Work done and irreversible entropy production in a suddenly quenched quantum spin chain with asymmetrical excitation spectra

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In this paper, we extend the studies on the emergent thermodynamics in a quenched quantum Ising chain (QIC) [R. Dorner, J. Goold, C. Cormick, M. Paternostro, and V. Vedral, Phys. Rev. Lett. **109**, 160601 (2012)] to a more general quantum spin chain with asymmetrical excitation spectra. We verify that the Jarzynski and Tasaki-Crooks relations are still tenable in this system. As an example, we discuss the behaviors of the work done and irreversible entropy production induced by a sudden quenching in the anisotropic *XY* chain in a transverse field with the *XZY-YZX* type of three-site interactions. Different from the QIC, this system has the phase transitions not only between two gapped phases, but also between gapped (or gapless) and gapless phases at zero temperature. We discuss the effects of quantum phase transitions on the work done and irreversible entropy production at low temperature.

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I. INTRODUCTION

It is well known that if a system is driven far away from thermal equilibrium its behavior cannot be described by near-equilibrium approximations. Thus fluctuation theorems may be used to discuss the evolution of the system [1–3]. In 1997, Jarzynski presented a relation [4] that shows how to determine free energy changes by measuring only the work performed on the system, without the need that the processes be quasistatic. By comparing probability distributions for the work spent in the original process with the time-reversed one, Crooks found a refinement of the Jarzynski relation, which is referred to as the Tasaki-Crooks relation [5]. Both these two relations became particularly useful for determining free energy differences. In other words, important equilibrium information can be extracted by studying the fluctuations in nonequilibrium work. In recent years, these relations, initially derived for classical systems, have been extended to quantum systems [2,6–13].

The simplest protocol to take a quantum system out of equilibrium is to change one of the system parameters, which is usually mentioned as *quantum quenching*. There is much work on the probability distribution function of the work done [8,10,13–16], the thermalization [17], and the quantum entanglement [11,18–24] of quantum systems following a quenching. Among them, it is particularly interesting when the change takes the system through a quantum phase transition (QPT) involving macroscopic changes in the state of the system at the initial and final points.

Recently, Dorner *et al.* gave an analytic demonstration of the fluctuation relations in a sudden quenched transverse quantum Ising chain (QIC) and found that near criticality a small change in the transverse field reflected in a sharp increase in irreversible entropy production [6]. Their derivation was based on the symmetrical excitation spectra. In this paper, we will extend the discussion about QIC to a comparatively general quantum spin chain with asymmetrical excitation spectra in order to find out whether the fluctuation relations depend on the symmetry of the excitation spectra and how the different QPTs affect the behaviors of the work done and irreversible entropy production. As an example, we discuss the behaviors of the work done and irreversible entropy production in the anisotropic *XY* chain in a transverse field with the *XZY-YZX* type of three-site interactions. Different from the QIC, this system has gapped and gapless phases [25,26], so that it may undergo three different QPTs at zero temperature: the QPT between two gapped phases, the QPT between gapped and gapless phases, and the QPT between two gapless phases. We discuss the effects of QPTs on the the work done and irreversible entropy production at low temperature.

This paper is organized as follows. In Sec. II, we give a brief description of the quantum Jarzynski and Tasaki-Crooks relations. In Sec. III, we extend the demonstration to the quantum spin chain with asymmetrical excitation spectra. In Sec. IV, we discuss the behaviors of the work done and irreversible entropy production in the anisotropic *XY* chain in a transverse field with the *XZY-YZX* type of three-site interactions. Section V gives a brief conclusion.

II. THE QUANTUM JARZYNSKI AND TASAKI-CROOKS RELATIONS

We consider a quantum system, whose state depends on some parameter λ, if the system equilibrates with the reservoir at the temperature β^{-1}. The partition function is

$$Z(\lambda) = \mathrm{Tr}[e^{-\beta H(\lambda)}]. \quad (1)$$

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At time $t = 0$, the system is in equilibrium with λ at an initial value λ_0. Then we take the parameter from its initial value λ_0 to a final value λ_τ at a later time $t = \tau$ and record the work done once λ_τ is reached. The initial and final Hamiltonians have the spectral decompositions $H(\lambda_0) = \sum_n E_n(\lambda_0)|n\rangle\langle n|$ and $H(\lambda_\tau) = \sum_m E_m(\lambda_\tau)|\widetilde{m}\rangle\langle\widetilde{m}|$ with eigenvalues $E_n(\lambda_0)$ $[E_m(\lambda_\tau)]$ and eigenvectors $|n\rangle$ ($|\widetilde{m}\rangle$), respectively. Following a sudden quenching ($\tau \to 0$), the transition probabilities are $p_n^0 p_{m|n}^\tau = e^{-\beta E_n(\lambda_0)}|\langle n|\widetilde{m}\rangle|^2/Z(\lambda_0)$. Accordingly, the work distribution is [7]

$$P_F(W) = \sum_{n,m} p_n^0 p_{m|n}^\tau \delta\{W - [E_m(\lambda_\tau) - E_n(\lambda_0)]\}. \tag{2}$$

The Fourier transform of the work distribution is [8]

$$\begin{aligned}\chi_F(u) &= \int dW e^{iuW} P(W) \\ &= \mathrm{Tr}[e^{iuH(\lambda_\tau)} e^{-iuH(\lambda_0)} e^{-\beta H(\lambda_0)}/Z(\lambda_0)],\end{aligned} \tag{3}$$

which is also referred to as the characteristic function in the *forward* process.

The Tasaki-Crooks relation is written in terms of the characteristic function as [5]

$$\frac{\chi_F(u)}{\chi_B(-u+i\beta)} = \frac{Z(\lambda_\tau)}{Z(\lambda_0)}. \tag{4}$$

χ_B, here, is the characteristic function in the *backward* process. The Jarzynski relation [4] is given as

$$\chi_F(i\beta) = \langle e^{-\beta W}\rangle = \frac{Z(\lambda_\tau)}{Z(\lambda_0)} = e^{-\beta\Delta F}. \tag{5}$$

These two fluctuation relations are demonstrated by Dorner *et al.* in a sudden quenched transverse QIC [6] by treating the quenching as a thermodynamic transformation. The two relations allow one to access the free energy difference ΔF between two states via nonequilibrium measurements, such as the work done, no matter how far the system is driven out of equilibrium. For the isothermal case, the average work $\langle W\rangle$ will exceed the free energy difference ΔF between initial and final states: $\langle W\rangle \geqslant \Delta F$. The deficit between the average work and the variation in free energy is the average irreversible work, $\langle W_{\rm irr}\rangle$. The irreversible entropy production is defined as

$$\Delta S_{\rm irr} = \beta\langle W_{\rm irr}\rangle = \beta(\langle W\rangle - \Delta F) \tag{6}$$

with

$$\begin{aligned}\langle W\rangle &= -i\frac{d\chi_F(u)}{du}\bigg|_{u=0} \\ &= \mathrm{Tr}\left[\frac{H(\lambda_\tau)e^{-\beta H(\lambda_0)}}{Z(\lambda_0)}\right] - \mathrm{Tr}\left[\frac{H(\lambda_0)e^{-\beta H(\lambda_0)}}{Z(\lambda_0)}\right].\end{aligned} \tag{7}$$

In the following, we will extend the discussion to a more general quantum spin chain.

III. A QUANTUM SPIN CHAIN WITH ASYMMETRICAL EXCITATION SPECTRA

Now we discuss a general quantum spin chain, whose Hamiltonian can be diagonalized to a quadratic form as

$$H = \sum_k \Lambda_k\left(\eta_k^\dagger\eta_k - \frac{1}{2}\right), \tag{8}$$

where k are the waves vectors, Λ_k are the quasiparticle excitation spectra, and η_k and $\eta_k^\dagger$ are fermionic annihilation and creation operators.

The quasiparticle excitation spectra Λ_k can always be written as $\Lambda_k = \varepsilon_k + a_k$. Here, ε_k are symmetrical while a_k are antisymmetrical in the momentum space, that is, $\varepsilon_k = \varepsilon_{-k}$ and $a_k = -a_{-k}$. Obviously, QIC is corresponding to the case of $a_k = 0$.

Therefore, the Hamiltonian H can be written as

$$\begin{aligned}H &= \sum_{k>0}\varepsilon_k(\eta_k^\dagger\eta_k + \eta_{-k}^\dagger\eta_{-k} - 1) + \sum_{k>0} a_k(\eta_k^\dagger\eta_k - \eta_{-k}^\dagger\eta_{-k}) \\ &= \sum_{k>0}[\varepsilon_k(\mathbf{n}_k + \mathbf{n}_{-k} - 1) + a_k(\mathbf{n}_k - \mathbf{n}_{-k})] \\ &= \sum_{k>0} H_k.\end{aligned} \tag{9}$$

Here, $\mathbf{n}_k$ and $\mathbf{n}_{-k}$ are particle number operators and $|n_k,n_{-k}\rangle$ ($n_k = 0,1$) are energy eigenstates of H_k.

From Eqs. (1) and (9), the partition function is

$$\begin{aligned}Z &= 2^N \prod_k \cosh\left(\frac{\beta\Lambda_k}{2}\right) \\ &= \prod_{k>0}(e^{-\beta\varepsilon_k} + e^{\beta\varepsilon_k} + e^{-\beta a_k} + e^{\beta a_k}).\end{aligned} \tag{10}$$

In the case of a sudden quenching, H changes to

$$H(\lambda_\tau) = \sum_k \widetilde{\Lambda}_k\left(\widetilde{\eta}_k^\dagger\widetilde{\eta}_k - \frac{1}{2}\right) = \sum_k(\widetilde{\varepsilon}_k + \widetilde{a}_k)\left(\widetilde{\eta}_k^\dagger\widetilde{\eta}_k - \frac{1}{2}\right). \tag{11}$$

$\widetilde{\eta}_k(\widetilde{\eta}_k^\dagger)$ and $\eta_k(\eta_k^\dagger)$ are related by the Bogoliubov transformations [25]:

$$\begin{aligned}\eta_k &= \cos\varphi_k\widetilde{\eta}_k + i\sin\varphi_k\widetilde{\eta}_{-k}^\dagger, \\ \eta_k^\dagger &= \cos\varphi_k\widetilde{\eta}_k^\dagger - i\sin\varphi_k\widetilde{\eta}_{-k}.\end{aligned} \tag{12}$$

Here, φ_k are the Bogoliubov angles.

Therefore, the relations of the eigenstates of $H_k(\lambda_0)$ and $H_k(\lambda_\tau)$ are

$$\begin{aligned}|0_k,0_{-k}\rangle &= i\cos\varphi_k|\widetilde{0}_k,\widetilde{0}_{-k}\rangle + \sin\varphi_k|\widetilde{1}_k,\widetilde{1}_{-k}\rangle, \\ |0_k,1_{-k}\rangle &= |\widetilde{0}_k,\widetilde{1}_{-k}\rangle, \\ |1_k,0_{-k}\rangle &= |\widetilde{1}_k,\widetilde{0}_{-k}\rangle, \\ |1_k,1_{-k}\rangle &= -i\sin\varphi_k|\widetilde{0}_k,\widetilde{0}_{-k}\rangle + \cos\varphi_k|\widetilde{1}_k,\widetilde{1}_{-k}\rangle.\end{aligned} \tag{13}$$

In the above, $|\widetilde{n}_k,\widetilde{n}_{-k}\rangle$ ($\widetilde{n}_k = 0,1$) are energy eigenstates of the final Hamiltonian $H_k(\lambda_\tau)$.

The characteristic function Eq. (3) takes the form

$$\begin{aligned}&\chi_F(u) \\ &= \frac{1}{Z(\lambda_0)}\prod_{k>0}\sum_{n=0,1} e^{-(iu+\beta)[\varepsilon_k(\mathbf{n}_k+\mathbf{n}_{-k}-1)+a_k(\mathbf{n}_k-\mathbf{n}_{-k})]} \\ &\quad\times\langle n_k,n_{-k}|e^{iu[\widetilde{\varepsilon}_k(\widetilde{\mathbf{n}}_k+\widetilde{\mathbf{n}}_{-k}-1)+\widetilde{a}_k(\widetilde{\mathbf{n}}_k-\widetilde{\mathbf{n}}_{-k})]}|n_k,n_{-k}\rangle \\ &= \prod_{k>0}\{e^{(iu+\beta)\varepsilon_k}[e^{-iu\widetilde{\varepsilon}_k}\cos^2\varphi_k + e^{iu\widetilde{\varepsilon}_k}\sin^2\varphi_k] \\ &\quad + e^{-(iu+\beta)\varepsilon_k}[e^{-iu\widetilde{\varepsilon}_k}\sin^2\varphi_k + e^{iu\widetilde{\varepsilon}_k}\cos^2\varphi_k] \\ &\quad + e^{(iu+\beta)a_k}e^{-iu\widetilde{a}_k} + e^{-(iu+\beta)a_k}e^{iu\widetilde{a}_k}\}/Z(\lambda_0).\end{aligned} \tag{14}$$

Under the mapping $\lambda_0 \leftrightarrow \lambda_\tau$, $\varepsilon_k \leftrightarrow \widetilde{\varepsilon}_k$, $a_k \leftrightarrow \widetilde{a}_k$, $\varphi_k \leftrightarrow -\varphi_k$, the backward characteristic function is easily obtained as

$$\begin{aligned}\chi_B(v) = \prod_{k>0}\{&e^{(iv+\beta)\widetilde{\varepsilon}_k}[e^{-iv\varepsilon_k}\cos^2\varphi_k + e^{iv\varepsilon_k}\sin^2\varphi_k]\\ &+ e^{-(iv+\beta)\widetilde{\varepsilon}_k}[e^{-iv\varepsilon_k}\sin^2\varphi_k + e^{iv\varepsilon_k}\cos^2\varphi_k]\\ &+ e^{(iv+\beta)\widetilde{a}_k}e^{-iva_k} + e^{-(iv+\beta)\widetilde{a}_k}e^{iva_k}\}/Z(\lambda_\tau). \qquad (15)\end{aligned}$$

After a simple algebra calculation, the Tasaki-Crooks relation can be derived as

$$\chi_B(-u+i\beta) = \frac{Z(\lambda_0)}{Z(\lambda_\tau)}\chi_F(u). \qquad (16)$$

Moreover,

$$\chi_F(i\beta) = \frac{Z(\lambda_\tau)}{Z(\lambda_0)} \qquad (17)$$

for $\chi_B(0) = 1$. Hence, the two relations are also tenable in this system although the excitation spectra of which are asymmetrical.

Furthermore, the analytic forms of the work done and irreversible entropy production for this spin chain are as follows:

$$\begin{aligned}\langle W\rangle = \sum_{k>0}\{&[\varepsilon_k - \widetilde{\varepsilon}_k\cos 2\varphi_k][e^{\beta\varepsilon_k} - e^{-\beta\varepsilon_k}]\\ &+ (a_k - \widetilde{a}_k)(e^{\beta a_k} - e^{-\beta a_k})\}\\ &\times \frac{1}{e^{-\beta\varepsilon_k} + e^{\beta\varepsilon_k} + e^{-\beta a_k} + e^{\beta a_k}}, \qquad (18)\end{aligned}$$

and

$$\Delta S_{\text{irr}} = \beta\langle W\rangle + \sum_k \ln\frac{\cosh\left[\frac{\beta\Lambda_k(\lambda_\tau)}{2}\right]}{\cosh\left[\frac{\beta\Lambda_k(\lambda_0)}{2}\right]}. \qquad (19)$$

IV. THE ANISOTROPIC *XY* CHAIN IN A TRANSVERSE FIELD WITH THE *XZY-YZX* TYPE OF THREE-SITE INTERACTIONS

As an example, we discuss the behaviors of the work done and irreversible entropy production induced by a sudden quenching in the anisotropic *XY* chain in a transverse field with the *XZY-YZX* type of three-site interactions, whose Hamiltonian is [26]

$$\begin{aligned}H = &-\sum_{n=1}^{N}\left(\frac{1+\gamma}{2}\sigma_n^x\sigma_{n+1}^x + \frac{1-\gamma}{2}\sigma_n^y\sigma_{n+1}^y + h\sigma_n^z\right)\\ &-\sum_{n=1}^{N}\left[\frac{\alpha}{2}\left(\sigma_{n-1}^x\sigma_n^z\sigma_{n+1}^y - \sigma_{n-1}^y\sigma_n^z\sigma_{n+1}^x\right)\right]. \qquad (20)\end{aligned}$$

Here, N is the number of spins in the chain, γ $(-1 \leqslant \gamma \leqslant 1)$ is the anisotropy parameter of the system, $\sigma^{x,y,z}$ are the Pauli matrices, h $(h \geqslant 0)$ is a uniform external transverse field, and $\alpha(\alpha > 0)$ is the *XZY-YZX* type of three-spin coupling constant.

By using the Jordan-Wigner transformation [27]

$$\begin{aligned}\sigma_n^x &= \prod_{i=1}^{n-1}(1-2c_i^\dagger c_i)(c_n^\dagger + c_n),\\ \sigma_n^y &= \frac{1}{i}\prod_{i=1}^{n-1}(1-2c_i^\dagger c_i)(c_n^\dagger - c_n), \qquad (21)\\ \sigma_n^z &= 2c_n^\dagger c_n - 1,\end{aligned}$$

the Hamiltonian (20) can be written by spinless fermion operators $c_n^\dagger$ and c_n. Then, through Fourier transform $c_n = \frac{1}{\sqrt{N}}\sum_k c_k e^{-ikn}$ and the Bogoliubov transformation of the fermionic operators [25]

$$c_k = \cos\theta_k\eta_k + i\sin\theta_k\eta_{-k}^\dagger, \qquad (22)$$

the Hamiltonian (20) can be written in the momentum space and be reduced to the diagonal form of Eq. (8) with

$$\Lambda_k = -\alpha\sin 2k + 2\sqrt{(\cos k + h)^2 + \gamma^2\sin^2 k}. \qquad (23)$$

Here,

$$\begin{aligned}\cos\theta_k &= \frac{h+\cos k - \sqrt{b_k}}{\sqrt{2[b_k - (h+\cos k)\sqrt{b_k}]}},\\ \sin\theta_k &= \frac{\gamma\sin k}{\sqrt{2[b_k - (h+\cos k)\sqrt{b_k}]}}, \qquad (24)\end{aligned}$$

with $b_k = \varepsilon_k^2/4$. Accordingly, $\varepsilon_k = 2\sqrt{(\cos k + h)^2 + \gamma^2\sin^2 k}$, and $a_k = -\alpha\sin 2k$. It is easy to obtain that in Eq. (12) the Bogoliubov angles $\varphi_k = \widetilde{\theta}_k - \theta_k$, where $\widetilde{\theta}_k$ corresponds to $H_k(\lambda_\tau)$.

There are three parameters in this system: h, γ, and α, so we study the behaviors of the work done and irreversible entropy production in the following three cases:

(a) α and γ are fixed, and thus $\lambda = h$, $a_k = \widetilde{a}_k$, and

$$\langle W\rangle = \sum_{k>0}\Delta h\cos 2\theta_k g_k(h_0); \qquad (25)$$

(b) α and h are fixed, and thus $\lambda = \gamma$, $a_k = \widetilde{a}_k$, and

$$\langle W\rangle = \sum_{k>0}\Delta\gamma\sin k\sin 2\theta_k g_k(\gamma_0); \qquad (26)$$

(c) h and γ are fixed, and thus $\lambda = \alpha$, $\varepsilon_k = \widetilde{\varepsilon}_k$, and

$$\langle W\rangle = \sum_{k>0}\Delta\alpha\sin 2k g_k'(\alpha_0). \qquad (27)$$

Here, $\Delta h = h_\tau - h_0$, $\Delta\gamma = \gamma_\tau - \gamma_0$, $\Delta\alpha = \alpha_\tau - \alpha_0$, $g_k(\lambda_0) = \frac{2(e^{\beta\varepsilon_k} - e^{-\beta\varepsilon_k})}{e^{-\beta\varepsilon_k} + e^{\beta\varepsilon_k} + e^{-\beta a_k} + e^{\beta a_k}}$, and $g_k'(\lambda_0) = \frac{e^{\beta a_k} - e^{-\beta a_k}}{e^{-\beta\varepsilon_k} + e^{\beta\varepsilon_k} + e^{-\beta a_k} + e^{\beta a_k}}$ with h_τ, γ_τ, and α_τ the parameters of $H(\lambda_\tau)$.

It is thus clear that $\langle W\rangle$ is proportional to $\Delta\lambda$ whether $\lambda = h, \gamma$, or α. In other words, the value of $\frac{\langle W\rangle}{\Delta\lambda}$ is entirely due to the properties of the initial state. In contrast, the relation between the irreversible entropy production ΔS_{irr} and $\Delta\lambda$ is not linear. If $\Delta\lambda \to 0$, $\Delta F = \Delta\lambda\frac{\partial F}{\partial\lambda}|_{\lambda=\lambda_0} = \langle W\rangle$, which means $\Delta S_{\text{irr}} = 0$. If $\Delta\lambda$ is not small enough, $\frac{\Delta S_{\text{irr}}}{\Delta\lambda}$ depends on not only the properties of the initial state but also the difference between the initial state and the final state.

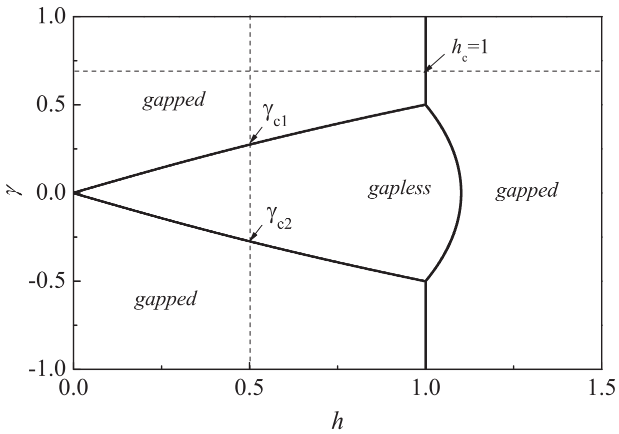

FIG. 1. The phase diagram at zero temperature for the *XY* chain with the *XZY-YZX* type of three-site interaction in a transverse field for $\alpha = 0.5$. The horizontal dashed line corresponds to $\gamma = 0.7$ and the vertical dashed line corresponds to $h = 0.5$.

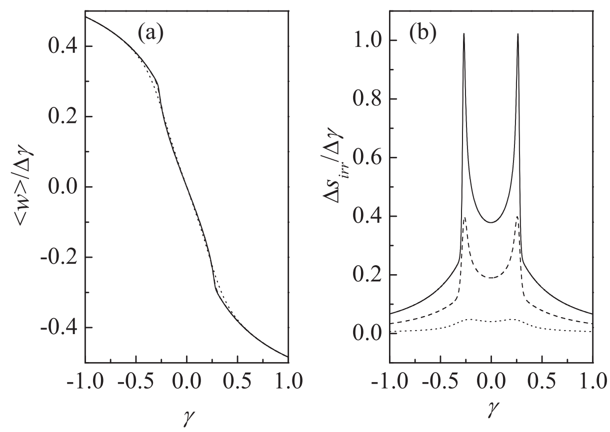

FIG. 3. (a) $\frac{\langle w\rangle}{\Delta\gamma}$ and (b) $\frac{\Delta s_{\rm irr}}{\Delta\gamma}$ as functions of γ for $\Delta\gamma = 0.01$, $h = 0.5$, $\alpha = 0.5$, and $N = 1000$. The solid, dashed, and dotted lines correspond to $\beta = 100,50,10$, respectively.

From the spectra (23), we get the phase diagrams at zero temperature shown as Fig. 1 for $\alpha = 0.5$ and $h \geqslant 0$ and Fig. 4 for $h = 0$ and $\alpha \geqslant 0$. In these diagrams, there are gapped and gapless phases [26]. Between gapped phases, the QPT is referred to as the *Ising transition* [28]. Between gapless and gapped phases, the QPT is similar to the *anisotropy transition* [27].

In the following, we show some numerical results of the work done and the irreversible entropy production. After numerical calculation, we find that $\frac{\langle W\rangle}{N}$ and $\frac{\Delta S_{\rm irr}}{N}$ do not change as N is changing when N is large enough, so we only discuss the quantities per site, that is, $\langle w\rangle = \frac{\langle W\rangle}{N}$ and $\Delta s_{\rm irr} = \frac{\Delta S_{\rm irr}}{N}$, and only show the results corresponding to $N = 1000$ without loss of generality.

First, we study the case of a small change. Figure 2 shows $\frac{\langle w\rangle}{\Delta h}$ and $\frac{\Delta s_{\rm irr}}{\Delta h}$ as functions of h for $\Delta h = 0.01$. As h is changing, the system undergoes a QPT between two gapped phases at $h_c = 1$ with $\alpha = 0.5$ and $\gamma = 0.7$ (see the horizontal dashed line in Fig. 1). Figure 3 shows $\frac{\langle w\rangle}{\Delta\gamma}$ and $\frac{\Delta s_{\rm irr}}{\Delta\gamma}$ as functions of γ for $\Delta\gamma = 0.01$. As γ is changing, the system undergoes a QPT between gapped and gapless phases at $\gamma_{c_{1,2}} \approx \pm 0.274$ with $\alpha = 0.5$ and $h = 0.5$ (see the vertical dashed line in Fig. 1). It is easy to be concerned that when the parameter λ ($\lambda = h$ or γ) changes through the critical lines $\frac{\langle w\rangle}{\Delta\lambda}$ and $\frac{\Delta s_{\rm irr}}{\Delta\lambda}$ are both continuous in the vicinity of the critical points, particularly where $\frac{\Delta s_{\rm irr}}{\Delta\lambda}$ achieves its extreme value. It can be understood that near the critical point the equilibrium state changes dramatically for small changes of λ, so that to drive the system across the critical point is difficult, and this is reflected in a sharp increase in irreversible entropy production. Besides, as h is changed, $\frac{\Delta s_{\rm irr}}{\Delta h}$ tends to diverge symmetrically at the critical point, while $\frac{\Delta s_{\rm irr}}{\Delta\gamma}$ changes more rapidly in one side of the critical points as γ is changed. We believe that this results from the different QPTs, even if they are both second order.

Now, we turn to discuss another interesting case. Figure 4 gives the phase diagram of this system for $h = 0$ at zero temperature. There is a special point $(\alpha,\gamma) = (1,0)$ (see the horizontal dashed line in Fig. 4). The two phases for $\alpha < 1$ and $\alpha > 1$ are both gapless. However, at zero temperature when

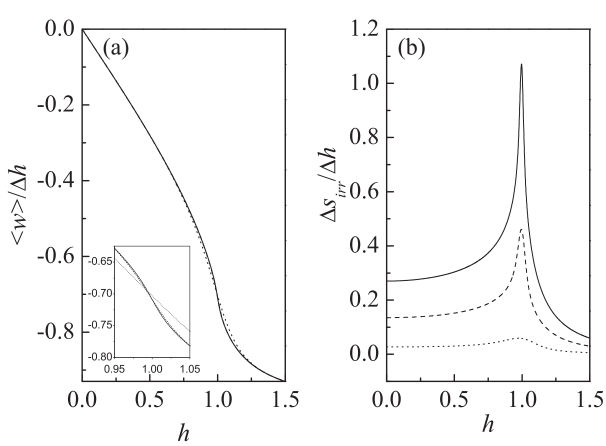

FIG. 2. (a) $\frac{\langle w\rangle}{\Delta h}$ and (b) $\frac{\Delta s_{\rm irr}}{\Delta h}$ as functions of h for $\Delta h = 0.01$, $\gamma = 0.7$, $\alpha = 0.5$, and $N = 1000$. The solid, dashed, and dotted lines correspond to $\beta = 100,50,10$, respectively.

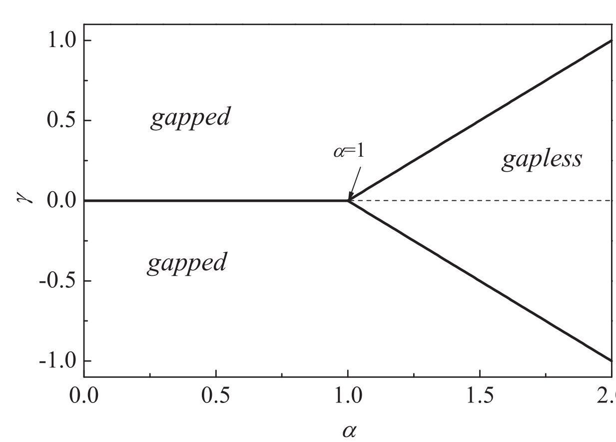

FIG. 4. The phase diagram at zero temperature for the *XY* chain with the *XZY-YZX* type of three-site interaction in a transverse field for $h = 0$. The horizontal dashed line corresponds to $\gamma = 0$.

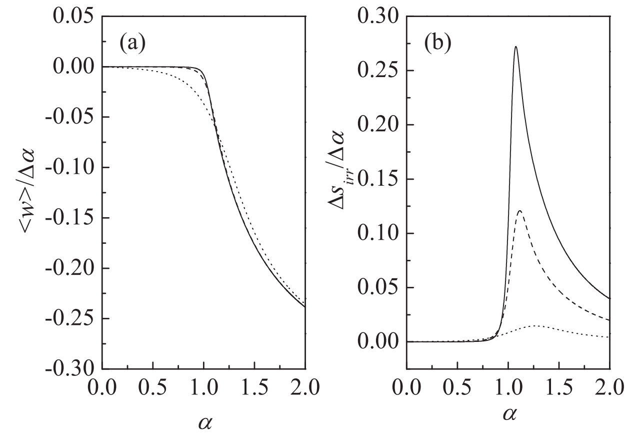

FIG. 5. (a) $\frac{\langle w\rangle}{\Delta\alpha}$ and (b) $\frac{\Delta s_{\rm irr}}{\Delta\alpha}$ as functions of α for $\Delta\alpha = 0.01$, $\gamma = 0$, $h = 0$, and $N = 1000$. The solid, dashed, and dotted lines correspond to $\beta = 100,50,10$, respectively.

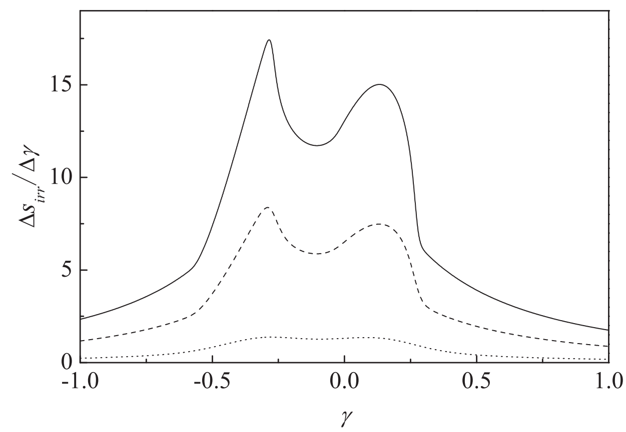

FIG. 6. $\frac{\Delta s_{\rm irr}}{\Delta\gamma}$ as a function of γ for $\Delta\gamma = 0.3$, $h = 0.5$, $\alpha = 0.5$, and $N = 1000$. The solid, dashed, and dotted lines correspond to $\beta = 100,50,10$, respectively.

$\alpha < 1$ the energy spectra have only two Fermi points, whereas when $\alpha > 1$ the energy spectra also have two additional Fermi points [29]. Figure 5 shows the variations of $\frac{\langle w\rangle}{\Delta\alpha}$ and $\frac{\Delta s_{\rm irr}}{\Delta\alpha}$ as α is changing for $\gamma = 0$ and $h = 0$. In the vicinity of the point $\alpha = 1$, $\frac{\langle w\rangle}{\Delta\alpha}$ and $\frac{\Delta s_{\rm irr}}{\Delta\alpha}$ are both continuous and $\frac{\Delta s_{\rm irr}}{\Delta\alpha}$ achieves its extreme value as well. This indicates that although the two phases are both gapless their properties are different. In addition, when $\beta \to \infty$, $\langle w\rangle = 0$ for $\alpha < 1$, which means in this region the energy of the system is irrelevant to α and the system is in a neutral equilibrium state.

Second, when the sudden change is large, $\frac{\langle w\rangle}{\Delta\lambda}$ is the same as that for a small $\Delta\lambda$, but the properties of $\frac{\Delta s_{\rm irr}}{\Delta\lambda}$ are different. For instance, Fig. 6 shows $\frac{\Delta s_{\rm irr}}{\Delta\gamma}$ varying as a function of γ for $\Delta\gamma = 0.3$, whose value is much larger than that for $\Delta\gamma = 0.01$, respectively. The reason is that the sudden quenching coincides with the distance between the states of the initial and final Hamiltonians. The more λ is changed, the more the irreversible entropy production is needed. Moreover, instead of being near the critical points, the sudden quenching between gapped and gapless phases occurs in the interval $\gamma \in (\gamma_{c_2} - \Delta\gamma, \gamma_{c_2}) \cup (\gamma_{c_1} - \Delta\gamma, \gamma_{c_1})$ approximately. γ_{c_1} and γ_{c_2} at zero temperature for $\alpha = 0.5$ and $h = 0.5$ are shown in Fig. 1. As the result, the irreversible entropy production is asymmetrical on either side of the critical points and changes no more sharply near the critical points, which is different from that for $\Delta\gamma = 0.01$.

Certainly, we notice that in the above cases the signature of quantum criticality decreases at higher temperatures with the emergence of thermal fluctuations, and the differences between different QPTs are also eliminated by thermal effects.

V. CONCLUSION

We have studied the statistics of the fluctuation relations, work done, and irreversible entropy production in a sudden quenched quantum chain, whose Hamiltonian can be diagonalized to a reduced form as $H = \sum_k \Lambda_k(\eta_k^\dagger\eta_k - \frac{1}{2})$, with asymmetrical excitation spectra. We verify that the Jarzynski and Tasaki-Crooks relations are still tenable in this system.

The anisotropic *XY* chain in a transverse field with the *XZY-YZX* type of three-site interactions is taken as an example to discuss the effects of the quenching, especially between different phases, on the work done and irreversible entropy production. It is found that, different from the irreversible entropy production, the average work $\langle W\rangle$ is proportional to $\Delta\lambda$ so that the essential characteristics of $\frac{\langle W\rangle}{\Delta\lambda}$ are entirely due to the properties of the initial state. The work done and the irreversible entropy production are both continuous. Moreover, the latter achieves its extreme value in the vicinity of the critical points between different phases and shows some difference for different QPTs at low temperature.

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