Experimental investigation of the elastic enhancement factor in a transient region between regular and chaotic dynamics

Michał Ławniczak, Małgorzata Białous, Vitalii Yunko, Szymon Bauch, and Leszek Sirko
Institute of Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warszawa, Poland
(Received 24 December 2014; published 26 March 2015)

We present the results of an experimental study of the elastic enhancement factor W for a microwave rectangular cavity simulating a two-dimensional quantum billiard in a transient region between regular and chaotic dynamics. The cavity was coupled to a vector network analyzer via two microwave antennas. The departure of the system from an integrable one due to the presence of antennas acting as scatterers is characterized by the parameter of chaoticity $\kappa = 2.8$. The experimental results for the rectangular cavity are compared with those obtained for a microwave rough cavity simulating a chaotic quantum billiard. The experimental results were obtained for the frequency range $\nu = 16$–18.5 GHz and moderate absorption strength $\gamma = 5.2$–7.4. We show that the elastic enhancement factor for the rectangular cavity lies below the theoretical value $W = 3$ predicted for integrable systems, and it is significantly higher than that obtained for the rough cavity. The results obtained for the microwave rough cavity are smaller than those obtained within the framework of random matrix theory, and they lie between them and those predicted within a recently introduced model of the two-channel coupling [V. V. Sokolov and O. V. Zhirov, arXiv:1411.6211 [nucl-th]].

DOI: 10.1103/PhysRevE.91.032925 PACS number(s): 05.45.Mt, 03.65.Nk

I. INTRODUCTION

The elastic enhancement factor was introduced more than 50 years ago by Moldauer [1], and since then it has been frequently considered in nuclear physics [2–4] and in other fields [5,6]. The elastic enhancement factor W_β is the ratio of variances of diagonal elements of the two-port scattering matrix $\hat{S}$ to off-diagonal elements of this matrix [4–6]. From an experimental point of view, the elastic enhancement factor W_β, where $\beta = 1$ or 2 is the symmetry index for systems with preserved and broken time-reversal symmetry, respectively, is especially interesting because it can be used to study realistic open systems also in the presence of absorption. The properties of the elastic enhancement factor W_β have been studied in several precisely controllable systems, such as microwave cavities [7–10] and networks [11–13]. The conjecture on the universality of the ratio of variances of the scattering elements in electromagnetic fields in the mode-stirred reverberating chambers (time-reversal invariant system) was put forward by Fiachetti and Michelson [7]. The universality of the elastic enhancement factor $W_{\beta=1}$ was also tested in the wave scattering experiments with microwave cavities simulating chaotic quantum billiards [8,9] in the presence of absorption. Dietz *et al.* [9] studied the universality of the elastic enhancement factor W_β with microwave cavities in the case of preserved and partially broken time-reversal symmetries. Quite recently, an extensive study of the elastic enhancement factor $W_{\beta=1}$ was published by Yeh *et al.* [10]. In that paper, the authors were also able to study the elastic enhancement factor for microwave cavities with time-reversal symmetry in a low absorption regime. The reciprocal quantity $\Xi = 1/W_{\beta=1}$ was considered theoretically and measured as a function of frequency for a chaotic microwave cavity with time-reversal symmetry [8,10].

The elastic enhancement factor W_β has also been studied for microwave irregular networks [14,15] simulating quantum graphs with preserved and broken time-reversal symmetry in the presence of moderate and large absorption strength defined as follows: $\gamma = 2\pi\Gamma/\Delta$, where Γ is the average resonance width and Δ is the mean level spacing [5,6], $5 \leqslant \gamma \leqslant 54.4$ [11–13]. Microscopically, the absorption strength $\gamma = \sum_c T_c$ can be modeled by means of a huge number of open, coupled to continuum channels "c," where $T_c = 1 - |\langle S_{cc}\rangle|^2$ and $\langle S_{cc}\rangle$ stands for the average S matrix [6]. The recent paper of Kharkov and Sokolov [4] showed that the elastic enhancement factor of open systems with a transient from the regular to chaotic internal dynamics depends on both the parameter of chaoticity κ and the openness η. The openness η is formally described by the same formula as the absorption strength γ [4].

It is important to point out that the elastic enhancement factor for systems with absorption, in a transient region between regular and chaotic dynamics, has not been studied experimentally yet. In this paper, we present the results of an experimental study of the elastic enhancement factor W_β [5,6] for microwave rectangular and rough cavities, coupled to the vector network analyzer through antennas, simulating, respectively, partially chaotic and chaotic two-dimensional (2D) quantum billiards with preserved time-reversal symmetry ($\beta = 1$) in the presence of moderate absorption.

The elastic enhancement factor W_β is defined by the relationship [5,6]

$$W_\beta = \frac{\sqrt{\mathrm{var}(S_{aa})\mathrm{var}(S_{bb})}}{\mathrm{var}(S_{ab})}, \tag{1}$$

where $\mathrm{var}(S_{ab}) \equiv \langle |S_{ab}|^2\rangle - |\langle S_{ab}\rangle|^2$ is the variance of the scattering matrix element S_{ab} of the two-port scattering matrix,

$$\hat{S} = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix}. \tag{2}$$

For small and intermediate values of the parameter γ, the elastic enhancement factor W_β might depend both on the parameter γ and on the coupling to the system [8]. However, for large absorption strength $\gamma \gg 1$, the elastic enhancement factor W_β can be approximated by the formula $W_\beta = 2/\beta$ [5,6,8]. Fiachetti [16] showed that in the case of a stochastic environment, which can be characterized by a statistically isotropic scattering matrix, the elastic enhancement factor should have the universal value $W_{\beta=1} = 2$. Recently, the two-channel problem (e.g., an experimental system with two ports "a" and "b") with internal absorption and time-reversal symmetry has been numerically considered by Sokolov and Zhirov [17]. It has been shown that for the equivalent channels "a" and "b" with transmission coefficients $T_a = T_b = T$, $0 \leqslant T \leqslant 1$, the elastic enhancement factor $W_{\beta=1}$ depends both on the transmission coefficient T and internal absorption, and it can take, respectively, values between 3 and 2. Hereafter, we will use the abbreviation $W \equiv W_{\beta=1}$.

II. MICROWAVE CAVITIES SIMULATING QUANTUM BILLIARDS

In the experiment, we used a microwave rectangular cavity to simulate a two-dimensional (2D) billiard in a transient region between regular and chaotic dynamics. A quantum chaotic billiard was simulating by a rough microwave cavity. If the excitation frequency ν is below $\nu_{\max} = c/2d$, where c is the speed of light in the vacuum and d is the height of the cavity, only the transverse magnetic TM_0 mode can be excited inside the cavity. Then, the analogy between microwave flat cavities and quantum billiards is based upon the equivalency of the Helmholtz equation describing the microwave cavities and the Schrödinger equation describing the quantum systems [18,19].

Absorption of the cavities can either be changed by changing the frequency range of the measurements, or more effectively, by the application of microwave absorbers. In this paper, we are only interested in moderate absorption, for which $W > 2$ [5,6,8] and which can be controlled by the choice of the microwave frequency range.

III. EXPERIMENTAL SETUP

Figure 1(a) shows the scheme of the rectangular microwave cavity, which was used for measuring of the two-port scattering matrix $\hat{S}$. The scattering matrix $\hat{S}$ of the cavity was measured in the frequency window 16–18.5 GHz. The vector network analyzer Agilent E8364B was connected through the HP 85133-616 and HP 85133-617 flexible microwave cables to the two microwave antennas that were introduced inside the cavity [holes A_1, A_2, A_3, A_4, and A_5 in Fig. 1(a)]. The antennas wires (diameter 0.9 mm) were protruded 3 mm into the cavity. The measurements were completed for 10 different positions of the antennas. The width of the rectangular cavity was $L_2 = 20$ cm. Different realizations of the cavity were created by the change of its length from $L_1 = 41.5$ to 36.5 cm in 25 steps of 0.2 cm length.

Figure 1(b) shows the scheme of the rough microwave cavity [20]. The cavity is composed of the two side wall segments. Segment (1) is described by the function $r(\theta) = r_0 + \sum_{i=2}^{M} a_i \sin(i\theta + \phi_i)$, where the mean radius $r_0 =$ 20.0 cm, $M = 20$, and $0 \leqslant \theta < \pi$. The amplitudes a_i and the phases ϕ_i are uniformly distributed on [0.084,0.091] cm and $[0, 2\pi]$, respectively.

Both the rectangular and rough cavities had the same height $d = 8$ mm, so that $\nu_{\max} = 18.7$ GHz. Also, in the case of the rough cavity, the two-port scattering matrix $\hat{S}$ was measured in the frequency range 16–18.5 GHz. The 3-mm-long antennas were introduced inside the cavity through the holes A_1 and A_2. To create different realizations of the rough cavity, a metallic perturber with area $A_p \simeq 9\,\text{cm}^2$ and perimeter $P_p \simeq 26$ cm [see panel (b)] was moved inside the cavity along the sidewalls using an external magnet. The linear size of the perturber $L_p \simeq 5$ cm was more than 2.5 times bigger than the microwave wavelength at 16 GHz.

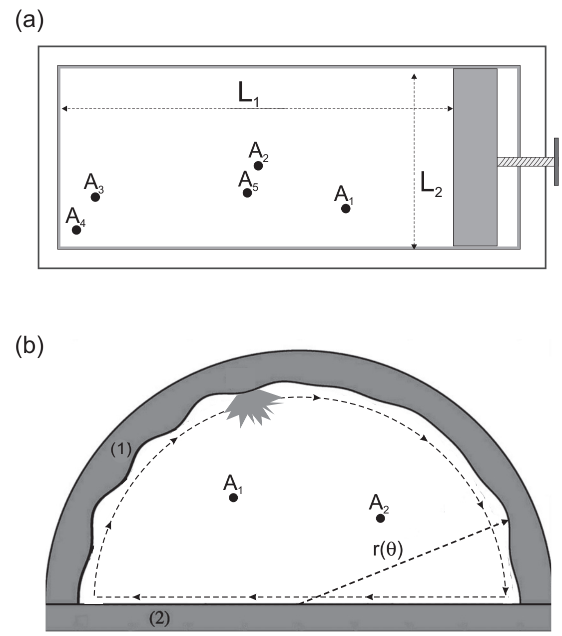

FIG. 1. (a) The rectangular microwave cavity and (b) the rough cavity, which were used for measuring the two-port scattering matrix $\hat{S}$. The rough cavity side wall segments are marked by (1) and (2) (see text). The scattering matrix $\hat{S}$ of the cavities was measured in the frequency window: 16–18.5 GHz. The vector network analyzer Agilent E8364B was connected to the microwave antennas, which were introduced inside the cavities [holes A_1, A_2, A_3, A_4, and A_5 in (a) and A_1 and A_2 in (b)] through the flexible microwave cables HP 85133-616 and HP 85133-617. The width of the rectangular cavity was 20 cm. The length of the cavity was changed from $L_1 = 41.5$ to 36.5 cm in 25 steps of 0.2 cm length. To create different realizations of the rough cavity, a metallic perturber [see panel (b)] was moved inside the cavity.

IV. THE NEAREST-NEIGHBOR SPACING DISTRIBUTIONS

The properties of microwave cavities were investigated using the nearest-neighbor spacing distribution $P(s)$. The distribution $P(s)$ for the microwave rectangular cavity obtained for the frequency range $\nu = 16$–17 GHz is shown in Fig. 2(a) (bars). The distribution $P(s)$ was averaged over 30 microwave cavity configurations. In this way, 2760 eigenfrequencies of the cavity were used in the calculations of the distribution $P(s)$.

Figure 2(a) shows that in spite of using short microwave antennas, the experimental distribution $P(s)$ departs from the Poisson distribution (broken line), which is characteristic of classically integrable systems. The distribution $P(s)$ is also different from the theoretical prediction for the Gaussian orthogonal ensemble (GOE) in RMT (full line) characteristic

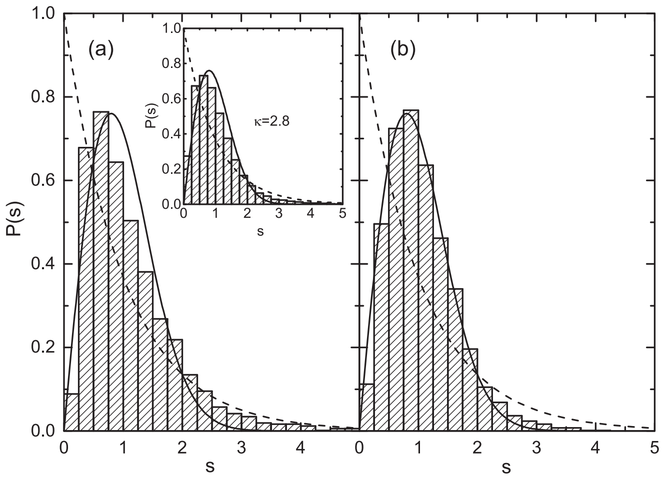

FIG. 2. (a) The nearest-neighbor spacing distribution $P(s)$ for the microwave rectangular cavity coupled to the external channels via antennas (bars) obtained for the frequency range $\nu = 16$–17 GHz. The experimental distribution $P(s)$ is compared to the Poisson distribution (broken line), which is characteristic for classically integrable systems, and to the theoretical prediction for GOE in RMT (full line). The inset shows the numerically reconstructed nearest-neighbor spacing distribution $P(s)$ for the chaoticity parameter $\kappa = 2.8$. (b) The nearest-neighbor spacing distribution $P(s)$ for the microwave rough cavity obtained for the frequency range $\nu = 6$–9 GHz (bars) is compared to the theoretical prediction for GOE (full line) and to the Poisson distribution (broken line).

of chaotic systems with time-reversal symmetry, showing the transition between integrability and chaos. For $0.25 \leqslant s \leqslant 0.75$, it is higher than the one for GOE in RMT, however for $s > 2.0$ it is closer to the Poisson distribution. This behavior is different from that investigated by Robnik and Veble [21] for irrational and rational rectangles where huge fluctuations and the departure of the distribution $P(s)$ from the Poisson one were reported for very small s.

The results obtained for the microwave rectangular cavity should be contrasted with the results obtained for the nearest-neighbor spacing distribution $P(s)$ for the rough cavity [bars in Fig. 2(b)], which shows much better agreement with the theoretical prediction for GOE in RMT (full line). In the case of the rough cavity, the distribution $P(s)$ was calculated on the basis of 3554 cavity eigenfrequencies. Some small discrepancies in the experimental $P(s)$ from the RMT prediction for $s \geqslant 2.25$ may be connected with either some unresolved resonances or fingerprints of nonuniversal behavior of the rough cavity. Similar discrepancies in the nearest-neighbor spacing distribution $P(s)$ for $s \simeq 2.5$ are also visible in the experimental results presented in the paper by Poli *et al.* [22]. Typical spectra of the rectangular cavity in the frequency range 16–17 GHz and the rough cavity in the frequency range 8–9 GHz are shown in Figs. 3(a) and 3(b), respectively.

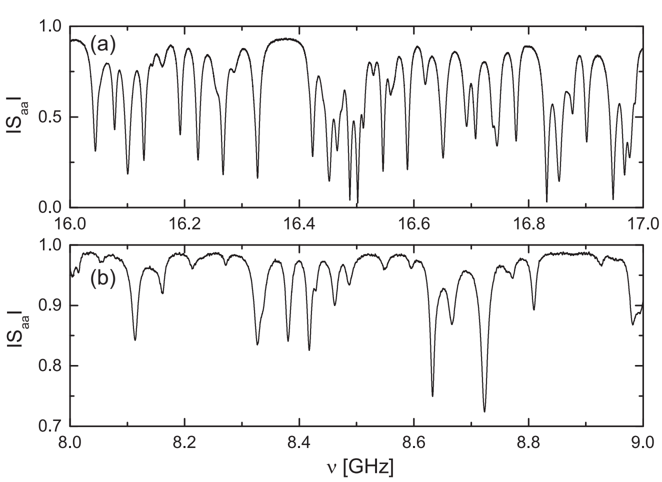

FIG. 3. Typical spectra of the rectangular cavity in the frequency range 16–17 GHz [panel (a)] and the rough cavity in the frequency range 8–9 GHz [panel (b)].

V. EXPERIMENTAL AND NUMERICAL RESULTS FOR THE ELASTIC ENHANCEMENT FACTOR

In Fig. 4(a), the elastic enhancement factor W of the two-port scattering matrix $\hat{S}$ of the microwave rectangular cavity is shown as a function of microwave frequency $\nu = 16$–18.5 GHz (full circles). Due to significant fluctuations of the enhancement factor W, the experimental points were obtained by averaging W over 250 different realizations of the cavity length and the antenna positions in the frequency window $(\nu - \delta\nu/2, \nu + \delta\nu/2)$, where $\delta\nu = 0.5$ GHz. The two black broken lines $W = 2$ and 3 show, respectively, the RMT limits for very strong and very low absorption.

The parameter γ for the microwave rectangular cavities depends on the microwave frequency, and it was changed from 5.2 to 7.4 with the increase of frequency ν from 16 to 18.5 GHz.

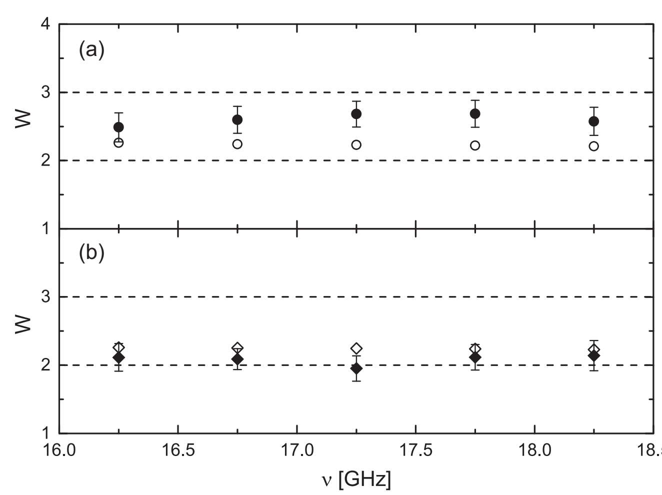

FIG. 4. (a) The elastic enhancement factor W of the two-port scattering matrix $\hat{S}$ of the rectangular cavity coupled to the external channels via antennas simulating a quantum system with the chaoticity parameter $\kappa = 2.8 \pm 0.5$ (full circles). The RMT results [5,6] are shown by empty circles. (b) The elastic enhancement factor W of the two-port scattering matrix $\hat{S}$ of the microwave rough cavity simulating a quantum chaotic system (full black rhombi). The RMT results are shown by empty rhombi. The measurements were done in the frequency window $\nu = 16$–18.5 GHz. The two black broken lines $W = 2$ and 3 show, respectively, the RMT limits for very strong and very low absorption. The latter limit $W = 3$ is also expected for the integrable systems.

According to Kharkov and Sokolov [4], the elastic enhancement factor W_β of the two-port scattering matrix $\hat{S}$ evaluated within the framework of RMT can be expressed by

$$W_\beta = 2 + \delta_{1\beta} - \gamma \int_0^\infty d\tau\, e^{-\gamma\tau} b_{2,\beta}(\tau,\kappa), \tag{3}$$

where $b_{2,\beta}(\tau,\kappa)$ is the spectral form factor. The parameter of chaoticity κ changes from $\kappa = 0$ for classically integrable systems to $\kappa \to \infty$ for chaotic systems. It is important to note that in the transition region $0 < \kappa < \infty$, the spectral form factor $b_{2,\beta}(\tau,\kappa)$ is currently known only for systems with broken time-reversal symmetry. For integrable systems with time-reversal symmetry, $b_{2,1}(\tau,\kappa = 0) = 0$, which immediately leads to $W = 3$.

Figure 4(a) shows that the elastic enhancement factor W of the two-port scattering matrix $\hat{S}$ of the rectangular cavity is below the theoretical value $W = 3$. This result, together with the complementary one for the experimental distribution $P(s)$ [see Fig. 2(a)], strongly suggests that the system simulated by the two-port microwave rectangular cavity due to scattering on the antennas departs from the integrable one. This phenomenon was predicted by Seba [23] and then thoroughly analyzed by Tudorovskiy *et al.* [24]. The influence of antennas on the widths of resonances in a two-dimensional rectangular microwave cavity, in a much lower frequency range than that investigated in this paper (i.e., from below 1 GHz up to 5.5 GHz), was studied by Barthélemy *et al.* [25]. In this experiment, relatively short (2-mm-long) antennas were used. To give an idea about the antennas' performance, 3-mm-long antennas used is our experiment were characterized in the frequency range 16–18.5 GHz by the antenna coupling $\frac{1}{2}(T_a + T_b) \simeq 0.75$. In the frequency range 4.5–5.5 GHz, which was considered in Ref. [25], the same antennas were characterized by much smaller antenna coupling, $\frac{1}{2}(T_a + T_b) \simeq 0.15$.

To estimate the chaoticity parameter κ for such a system, we reconstructed the nearest-neighbor spacing distribution shown in Fig. 2(a) using the random matrix Potter-Rosenzweig model described in Ref. [26], where the matrix a_{ij} is defined as follows:

$$a_{ij} = g_{ij}[\delta_{ij} + \lambda(1 - \delta_{ij})], \tag{4}$$

where g_{ij} denotes a symmetric matrix that belongs to GOE matrices. λ is the transition parameter. The off-diagonal elements g_{ij} are independently Gaussian-distributed with the same variance $\mathrm{var}(g_{ij}) = 1$ and the mean zero. The diagonal elements g_{ii} are independently distributed with the variance $\mathrm{var}(g_{ii}) = 2$.

For the matrices a_{ij} of the size $N \times N$, we found out that the parameter λ can be approximated by $\lambda = \kappa/N$. The fit of the numerical nearest-neighbor spacing distribution $P(s)$, calculated on the basis of 100 realizations of 200×200 matrices, to the experimental one yields the chaoticity parameter $\kappa = 2.8 \pm 0.5$. The inset in Fig. 2(a) shows the numerically reconstructed nearest-neighbor spacing distribution $P(s)$. Unfortunately, even knowing the chaoticity parameter κ, we are not able to compare our experimental results with the theoretical ones since the explicit form of the spectral form factor $b_{2,1}(\tau,\kappa)$ is not known. Though the paper [4] suggests that the behavior of the enhancement factor for systems with time-reversal symmetry should be similar to the behavior for systems with broken time-reversal symmetry, this has yet to be proven. Just for completeness of the presentation in Fig. 4(a), we also show the RMT results predicted by Eq. (3) (empty circles) with the spectral form factor $b_{2,1}(\tau,\kappa \to \infty)$ defined by Eq. (6).

In Fig. 4(b), the elastic enhancement factor W of the two-port scattering matrix $\hat{S}$ of the microwave rough cavity simulating a quantum chaotic system is shown (full black rhombi) as a function of microwave frequency $\nu = 16$–18.5 GHz. The results were averaged over 105 different perturber positions in the frequency window $\delta\nu = 0.5$ GHz. It is important to note that the theoretical and experimental investigations of rough billiards (cavities) [27–29] showed that for lower energies (frequencies) there exist regimes of localization and Wigner ergodicity, and only for higher energies (frequencies) do billiards (cavities) become fully chaotic. This fully chaotic regime is called the regime of Shnirelman ergodicity. For the rough cavity used in the experiment, the regime of Shnirelman ergodicity extends for $\nu > 9.9$ GHz. The presence of the perturber causes that even for lower frequencies $\nu = 6$–9 GHz, the nearest-neighbor spacing distribution $P(s)$ is close to the theoretical prediction for GOE in RMT.

The parameter $\gamma = \frac{1}{2}(\gamma^{(a)} + \gamma^{(b)})$ was estimated by adjusting the theoretical mean reflection coefficients parametrized by the parameters $\gamma^{(j)}$,

$$\langle R \rangle_{\mathrm{th}}^{(j)} = \int_0^1 dR\, R P(R), \tag{5}$$

to the experimental ones $\langle R \rangle^{(j)}$ obtained after eliminating the direct processes [30,31]. The index $j = a,b$ denotes the port a or b. In the calculations of $\langle R \rangle_{\mathrm{th}}^{(j)}$, we used the analytic expression for the distribution $P(R)$ of the reflection coefficient R given in Ref. [32]. We found out that using the same antennas, 3 mm long, as in the case of the rectangular cavity, the value of the parameter γ was changed from 5.3 to 6.8 with the increase of microwave frequency ν from 16 to 18.5 GHz, respectively. Taking into account that the microwave antennas act as single scattering channels, the absorption strength γ can be expressed as a sum of the transmission coefficients: $\gamma = \sum_c T_c = T_a + T_b + \alpha$, where α represents the internal absorption of the cavity [9]. The values of the absorption strength γ and the transmission coefficients T_a, T_b, and α are shown in Table I.

Figure 4(b) shows that the experimental results obtained for the rough cavity are below the theoretical ones predicted for W by Eq. (3) within the framework of RMT (empty rhombi). For chaotic systems ($\kappa \to \infty$) with the symmetry index $\beta = 1$, the spectral form factor $b_{2,1}(\tau,\kappa \to \infty)$ in Eq. (3) has the

TABLE I. The absorption strength γ, the transmissions coefficients T_a,T_b, and the internal absorption of the cavity α in the frequency range $\delta\nu$.

$\delta\nu$ (GHz)	γ	T_a	T_b	α
16.0–16.5	5.32	0.57	0.61	4.14
16.5–17.0	5.82	0.67	0.64	4.51
17.0–17.5	6.29	0.61	0.78	4.90
17.5–18.0	6.55	0.73	0.84	4.98
18.0–18.5	6.82	0.74	0.84	5.24

form [5,6]

$$b_{2,1}(\tau,\kappa \to \infty) = [1 - 2\tau + \tau \log(1 + 2\tau)]\Theta(1 - \tau) + \left[\tau \log \frac{2\tau + 1}{2\tau - 1} - 1\right]\Theta(\tau - 1), \quad (6)$$

where $\Theta(\cdot)$ is the Heaviside step function.

It is important to point out that the recent numerical results presented in Ref. [17] for the two-channel problem with absorption give better agreement with the experimental ones. In the case of the two equivalent channels with $0.6 \leqslant T \leqslant 0.8$ and the internal absorption $\alpha = 3$, the theory predicts W to be between 2.08 and 2.03 (see Figure 5 in Ref. [17]). In the experiment, the internal absorption $\alpha \simeq 5$ was larger than those considered in the theoretical calculations, therefore one should expect even smaller theoretical values of the elastic enhancement factor W. For comparison, the experimental elastic enhancement factor W is scattered between 2.1 and 1.95.

VI. CONCLUSIONS

The elastic enhancement factor W was experimentally studied for microwave rectangular and rough cavities simulating partially chaotic (characterized by the transient parameter $\kappa = 2.8$) and chaotic two-dimensional quantum billiards, respectively. Both systems were characterized by similar, moderate absorption strengths, $\gamma = 5.2$–7.4 and 5.3–6.8, respectively. We show that the results obtained for the rectangular cavity lie below the theoretical prediction for integrable systems, $W = 3$, however they are significantly higher than those obtained for the microwave rough cavity. The results obtained for the microwave rough cavity are smaller than those obtained within the framework of RMT, and they lie between them and the results predicted within a model of the two-channel coupling recently introduced by Sokolov and Zhirov [17]. Our experimental results suggest that the elastic enhancement factor can be used as a measure of internal chaos that can be especially useful for systems with significant absorption or openness.

ACKNOWLEDGMENTS

We are very grateful to V. Sokolov for fruitful discussions. This work was partially supported by the Ministry of Science and Higher Education Grants No. N202 130239 and No. UMO-2013/09/D/ST2/03727.

[1] P. A. Moldauer, Phys. Rev. **123**, 968 (1961); **135**, B642 (1964).
[2] W. Kretschmer and M. Wangler, Phys. Rev. Lett. **41**, 1224 (1978).
[3] J. J. M. Verbaarschot, Ann. Phys. (NY) **168**, 368 (1986).
[4] Y. Kharkov and V. Sokolov, Phys. Lett. B **718**, 1562 (2013).
[5] Y. V. Fyodorov, D. V. Savin, and H.-J. Sommers, J. Phys. A **38**, 10731 (2005).
[6] D. V. Savin, Y. V. Fyodorov, and H.-J. Sommers, Acta Phys. Pol. A **109**, 53 (2006).
[7] C. Fiachetti and B. Michielsen, Electron. Lett. **39**, 1713 (2003).
[8] X. Zheng, S. Hemmady, T. M. Antonsen, Jr., S. M. Anlage, and E. Ott, Phys. Rev. E **73**, 046208 (2006).
[9] B. Dietz, T. Friedrich, H. L. Harney, M. Miski-Oglu, A. Richter, F. Schäfer, and H. A. Weidenmüller, Phys. Rev. E **81**, 036205 (2010).
[10] J.-H. Yeh, Z. Drikas, J. Gil Gil, S. Hong, B. T. Taddese, E. Ott, T. M. Antonsen, T. Andreadis, and S. M. Anlage, Acta Phys. Pol. A **124**, 1045 (2013).
[11] M. Ławniczak, S. Bauch, O. Hul, and L. Sirko, Phys. Rev. E **81**, 046204 (2010).
[12] M. Ławniczak, S. Bauch, O. Hul, and L. Sirko, Phys. Scr. **T143**, 014014 (2011).
[13] M. Ławniczak, S. Bauch, O. Hul, and L. Sirko, Phys. Scr. **T147**, 014018 (2012).
[14] O. Hul, S. Bauch, P. Pakoński, N. Savytskyy, K. Życzkowski, and L. Sirko, Phys. Rev. E **69**, 056205 (2004).
[15] O. Hul, M. Ławniczak, S. Bauch, A. Sawicki, M. Kuś, and L. Sirko, Phys. Rev. Lett **109**, 040402 (2012).
[16] B. Michielsen, F. Isaac, I. Junqua, and C. Fiachetti, arXiv:math-ph/0702041.
[17] V. V. Sokolov and O. V. Zhirov, arXiv:1411.6211 [nucl-th].
[18] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).
[19] R. Blümel, P. M. Koch, and L. Sirko, Found. Phys. **31**, 269 (2001); L. Sirko, P. M. Koch, and R. Blümel, Phys. Rev. Lett. **78**, 2940 (1997).
[20] O. Hul, N. Savytskyy, O. Tymoshchuk, S. Bauch, and L. Sirko, Phys. Rev. E **72**, 066212 (2005).
[21] M. Robnik and G. Veble, J. Phys. A **31**, 4669 (1998).
[22] Ch. Poli, G. A. Luna-Acosta, and H.-J. Stöckmann, Phys. Rev. Lett. **108**, 174101 (2012).
[23] P. Seba, Phys. Rev. Lett. **64**, 1855 (1990).
[24] T. Tudorovskiy, U. Kuhl, and H.-J. Stöckmann, New J. Phys. **12**, 123021 (2010).
[25] J. Barthélemy, O. Legrand, and F. Mortessagne, Phys. Rev. E **71**, 016205 (2005).
[26] F. Leyvraz, J. Quezada, T. H. Seligman, and M. Lombardi, Phys. Rev. Lett. **67**, 2921 (1991).
[27] K. M. Frahm and D. L. Shepelyansky, Phys. Rev. Lett. **79**, 1833 (1997).
[28] Y. Hlushchuk, L. Sirko, U. Kuhl, M. Barth, and H.-J. Stöckmann, Phys. Rev. E **63**, 046208 (2001).
[29] N. Savytskyy, O. Hul, and L. Sirko, Phys. Rev. E **70**, 056209 (2004).
[30] S. Hemmady, X. Zheng, J. Hart, T. M. Antonsen, E. Ott, and S. M. Anlage, Phys. Rev. E **74**, 036213 (2006).
[31] M. Ławniczak, O. Hul, S. Bauch, P. Seba, and L. Sirko, Phys. Rev. E **77**, 056210 (2008).
[32] D. V. Savin, H.-J. Sommers, and Y. V. Fyodorov, JETP Lett. **82**, 544 (2005).