Evidence for nonuniversal scaling in dimension-four Ising spin glasses

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The critical behavior of the Binder cumulant for Ising spin glasses in dimension four is studied through simulation measurements. Data for the bimodal interaction model are compared with those for the Laplacian interaction model. Special attention is paid to scaling corrections. The limiting infinite size value at criticality for this dimensionless variable is a parameter characteristic of a universality class. This critical limit is estimated to be equal to 0.523(3) in the bimodal model and to 0.473(3) in the Laplacian model.

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I. INTRODUCTION

For standard second order transitions, renormalization group theory (RGT) provides an elegant and detailed explanation of universality. Thus in the family of simple ferromagnets, within a universality class of models having space dimension d and spin dimensionality n, all models have identical critical properties corresponding to an isolated fixed point in the renormalization group flow. The only documented exceptions all appear to be cases of specific spin models in dimension two (discussed, for instance, in Ref. [1]); for these models the critical behavior is more complicated, with critical exponents varying continuously when a control parameter is modified. The corresponding renormalization group scenario consists of a line of fixed points rather than an isolated fixed point, with motion along the line produced by a marginal operator.

The Ising spin glasses (ISGs) which we will consider have symmetric (positive and negative) random near neighbor interactions rather than the regular interactions with fixed sign of a simple ferromagnet; the theoretical situation for critical behavior in ISGs is far less advanced than for the standard models. The ISG upper critical dimension is known to be six, but it was found 30 years ago that the ϵ expansions in ISGs are not fully predictive since the first few orders have a nonconvergent behavior and higher orders are not known [2]. This can be taken as an indication that a fundamentally different theoretical approach is required for RGT at spin glass transitions, and indeed, "classical tools of RGT analysis are not suitable for spin glasses" [3–5], although no explicit theoretical predictions have been made so far concerning the important question of universality in these systems.

Claims of universality in ISGs have been made repeatedly based on numerical data [6–10]. Here, from a detailed analysis of numerical simulation measurements on ISGs in dimension four we come to the empirical conclusion that, on the contrary, the critical properties of these systems depend on the form of the interaction distribution. A breakdown of universality at a continuous spin glass transition for a dimension well above two may be a symptom of the need for a novel RGT approach in this class of models.

The ISG Hamiltonian is

$$\mathcal{H} = -\sum_{ij} J_{ij} S_i S_j, \qquad (1)$$

with the near neighbor symmetric distributions normalized to $\langle J_{ij}^2\rangle = 1$. We use the inverse temperature $\beta = 1/T$ as the thermal parameter. The Ising spins sit on simple hypercubic lattices with periodic boundary conditions. The spin overlap parameter is defined by

$$q = \frac{1}{L^d}\sum_i S_i^A S_i^B, \qquad (2)$$

where A and B indicate two copies of the same system. We have studied in dimension four the bimodal model with a $\pm J$ interaction distribution and the Laplacian model with a $P(J_{ij}) \sim \exp(-|J_{ij}|)$ interaction distribution.

Simulations in ISGs are much more laborious than the equivalent simulations in simple ferromagnets because equilibration is slow and averages must be taken over large numbers of samples. The simulations were carried out using the exchange Monte Carlo method [11] for equilibration using so-called multispin coding. In the bimodal model measurements were made on 2^{14} individual samples (or J_{ij} realizations) for $3 \leqslant L \leqslant 7$, on 2^{13} samples for $8 \leqslant L \leqslant 12$, and on 2^{12} samples for $L = 13$ and $L = 14$. For the Laplacian model, measurements were made on 2^{13} samples for $3 \leqslant L \leqslant 12$. After every sweep an exchange was attempted with a success rate of at least 30%. At least 40 temperatures were used, forming a geometric progression reaching down to $\beta_{\max} = 0.55$ in the bimodal case and $\beta_{\max} = 0.70$ in the Laplacian case.

This ensures that our data span the critical temperature region which is essential for the finite size scaling (FSS) fits. Near the critical temperature the β step length was at most 0.03. The various systems were deemed to have reached equilibrium when the sample average susceptibility for the lowest temperature showed no trend between runs. For example, in the Laplacian case for $L = 12$ this means about 200 000 sweep-exchange steps.

After equilibration, at least 200 000 measurements were made for each sample for all sizes, taking place after every sweep-exchange step. We registered the energy $E(\beta,L)$; the correlation length $\xi(\beta,L)$; the spin overlap moments $\langle|q|\rangle$, $\langle q^2\rangle$, $\langle|q|^3\rangle$, $\langle q^4\rangle$; and the corresponding link overlap q_ℓ moments, where the link overlap is defined as

$$q_\ell = \frac{1}{dL^d}\sum_{ij} S_i^A S_j^A S_i^B S_j^B. \qquad (3)$$

In addition, some correlations $\langle E(\beta,L),U(\beta,L)\rangle$ between the energy and observables $U(\beta,L)$ were also registered so that thermodynamic derivatives could be evaluated using the relation $\partial U(\beta,L)/\partial\beta = \langle U(\beta,L),E(\beta,L)\rangle - \langle U(\beta,L)\rangle\langle E(\beta,L)\rangle$ (see, e.g., Ref. [12]). Bootstrap analyses of the errors in the derivatives as well as in the observables $U(\beta,L)$ themselves were carried out.

II. TESTING UNIVERSALITY

Jörg and Katzgraber [13] used an elegant scaling display of raw numerical data to test for universality in ISGs. They plotted the ratio $y(\beta,L) = g(\beta,2L)/g(\beta,L)$ against $x(\beta,L) = g(\beta,L)$, where

$$g(\beta,L) = \frac{1}{2}\left(3 - \frac{[\langle q^4\rangle]}{[\langle q^2\rangle]^2}\right) \tag{4}$$

is the Binder cumulant for inverse temperature β and lattice size L, with q being the spin glass order parameter of Eq. (2) and $[\cdots]$ denoting the average taken over the samples. They studied numerically two ISGs in dimension four, one with a Gaussian interaction distribution and one with a diluted bimodal distribution. Over the range of temperatures used for the measurements, which extended well into the ordered phase, the scaled data points were independent of L and followed the same curve for the two systems to within the statistics. Jörg and Katzgraber concluded that these results were evidence of universality in ISGs.

In Fig. 1 we show the same scaling plot as that of Ref. [13] in dimension four but using instead standard bimodal interactions and compare them to Laplacian interactions. The temperatures span the critical temperatures.

For the Laplacian ISG, our data show scaling with no correction term to within the statistics; the scaling curves are almost indistinguishable from those for the models of Ref. [13]. The bimodal data, on the other hand, show a strong L dependence due to large finite size scaling corrections; the scaling curve $y(x)$ moves continuously to the right with increasing L. With a natural extrapolation the thermodynamic (large L) limit scaling curve for the bimodal interaction ISG will lie well to the right of the L-independent Laplacian curve, so the two models appear not to be in the same universality class.

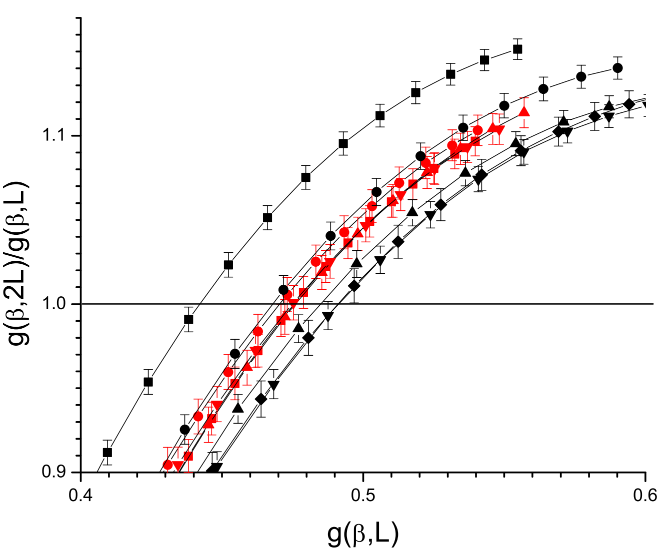

FIG. 1. (Color online) Scaling plots for the Binder cumulant of 4D ISGs, $g(\beta,2L)/g(\beta,L)$ vs $g(\beta,L)$. Red (gray): Laplacian interactions; black: bimodal interactions. Squares, circles, triangles, inverted triangles, and diamonds are for $L = 3,4,5,6,7$, respectively.

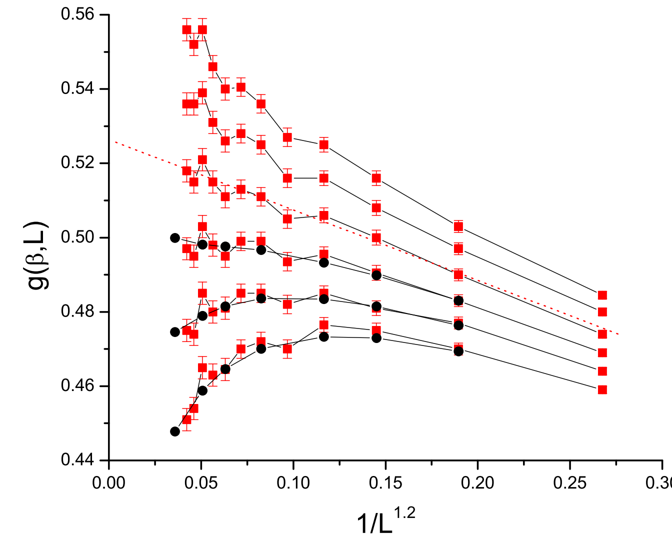

FIG. 2. (Color online) The Binder cumulant of the 4D bimodal ISG near criticality. Inverse temperatures $\beta = 0.510$, 0.5075, 0.505, 0.5025, 0.500, 0.4975 are shown from top to bottom. Red squares: present data. Black circles: read from Ref. [14]. The two data sets are consistent. The dashed straight line indicates criticality.

Standard finite size scaling expressions which include a single leading conformal correction term lead to a size dependence $\beta_c - \beta_{\mathrm{cross}}(L) = AL^{-(\omega+1/\nu)}$, where β_{cross} is the crossing point where $g(\beta,2L) = g(\beta,L)$ [represented by $y(x) = 1$ in Fig. 1], where ω is the correction-to-scaling exponent. In the dimension-four bimodal ISG, ω has been estimated by simulations to be 1.04(10) [14]. From high temperature series expansion (HTSE) measurements $\theta = \omega\nu \approx 1.5$ [15], so $\omega \approx 1.3$. A natural extrapolation of the present bimodal data to infinite L assuming $\omega \approx 1.2$ gives a thermodynamic limit estimate which is certainly considerably larger than the Laplacian crossing point limit.

Data near criticality for the bimodal and Laplacian ISGs are shown in a different form in Figs. 2 and 3, respectively. Near criticality,

$$g(\beta,L) = g_c + AL^{-\omega} + B(\beta-\beta_c)L^{1/\nu}. \tag{5}$$

The bimodal data are consistent with $\beta_c = 0.505(1)$, $\omega \approx 1.2$, and $g_c = 0.523(3)$. The Laplacian data are consistent with $\beta_c = 0.622(1)$, $g_c = 0.473(3)$, and a negligible correction. The g_c values estimated for the Gaussian and dilute bimodal models in Ref. [10] are 0.470(5) and 0.472(2), which are similar to the Laplacian value.

The bimodal β_c value is confirmed independently by thermodynamic derivative data on dimensionless observables $U(\beta,L)$. The $U(\beta,L)$ curve becomes steeper and steeper with increasing L and tends to a step function centered on β_c in the large L limit. Calling the peak in the derivative

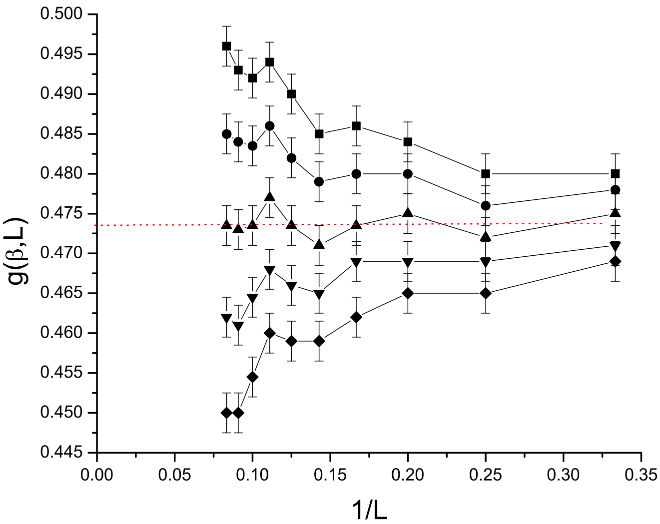

FIG. 3. (Color online) The Binder cumulant of the 4D Laplacian ISG near criticality. Inverse temperatures $\beta = 0.626, 0.624, 0.622, 0.620, 0.618$ are shown from top to bottom. The dashed straight line indicates criticality.

$D_m(L) = [\partial U(\beta,L)/\partial\beta]_{\max}$ and its location (the pseudocritical temperature) β_m, the inverse of the derivative peak height $x(L) = 1/D_m(L)$ and the corresponding inverse temperature location shift $\beta_c - \beta_m$ both scale as $L^{-1/\nu}(1 + aL^{-\omega})$ [12]. So at large L, the points $y(L) = \beta_m(L)$ plotted against $x(L)$ extrapolate linearly to $y(\infty) = \beta_c$ at $x(\infty) = 0$. An example of this type of plot with the dimensionless observable

$$W_q(\beta,L) = \frac{1}{\pi - 2}\left(\pi\frac{[\langle|q|\rangle]^2}{[\langle q^2\rangle]} - 2\right) \qquad (6)$$

is shown for the bimodal model in Fig. 4. From such plots an independent estimate $\beta_c = 0.505(1)$ is obtained for the bimodal model in four dimensions [16].

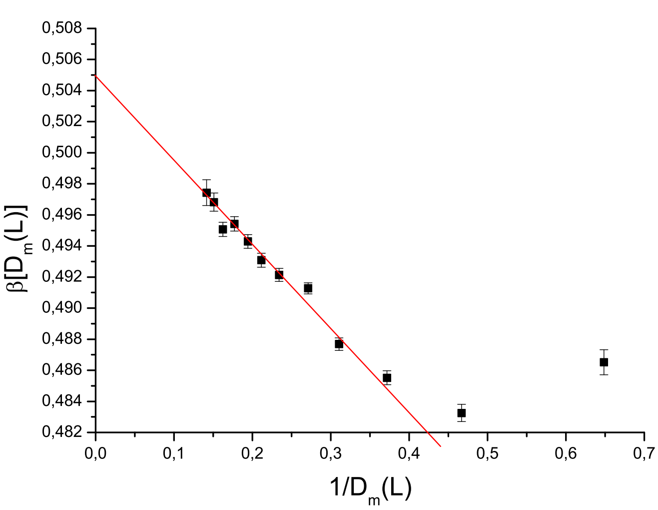

FIG. 4. (Color online) Scaling plot for the bimodal model $W_q(\beta,L)$ derivative peak location β_m against the inverse derivative peak height $1/D_m(L)$.

III. CORRECTIONS TO SCALING

For the $g(\beta_c,L)$ bimodal values to extrapolate finally to a limiting g_c value at infinite L consistent with that of the Laplacian model, putative bimodal ISG data for very large L (data inaccessible with current numerical resources) would have to bend back to the left in Fig. 1 or to sharply bend down in Fig. 2 (in an unlikely looking way), instead of extrapolating in a natural way to the large L critical limit estimated above. A necessary condition for this "backbending" is the presence of a hypothetical further correction term which begins to influence the data only at $L > 14$ and so has an extremely small exponent (and a prefactor A of the opposite sign).

We can search for potential candidate terms for the hypothetical backbending. In addition to the conformal correction, in principle, there can also be an analytic correction. This term would have an exponent $\omega_a \approx 2$, as in [17] where in the site percolation context "the subleading analytical corrections for most operators go as $L^{-\gamma/\nu} \approx L^{-2}$,"so such an analytic correction cannot play the role of the hypothetical very small exponent term. Turning back to the conformal corrections, the first term in the RGT ϵ expansion for the ISG leading irrelevant operator exponent is $\theta(d) = 6 - d$ [18,19] (see Ref. [20] for the analogous site percolation ϵ expansion). Leading ϵ-expansion terms in ISGs give useful qualitative indications for other critical exponents, and it turns out that the ϵ-expansion values for $\theta(d): \theta(5) \sim 1$, $\theta(4) \sim 2$, $\theta(3) \sim 3$ are qualitatively consistent with published effective $\theta(d)$ and $\omega(d) = \theta(d)/\nu(d)$ values from simulations and from quite independent HTSE results [15,21]. Bimodal ISG FSS estimates in three dimensions are $\omega(3) = 1.12(10)$ and $\nu(3) = 2.56(4)$ [8], so $\theta(3) = \omega\nu \approx 3$. We have seen that in four dimensions FSS estimates are [14] $\theta(4) \approx 1.15$ or $\theta(4) \approx 1.35$ [16]. From FSS data for different 5d ISG models $\omega(5) \approx 1$ and $\nu(5) \approx 0.75$ [16], so $\theta(5) \approx 1$. HTSE estimates in 4D ISG models are $\theta(4) \approx 1.4$ [15] and in 5d ISG models are $\theta(5) \approx 1.0$ [15,21].

These estimates are all broadly compatible with $\theta(d) \approx 6 - d$. Even though it is hard to pin down an exact value for $\omega(4)$, consistency definitively excludes a hypothetical leading conformal correction term in the 4D bimodal ISG with an exponent $\omega_b(4)$ much smaller than 1. A correction term with $\omega \approx 1.2$ for the bimodal ISG can be confidently identified with the leading conformal correction. By definition, no conformal correction term with a smaller exponent exists. It can be concluded that there is no backbending correction and that the natural extrapolations of the bimodal model data to the large L limit with $\omega \approx 1.2$ are valid.

IV. CONCLUSIONS

Systems in the same universality class must have identical values for the infinite size critical limit of a dimensionless parameter such as the Binder cumulant g_c. The observation of a critical limit for the bimodal ISG which is very different from those of the other three models disproves universality in these 4D ISGs.

From the existing data there appear to be two possible scenarios: two classes of ISGs (such as models with continuous distributions and those with discrete distributions) or, alternatively, ISG exponents which vary continuously with a parameter such as the kurtosis of the interaction distribution. It would be of interest for statistical physics in general to obtain further information on the question. Claims of universality for ISGs in other dimensions should be reexamined critically.

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