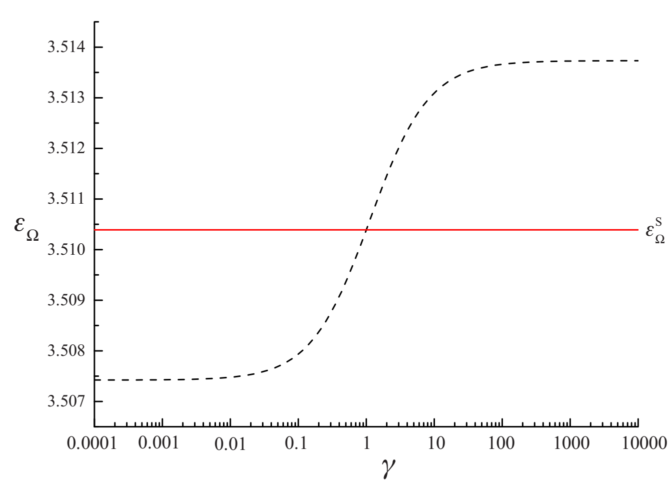


FIG. 9. (Color online) Optimal COPs with different specific heat ratios where $\varepsilon_C = 5.2$ and $\tau_c/\Psi_c = \tau_h/\Psi_h = 1$.

$T_{c0}$, we have

$$\left[\frac{\varphi}{1-\frac{1}{\gamma}\frac{\tau_c/\Psi_c(\varphi-1)}{\tau_h/\Psi_h}}\right]^2=\frac{\varepsilon_C+2}{\varepsilon_C+1}, \tag{28}$$

and the COP can be written as

$$\varepsilon=\frac{1}{\frac{\varepsilon_C+1}{\varepsilon_C}\frac{\varphi}{1-\frac{I_s}{\gamma}\frac{\tau_c/\Psi_c(\varphi-1)}{\tau_h/\Psi_h}}-1}. \tag{29}$$

Combining Eqs. (28) and (29), we have the same expression as Eq. (27). It is independent of the specific heat ratio and the heat conductance. Under the short contact time limit, the heat absorbing and releasing processes are nearly isothermal. The heat capacities have no impact on the teperature changes during the heat exchanging processes. Thus the endoreversible Carnot refirgerator model is recovered.

Long dimensionless contact time limit: Under the conditions where $t/\Psi\to\infty$, similar to heat engines, the exponential terms $\exp(-t/\Psi)$ can be eliminated; therefore by maximizing Eq. (24) with respect to $T_{c0}$, we have

$$\varphi^{1/\gamma+1}=\frac{\varepsilon_C+2}{\varepsilon_C+1}, \tag{30}$$

and the COP can be written as

$$\varepsilon=\frac{\varphi-1}{\gamma\frac{\varepsilon_C+1}{\varepsilon_C}(\varphi^{1/\gamma+1}-\varphi)-\varphi+1}. \tag{31}$$

The solution of Eq. (30) gives the optimal $\varphi$, then substituting it into Eq. (31), we have

$$\varepsilon_\Omega=\frac{[(\varepsilon_C+2)/(\varepsilon_C+1)]^{\gamma/(1+\gamma)}-1}{\gamma\frac{\varepsilon_C+1}{\varepsilon_C}(\varepsilon_C+2)/(\varepsilon_C+1)-\left(\gamma\frac{\varepsilon_C+1}{\varepsilon_C}+1\right)[(\varepsilon_C+2)/(\varepsilon_C+1)]^{\gamma/(1+\gamma)}+1}. \tag{32}$$

As mentioned above, the general lower and upper bounds of the COP for general refrigerators under the maximum $\Omega$ criterion can be obtained by applying the asymmetric limits $\gamma\to 0$ and $\gamma\to\infty$ we have

$$\varepsilon_\Omega^-=\frac{\varepsilon_C\ln\frac{\varepsilon_C+2}{\varepsilon_C+1}}{1-\varepsilon_C\ln\frac{\varepsilon_C+2}{\varepsilon_C+1}}=\frac{2}{3}\frac{1}{\frac{1}{\varepsilon_C}-\frac{1}{18\varepsilon_C^2}+\frac{1}{8\varepsilon_C^3}+O\left(1/\varepsilon_C^4\right)}, \tag{33}$$

and

$$\varepsilon_\Omega^+=\frac{\varepsilon_C}{(\varepsilon_C+1)(\varepsilon_C+2)\ln\frac{\varepsilon_C+2}{\varepsilon_C+1}-\varepsilon_C}=\frac{2}{3}\frac{1}{\frac{1}{\varepsilon_C}-\frac{1}{9\varepsilon_C^2}+\frac{1}{6\varepsilon_C^3}+O\left(1/\varepsilon_C^4\right)}. \tag{34}$$

In addition, the lower bound obtained through the low-dissipation model [27] is $\varepsilon_\Omega^{LD-}=2\varepsilon_C/3$, and the upper bound is given by

$$\varepsilon_\Omega^{LD+}=\frac{3+2\varepsilon_C}{4+3\varepsilon_C}\varepsilon_C=\frac{2}{3}\frac{1}{\frac{4}{\varepsilon_C}-\frac{4}{6\varepsilon_C^2}+\frac{4}{4\varepsilon_C^3}+O\left(1/\varepsilon_C^4\right)}. \tag{35}$$

It is clear all the COPs at maximum trade-off criterion are equivalent to the second order of $1/\varepsilon_C$ with the coefficient of $2/3$ and model-dependent differences appears at second and higher terms.

TABLE I. Efficiency and COP bounds under different specific heat ratios.

| Specific heat ratio<br>Efficiency–COP | $\gamma<1$ | $\gamma=1$ | $\gamma>1$ |
|---|---|---|---|
| $\eta_\Omega$ | $\eta_\Omega^s<\eta_\Omega<\eta_\Omega^+$ | $\eta_\Omega=\eta_\Omega^s$ | $\eta_\Omega^-<\eta_\Omega<\eta_\Omega^s$ |
| $\varepsilon_\Omega$ | $\varepsilon_\Omega^-<\varepsilon_\Omega<\varepsilon_\Omega^s$ | $\varepsilon_\Omega=\varepsilon_\Omega^s$ | $\varepsilon_\Omega^s<\varepsilon_\Omega<\varepsilon_\Omega^+$ |

## V. CONCLUSIONS

In conclusion, we have conducted as analysis of the efficiency and COP at maximum trade-off criterion for general heat engines and refrigerators with nonisothermal heat exchanging processes. Under the situations where the specific heat stays constant during the cycle, the bounds of the efficiency and COP are found to be the same with those obtained through the endoreversible Carnot ones, and are independent of the cycle time durations and the heat conductance. In addition when the dimensionless contact times approach zero, the endoreversible Carnot models are recovered. We should mention that, in the endoreversible model, the upper bound of the efficiency or COP is obtained by maximizing the objective function with respect to the time durations of the heat absorbing and releasing processes respectively. By using the heat transfer law, maximizing the objective function with respect to the time durations of the heat absorbing and releasing processes becomes maximizing it with respect to the temperatures of the isothermal heat absorbing and releasing processes. That is the inherent correlation for the two optimization spaces under the short time limits. In the situations where the dimensionless contact times approach infinite, the general upper and lower bounds of the efficiency and COP have been proposed under the asymmetric specific heat ratio limits.

Furthermore, the efficiency and COP bounds of different kinds of heat engines (such as Brayton, Otto, Diesel, and Atkinson cycles) and refrigerators (such as reversed Brayton, Otto, Diesel, and Atkinson cycles) have been analyzed. More general results can be seen in Table I.

## ACKNOWLEDGMENT

The work was supported by the National Key Basic Research Program of China (973 Program) (Grant No. 2013CB228302).

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