Entropic determination of the phase transition in a coevolving opinion-formation model

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We study an opinion formation model by the means of a coevolving complex network where the vertices represent the individuals, characterized by their evolving opinions, and the edges represent the interactions among them. The network adapts to the spreading of opinions in two ways: not only connected agents interact and eventually change their thinking but an agent may also rewire one of its links to a neighborhood holding the same opinion as his. The dynamics, based on a global majority rule, depends on an external parameter that controls the plasticity of the network. We show how the information entropy associated to the distribution of group sizes allows us to locate the phase transition between a phase of full consensus and another, where different opinions coexist. We also determine the minimum size of the most informative sampling. At the transition the distribution of the sizes of groups holding the same opinion is scale free.

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I. INTRODUCTION

The behavior of large communities of individuals may be studied using the concepts and methods of statistical physics [1]. We follow this line to consider the case of the build up of opinion groups in a population. Opinions spread within a population via person-to-person contacts where they are subject to controversy and discussion. Any two agents holding different opinions may, after being in contact, either keep their previous opinions or change them and eventually coincide. In this process agents with the same thinking may become more numerous, constituting large opinion groups while opinions held by few agents may lose relevance and eventually disappear.

The social changes involved in the spread of opinions and the formation of opinion groups can be studied by mapping this problem into the evolution of a social graph in which each node represents an agent characterized by a scalar variable representing its opinion, while the links represent the contacts (interactions) among the agents.

Early works that study the case of binary opinions use the framework of the Ising model, such as, for example, in Ref. [2–4]. The case of three-opinion states has also been studied, mapping it into a Blume-Emery-Griffith model [5]. On the other extreme, as may be the case of religious beliefs, opinions may actually appear in the way of a continuum spectrum, reflecting the shades and very refined differences of interpretation that individuals may show on a particular subject. In this case, opinions are described by continuous variables. The *bounded confidence model* considers the situation where the interaction between two agents depend on how similar their opinions already are [6]. An intermediate situation is considered in the continuous opinion, discrete actions (CODA) model [7], which interpolates between discrete actions, taken at some stage of the dynamics, based on evolving continuous opinion variables. Another intermediate situation, appears when several discrete opinions are possible, as for an election with many candidates. In this case, the opinions are described by a variable that can take many discrete values as in Ref. [8].

While people exchange opinions on a personal basis, it is also true that agents that agree, naturally tend to gather in closer communities while those with different opinions, segregate. In studying the formation of these groups, the problem is what comes first: either opinions spread over the topology of the network forming clusters of agreeing individuals, or a change in the topology brings together agents having the same opinion that were not in contact before. While early studies dealt with evolving opinions on fixed networks with different topologies, the situation where the opinions and the network contacts may change on time scales that are comparable, starts to be considered. A recent review of these *adaptive networks* may be found in Ref. [9].

Recently this point has been studied introducing the coevolution of nodes and links. Two mechanisms that mutually interfere with each other are considered: one is the change of the individual opinions by the successive interactions with other agents, and the other is the change in the structure of the neighborhood of each agent, thus conditioning its possible interactions [8,10]. The coevolution of both adaptation mechanisms may be controlled by an external parameter, as in Ref. [8] where a change in the opinion is produced with probability $1-\Phi$, ($\Phi \in [0,1]$), while the topology of the network is changed with probability Φ. Alternatively, the coevolution of nodes and links may depend on a dynamical variable as in Ref. [11] or finally both dynamics may be independent [10].

In order to study any complex system, it is very common to sample its states, either as part of a measurement process on a real system or in the statistical treatment of the model supposed to describe it. The fact that this sampling is always incomplete has been pointed out in recent work [12]. In this work a theoretical framework is given for the use of information entropy to characterize the sampling of the complex system, in the case where a function to be optimized may be identified.

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In the present work we apply this approach to the study of an opinion formation model where the dynamics of the nodes and that of the network coevolve on the same time scale. Our model studies a dynamics based on a *global majority rule*. This choice represents the situation where, after discussion, two agents finally choose to agree on the best accepted opinion of the two. This differs from the *invasive imitation dynamics* where the active agent simply copies the opinion of one of its neighbours, chosen at random, with a given probability. Our choice also differs from the *local* majority rule, where the active agent is only influenced by its nearest neighbors.

The aim of this work is to show how the ideas described in Ref. [12] can be used to study the phase transition of the model, and to compare the relevance of different samplings in terms of the amount of information that they give about the behavior of the system. Moreover, it points out the importance of a pertinent choice of variables in order to have access to this information. This method also allows us to determine the minimum size of the most informative sample.

II. THE MODEL

We consider a society of N agents, each one having an opinion that is labeled by an integer variable, $\omega_i = 1, \ldots, \Omega$.

No ordering or metrics are assigned to the opinion labels. We describe the community as a graph in which each node represents an individual. The interaction among agents is only allowed when the corresponding nodes are neighbors, i.e., they are joined by a link of the graph. The total number M of links of the graph, as well as the opinions, are initially distributed at random.

Starting from that initial configuration, the social graph is allowed to evolve. Such evolution takes place in discrete time steps in which links and opinions are assumed to coevolve.

At each step a node is chosen at random, we call it *the active agent*. Then, one randomly chooses one of its neighbors among those holding a different opinion. With probability $1 - \Phi$, the active agent confronts its opinion with the chosen neighbor. The result of such interaction is obtained applying a *global majority rule* by which the node holding the opinion with fewer supporters of the two adopts that of its counterpart. With probability Φ, rewiring takes place. This means that the link joining the active agent to the chosen neighbor is cut and the active agent is reconnected at random to any other agent of the system that holds its opinion.

According to this algorithm, in each step, either an opinion or a link, is changed. When $\Phi \approx 0$ opinions are changed very often and the topology of the network remains essentially unchanged, while if $\Phi \approx 1$ the opposite happens: opinions are left unchanged but the topology of the graph is largely modified. This procedure keeps the total number of links, M, constant. In either case links between agents having different opinions are gradually eliminated and replaced by links between agents with the same thinking. The adaptation process therefore converges to a situation in which there are no links between agents with different opinions.

In this model, as in Ref. [8], the two coevolving dynamics are related by the external parameter Φ. Nevertheless, the evolution rules for the nodes and for the links are different. Here the opinions evolve following a global majority rule, which stands for the discussion between the two agents, who finally choose to agree on the opinion that is most popular in the society at that moment. On the other hand, the rewiring allows the agent to connect to a neighborhood already favorable to its own opinion.

We will see that this dynamics not only accelerates convergence, as can be expected, but also changes the nature of the critical behavior of the system. This global majority rule (notice that it also differs from a *local* majority rule, where the active agent is only influenced by its nearest neighbors) may be interpreted as a simple model for recommendation networks, where the popularity of an opinion in the society (for instance, the quality of a product or a service) is an argument taken into account by the agent to build his own opinion.

During the adaptation process, opinion groups may change size by either growing or dwindling, causing eventually some opinions to disappear. Once the number of links between agents with different opinions vanishes, the social graph remains segmented into a set of disconnected subgraphs, each one with agents supporting the same opinion. This does not mean that each opinion is represented by a connected graph: agents with a same opinion may occupy the nodes of several disconnected subgraphs.

Once the convergence of each realization is achieved, each agent has acquired one opinion $\omega_i \in [1,2,\ldots,\Omega]$. So the N agents are distributed in several opinion groups with $k(\omega)$ adherents $[0 \leqslant k(\omega) \leqslant N]$. The opinions that have disappeared in the final state correspond to $k(\omega) = 0$. Therefore, $\sum_{\omega=1}^{\Omega} k(\omega) = N$. The number of opinion groups with size $k(\omega) = k$ is

$$m_k = \sum_{\omega=1}^{\Omega} \delta_{k,k(\omega)}. \tag{1}$$

Since $\sum_{k=0}^{N} m_k = \Omega$, we can define the probability of finding a group with k members as $P(m_k) = m_k/\Omega$. Notice that these m_k groups do not necessary hold the same opinion and that m_0, the number of groups with no member, counts the number of initial opinions that eventually disappear in the coevolution.

Following the ideas of Ref. [12], we calculate the information entropy contained in the distributions of the variables sampled in this model. Studying the evolution of the corresponding entropy one can determine the most informative sampling.

In the present case, a state of the system is given by the outcome of the adaptation process providing a distribution of the opinions across the population. The probability distribution, $P(\omega)$, that a randomly chosen agent has the opinion ω encodes part of the information that can be extracted by sampling the system. In the large N limit, this is defined by

$$P(\omega) = \frac{k(\omega)}{N}. \tag{2}$$

Correspondingly, it is possible to define the opinion entropy S_ω as

$$S_\omega = -\sum_{\omega=1}^{\Omega} P(\omega) \log[P(\omega)]. \tag{3}$$

One can also be interested in the probability $P(k)$ that an agent taken at random belongs to a group of size k, which in the large N limit is

$$P(k) = \frac{km_k}{N} \tag{4}$$

with the corresponding information entropy:

$$\begin{aligned} S_k &= -\sum_{k=1}^{N} P(k)\log[P(k)] \\ &= -\sum_{k=1}^{N} \frac{km_k}{N}\log\left[\frac{km_k}{N}\right] \\ &= -\sum_{k=1}^{N} \frac{km_k}{N}\log(m_k) - \sum_{k=1}^{N}\frac{km_k}{N}\log\left[\frac{k}{N}\right]. \end{aligned} \tag{5}$$

Replacing the value of m_k from Eq. (1) and changing the sums over k to sums over ω one obtains

$$S_k = -\sum_{k=1}^{N} \frac{km_k}{N}\log(m_k) - \sum_{\omega=1}^{\Omega}\frac{k(\omega)}{N}\log\left[\frac{k(\omega)}{N}\right] \tag{6}$$

$$= -\sum_{k=1}^{N} \frac{km_k}{N}\log(m_k) + S_\omega, \tag{7}$$

thus, $S_k \leqslant S_\omega$.

Within the present model there is some degree of ambiguity concerning the way to perform averages when N_r realizations are sampled in order to obtain statistically significant results. All the above derivations are valid for each realization separately; thus if each realization is labeled by r, one can rewrite the above equations as

$$S_\omega(r) = -\sum_{\omega=1}^{\Omega} P(\omega,r)\log[P(\omega,r)], \tag{8}$$

$$S_k(r) = -\sum_{k=1}^{N} \frac{km_k(r)}{N}\log\left[\frac{km_k(r)}{N}\right], \tag{9}$$

and averages can trivially be defined by

$$\overline{S_\omega} = \frac{1}{N_r}\sum_r S_\omega(r), \tag{10}$$

$$\overline{S_k} = \frac{1}{N_r}\sum_r S_k(r). \tag{11}$$

Average entropies also fulfill $\overline{S_k} < \overline{S_\omega}$.

However, there is a second possibility, namely to work with the distribution of group sizes measured over the N_r realizations. Then the average number of groups of a given size is

$$\overline{m_k} = \frac{1}{N_r}\sum_r m_k(r). \tag{12}$$

One can then define the information entropy associated to this global distribution of group sizes $\overline{m_k}$ as

$$S_{\langle k\rangle} = -\sum_{k=1}^{N} \frac{k\overline{m_k}}{N}\log\left[\frac{k\overline{m_k}}{N}\right] \tag{13}$$

$$= -\sum_{k=1}^{N} \frac{k\overline{m_k}}{N}\log(\overline{m_k}) + \overline{S_\omega}. \tag{14}$$

In this case $S_{\langle k\rangle} \neq \overline{S_k}$ showing that the result of calculating the group entropy of each realization and averaging over the sample (average of group size entropies over the N_r realizations, $\overline{S_k}$) is different from that of measuring the group probability distribution over all the realizations of the sample and calculating the entropy associated with the distribution so obtained ($S_{\langle k\rangle}$). Moreover, with this redefinition of the group size entropy the relation $S_{\langle k\rangle} < \overline{S_\omega}$ does not hold.

III. RESULTS

We have studied systems of $N = 400, 800, 1600, 3200$, and 6400 agents with a random initial distribution of M links. We have studied different connectivities of average degree $c = 4, 8$, and 12. The total number of initial opinions, Ω, goes from very low values ($\Omega = 2$) to very high values ($\Omega = 640$). Averages are typically taken over $N_r = 5000$ realizations, except for the largest sizes or the highest connectivities, where we have averaged over 1000 realizations. In the transition region we have performed up to 10 000 realizations for all the sizes in order to investigate the evolution of different entropies with the size of the sampling. Most of the results presented in this article correspond to the case $c = 4$.

The topology of the resulting social graph critically depends upon the value of Φ. When $\Phi \approx 0$, opinion changes are enhanced, the social graph approaches a *consensus state* in which a vast majority of agents merges into a single giant opinion group. As Φ grows, a richer spectrum of sizes takes place until a moment in which the probability distribution of the sizes of opinion groups approaches a power law. The same situation was found in Ref. [8] and can be assimilated to a dynamical phase transition. For even larger values of Φ the probability distribution of the sizes of opinion groups changes into a bell-type distribution that corresponds to the initial random assignment of opinions. The reason for this behavior is that for very large Φ, rewiring is dominant, and opinions do not evolve much.

In Fig. 1 we show the results for the average opinion entropy, $\overline{S_\omega}$, the average of group size entropies $\overline{S_k}$ and the global entropy of group sizes, $S_{\langle k\rangle}$, as a function of the adaptation parameter Φ. The plot of $\overline{S_\omega}$ is easy to interpret: when $\Phi \approx 0$ only a very few opinions survive because the system approaches the consensus state in which many opinions are left without any agent to support them, leading to a very low value of $\overline{S_\omega}$. $\overline{S_\omega}$ grows with Φ because the number of surviving opinions increases as consensus disappears. In the limit of high rewiring individual opinions are left essentially unchanged with respect to the initial random assignment.

$\overline{S_k}$, the average entropy associated to the probability that an agent taken at random belongs to a group of size k, is

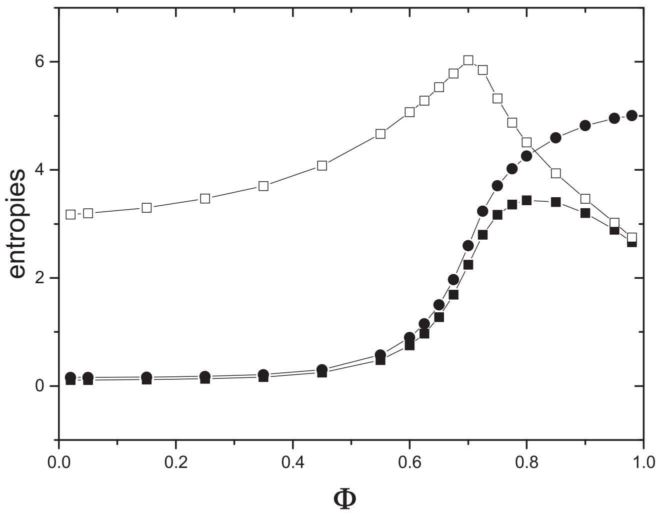

FIG. 1. Entropies corresponding to the probability distributions of opinions and sizes of opinions groups. The social graph has a total of $N = 1600$ agents and and $M = 3200$ links. They share a total of $\Omega = 160$ opinions. Averages are made over 5000 realizations. Open boxes correspond to $S_{\langle k \rangle}$ and filled boxes (circles) correspond to $\overline{S_k}$ ($\overline{S_\omega}$).

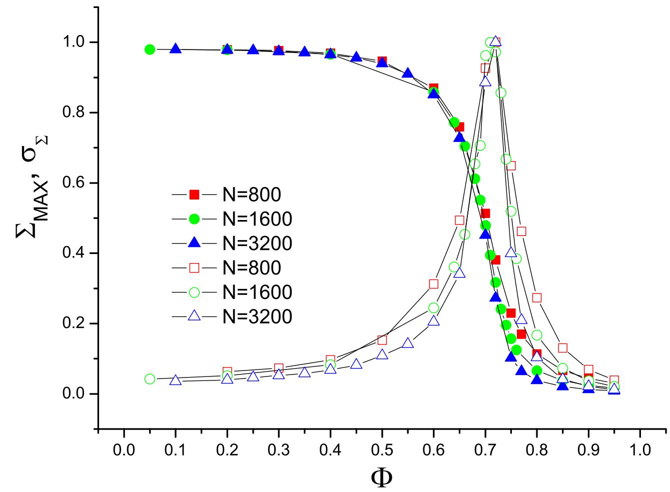

FIG. 2. (Color online) Average size (full symbols) and normalized dispersion (open symbols) of the largest cluster as a function of Φ, for different sizes.

also, for similar reasons, a growing function of Φ, except for the fact that for $\Phi = 1$, where only rewiring is possible, its value must fluctuate around the one corresponding to the initial distribution of opinion groups. This is also the case for $S_{\langle k \rangle}$, so for $\Phi = 1$, both entropies are coincident. This can be understood if one bears in mind that for this value of Φ, the number of supporters of each opinion is left unchanged, and therefore all the opinions are expected to have a number of followers that fluctuates around N/Ω, as in the initial distribution.

The curve for $S_{\langle k \rangle}$, where the entropy is evaluated using the group size distributions of all the realizations in the sample, is particularly interesting. Starting at a low value for $\Phi \approx 0$, where there are very few possible group sizes (consensus state), it develops a sharp maximum for a particular value, Φ_c. This maximum signals the occurrence of a phase transition between a consensus state and a fragmented one, where groups holding different opinions coexist. This is confirmed by the behavior of the order parameter Σ_{Max}, the normalized size of the maximum cluster. Figure 2 shows the sudden collapse of Σ_{Max} at Φ_c, along with the peak of its dispersion, σ_Σ, for different sizes (its height has been normalized in order to plot all the curves together).

In Fig. 3 we plot the probability distributions of the group sizes, calculated over the N_r realizations of the sampling, for different values of Φ.

For $\Phi \approx 0.05$ the distribution decays fast for very small group sizes an displays a significant peak for a group sizes of order N, showing that in most of the samples there is a large dominant group, corresponding to the consensus state, which coexists with some minority groups of different opinions. On the other extreme, the distribution for $\Phi \approx 0.95$ corresponds instead to a bell-shaped distribution with its maximum located at $k \approx N/\Omega$. In the transition region, the distribution function of group sizes may be fit to a power law, $P(m_k) \propto k^{-\alpha}$, as is shown by the inset, which displays the group size distributions curves in the transition region [$\Phi \in (0.71, 0.75)$]. It can be observed that the curve corresponding to the lower bound shows a reminiscence of the peak of the consensus state (large k values).

A precise determination of the exponent α is out of the scope of this work. On one hand, the value of the exponent is very sensitive to the precision in the determination of Φ_c and on the other, the finite size cutoff only allows for a fitting domain that is hardly over two decades. Much larger sizes of each realization will be needed in order to overcome this problem. Nevertheless, we have studied the variation of α in the critical region. From the values shown in Table I we can see that for the model studied here, the exponent lies below the

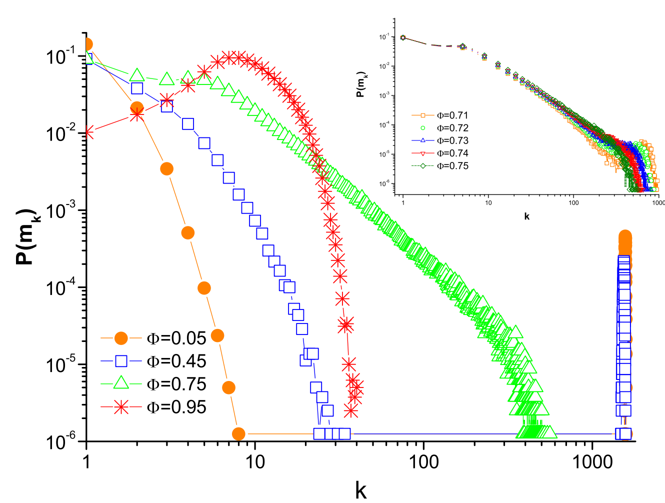

FIG. 3. (Color online) Distributions of group sizes for $N = 1600$ agents average degree $c = 4$ and $\Omega = 160$ initial opinions. The results correspond to a sampling of $N_r = 10\,000$ realizations. In the inset, for the sake of clarity the plot shows only one every four measured points. Notice that for $\Phi = 0.71$ a trace of the peak at high k, still remains.

TABLE I. Values of the critical exponent of the global distribution of cluster sizes, measured over the whole sampling, assuming a power law fit (extreme low and high values excluded) in the critical region.

Φ	α
0.71	2.4 ± 0.015
0.715	2.36 ± 0.02
0.72	2.2 ± 0.01
0.725	2.18 ± 0.02
0.73	2.03 ± 0.01
0.74	1.97 ± 0.09
0.75	2.03 ± 0.01

one corresponding to mean field percolation universality class ($\alpha = 2.5$), while for the invasive dynamics studied in Ref. [8] it is clearly above ($\alpha = 3.5$).

The value of Φ_c increases with the connectivity, c, of the network, we have found $\Phi_c \approx 0.85$ for $c = 8$ and $\Phi_c \approx 0.9$ for $c = 12$, indicating that when the connectivity is large, consensus is always reached within this model.

Interestingly, the plot of the number of adaptation steps required for the social graph to converge to a stationary state presents a well-developed peak in the critical region, as is shown in Fig. 4. Adaptation steps bear a close relationship with computing time; however, it is not a practical measure of the convergence time because the computing time required by a single adaptation step strongly depends upon the value of Φ. Nevertheless this peak in the number of adaptation steps is consistent with the power law behavior of the distribution of group sizes and is reminiscent of the critical slowing down observed in equilibrium critical phenomena, where the correlation time is related to the divergence of the correlation length, revealing the existence of fluctuations at all scales. Here, instead of domains of all sizes, as in magnetic models we have broad distribution of the sizes of groups of agents holding equal opinions.

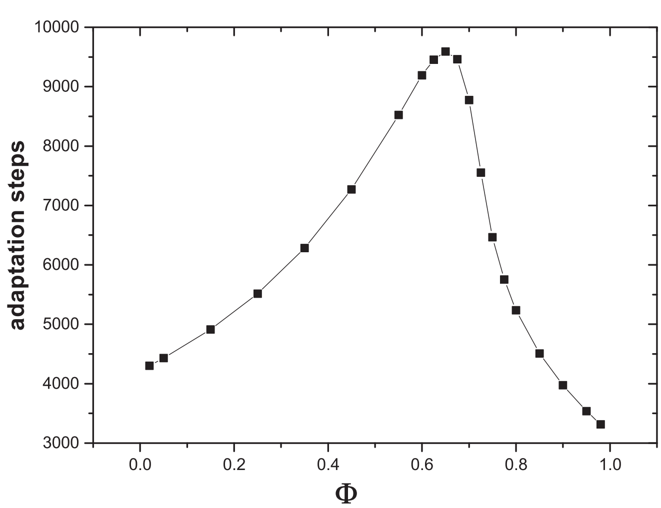

FIG. 4. Plot of the average number of adaptation steps required to reach a social graph without links between agents having different opinions for a system with $N = 1600$, $\Omega = 160$. Averages are made over 5000 realizations.

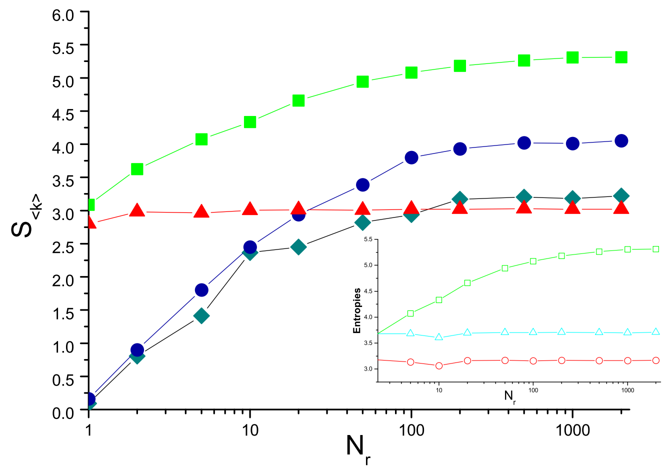

FIG. 5. (Color online) Dependence of $S_{\langle k \rangle}$ on the size of the sampling for different values of Φ above, below, and at the critical region, for a system of $N = 1600$ agents and $\Omega = 160$ opinions; diamonds: $\Phi = 0.05$, circles: $\Phi = 0.45$, squares: $\Phi = 0.75$, triangles: $\Phi = 0.95$. In the inset, the behavior of the three entropies defined here, with the size of the sampling, in the critical region ($\Phi = 0.75$); circles: $\overline{S_k}$, tirangles: $\overline{S_\omega}$, squares: $S_{\langle k \rangle}$.

Figure 5 shows that $S_{\langle k \rangle}$ grows with the size of the sampling, N_r, until saturation, except for very large values of Φ (corresponding to unchanged opinion groups), where it remains constant. This allows us to determine the number of realizations that gives the most informative sampling. Beyond that number, computing more realizations will not bring additional information. As expected [12], the most informative sampling (the one with the largest entropy) corresponds to Φ in the transition region.

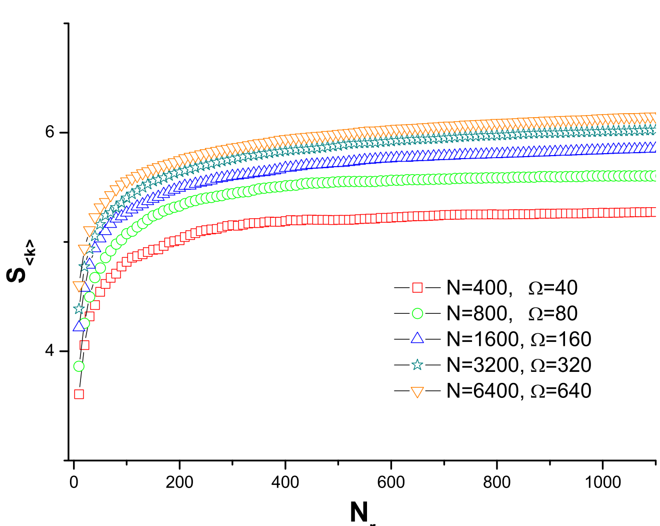

FIG. 6. (Color online) $S_{\langle k \rangle}$ as a function of the number of realizations N_r, for different system sizes in the critical region ($\Phi = 0.72$), where all the samples have the same ratio $N/\Omega = 10$. In order to detail the transitory regime, the N_r axis shows samplings up to $N_r = 1000$ realizations. We have calculated $S_{\langle k \rangle}$ over samplings containing up to $N_r = 10\,000$ realizations and the entropy remains constant.

The inset of Fig. 5 shows the behavior of the different entropies calculated here, in the critical region. While $S_{\langle k\rangle}$ increases with the size of the sampling until saturation, as described in the inset of the Fig. 2 (right) of Ref. [12], the other average entropies $\overline{S_\omega}$ and $\overline{S_k}$ remain constant. This means that if one calculates the entropy for each realization and averages over many realizations, the fact of increasing the sampling size will not bring any new information.

Moreover, the system is not self-averaging. If it were, a sampling consisting of large size networks (large N) and relatively few realizations (small N_r) should give results similar to those issued from a sampling consisting of many realizations N_r of smaller systems (provided that N is still reasonably big so as not to be of the same order of magnitude as the number of initial opinions). Figure 6 shows that this is not the case. The entropy in the critical region is measured for several systems sizes, as a function of N_r. All the sizes reach saturation at approximately the same value of N_r (obviously the saturation value of $S_{\langle k\rangle}$ does depend on N).

IV. CONCLUSIONS

We have studied the phase transition in a coevolutionary opinion model, using the information entropies associated to the distribution of different variables. In this model, the opinions of the agents and the topology of the social network evolve on the same time scale. This coevolution is controlled by an external adaptation parameter Φ, which controls the plasticity of the network by allowing us to switch between opinion updates and link rewiring.

Our results show that the system undergoes a phase transition between a consensus phase and a fragmented one, where several opinions coexist, in agreement with Ref. [8]. However, the dynamics used here induces a critical behavior that is different from the one found in Ref. [8]. First, we obtain a different critical value of the rewiring parameter, $\Phi_c \approx 0.72$. This can be easily understood: in both models the imitation probability is $1 - \Phi$, but as the global majority rule is more efficient in creating and sustaining consensus than the fact of simply copying the neighbors' opinion, the consensus state may remain in this model, up to values of Φ that are higher than the critical value obtained in Ref. [8], namely, $\Phi_c \approx 0.46$. We have made tests using the minority rule instead, and we have found that in these cases the convergence is severely hampered and even impeded, leading to a frustrated situation. Moreover, in the critical region, the distribution of group sizes fits a power law with an exponent that lies below the value corresponding to mean field percolation universality class ($\alpha = 2.5$), leading to a fat tail distribution, at a difference with the findings of Ref. [8], where ($\alpha = 3.5$) is found.

We show that Φ_c may be located using the information entropy associated to the distribution of group sizes measured over the whole sampling of N_r realizations.

We have found that, within this model, the entropy of the distribution of groups sizes is not self-averaging: large systems need as many realizations as small systems in order to reach the most informative regime (where entropy saturates).

In particular, the way in which the average over the different realizations is calculated is far from being irrelevant. This phenomenon is reminiscent of what is observed in disordered magnetic systems, where for instance, the response functions, as the specific heat or the susceptibility of each realization, show a well-developed peak which it is located at a different temperature for each realization [13–15].

As usual, when dealing with complex systems, the question of the choice of the variables to be sampled in order to get the most complete information on the system is crucial. Here the correct sampling is given by $P(\overline{m}_k)$ and its corresponding information entropy $S_{\langle k\rangle}$, which improves with the number of realizations until saturation, as it is shown in Ref. [12], which allows us to determine the size of the most informative sampling. On the other hand, the average entropy $\overline{S_k}$ will not give more information if we increase the number of realizations.

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