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# Marangoni instability of a liquid film flow with viscous dissipation

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A linear stability analysis of a thin liquid film flowing over a plate is performed. The plate is considered as impermeable and adiabatic. The upper surface of the film is assumed to be a free boundary with a non-negligible surface tension, characterized by a Robin thermal boundary condition. The thermoconvective instability is generated by the interplay between the heating due to viscous dissipation and the temperature-dependent surface tension at the free boundary. A basic parallel flow, arbitrarily oriented, is assumed and the basic temperature profile is determined analytically. In order to investigate the linear stability of the system, the normal mode method is employed. A system of ordinary differential equations defining an eigenvalue problem is thus obtained. The case of longitudinal rolls, where the base flow velocity is parallel to the axis rolls, is solved both analytically and numerically. Other possible inclinations of the base flow are investigated by means of a numerical procedure based on combining the Runge-Kutta and the shooting methods.

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## I. INTRODUCTION

The flow generated by density variations induced by heating from below is the well-known Rayleigh-Bénard convection [1] and it represents one of the most studied problems in natural convection. In such cases, buoyancy is the main mechanism responsible for triggering the instability that drives the flow. However, there are two important exceptions where buoyancy is essentially negligible: microgravity and microscale fluid systems. Under either of these conditions, surface tension gradients at the interface between different fluids induced by temperature variations drive the flow instability. This phenomenon, known as Marangoni convection, was discovered in an experimental study [2] that was explained theoretically using linear stability analysis soon afterward [3]. In fact, both mechanisms have already been included simultaneously to analyze their combined effect on the onset of what is known today as Rayleigh-Bénard-Marangoni convection [4]. These studies led to the conclusion that the convection patterns observed earlier [1] in thin, heated liquid films were apparently due to surface tension gradients instead of density gradients.

One of the simplest models for Marangoni convection is employed in these studies. It considers a single layer of liquid that shares an immiscible and nondeformable interface with another layer of gas whose bulk effects are negligible [3,4]. Extensions to include a deformable interface have been considered, where both stationary [5] and oscillatory [6] onsets of convective instability were analyzed. These linear stability studies have been extended to include heat conduction from the finite-width plate that supports the fluid layer [7], the effects of a bounding wall placed above a thin air layer that interfaces with the liquid film [8], and many other effects. Single-layer models provide useful results, especially when used to predict the behavior of short-wavelength disturbances [9]. An extensive review of such models can be found in any of several books published on this topic [10–12].

Despite being a well-known research topic, there are still many important features to be investigated. For instance, only very recently has the effect of a mean flow been included to enable the investigation of mixed convection [13], in what is known as Poiseuille-Bénard-Marangoni convection. This work still used the classical single-layer model with an immiscible and nondeformable interface. A quite recent investigation of Bénard-Marangoni instability in an annular fluid region has been presented by Hoyas *et al.* [14]. Experimental studies of this physical effect have been carried out by Riley and Neitzel [15] and by Minetti and Buffone [16].

The present paper proposes yet another extension for this model, which is the onset of Poiseuille-Marangoni convection driven by viscous dissipation. Both Poiseuille-Rayleigh-Bénard [17] and Couette-Rayleigh-Bénard [18] convection induced by viscous dissipation have been studied in the literature. However, consideration of viscous dissipation as a driver for interface instabilities is lacking. We mention that, although the effect of viscous dissipation can be considered as negligible in purely buoyant flows described according to the Oberbeck-Boussinesq approximation [19], in the presence of an externally imposed throughflow (mixed convection), viscous dissipation may be an active source of thermal instability [20].

In the forthcoming sections, the flow of a Newtonian fluid film on an impermeable adiabatic plate is studied by taking into account the internal heating by viscous dissipation. The upper free boundary of the flowing film is subject to a temperature-dependent surface tension, so the Marangoni effect arises as a possible cause of thermoconvective instability. The linear stability analysis of a basic stationary and parallel flow is carried out relative to longitudinal, oblique, and transverse normal modes.

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## II. MATHEMATICAL MODEL

A Newtonian fluid film flowing over a plate is studied. The thermoconvective instability of this flow is assumed to be generated by the temperature-dependent surface tension acting on the upper surface. The effect of viscous dissipation is taken into account as an internal heat source. The plate, i.e., the lower boundary, is considered as impermeable and adiabatic. The upper boundary is a free surface thermally constrained by a Robin boundary condition. On assuming a constant density $\rho$, the governing equations of local mass, momentum, and energy balance are given by

$$\begin{aligned}&\nabla \cdot \mathbf{u} = 0,\\ &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u},\\ &\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \alpha \nabla^2 T + \frac{2\nu}{c}\mathfrak{D}_{ij}\mathfrak{D}_{ij},\end{aligned} \tag{1}$$

where summation over repeated indices is implied, $\mathbf{u} = (u,v,w)$ is the velocity vector, $t$ is the time, $P$ is the pressure, $\nu$ is the kinematic viscosity, $T$ is the temperature, $\alpha$ is the thermal diffusivity, and $c$ is the specific heat. The viscous dissipation contribution in the local energy balance is expressed by means of the strain tensor $\mathfrak{D}_{ij}$, namely,

$$\mathfrak{D}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2}$$

where $\mathbf{x} = (x,y,z)$ is the position vector expressed by Cartesian coordinates. The hydrodynamic boundary conditions on the free boundary are thus expressed by equating the shear stress and the surface tension gradient

$$\tau_{zx} = \frac{\partial S}{\partial x}, \quad \tau_{zy} = \frac{\partial S}{\partial y} \quad \text{for} \quad z = H, \tag{3}$$

where $\tau$ is the shear stress, $H$ is the film thickness, and the surface tension $S$ is assumed to be a linear function of the temperature $S = S_0 - \sigma(T - T_0)$. The symbol $\sigma$ is a parameter defined as the negative surface tension variation with temperature $-\partial S/\partial T$. Since the upper boundary is assumed to be an impermeable and nondeformable interface, the boundary conditions that characterize the system are

$$\begin{aligned}&u = v = w = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{for} \quad z = 0\\ &\frac{\partial u}{\partial z} = -\frac{\sigma}{\mu}\frac{\partial T}{\partial x}, \quad \frac{\partial v}{\partial z} = -\frac{\sigma}{\mu}\frac{\partial T}{\partial y}, \quad w = 0 \quad \text{for} \quad z = H,\\ &\frac{\partial T}{\partial z} + \frac{h}{k}(T - T_\infty) = 0 \quad \text{for} \quad z = H,\end{aligned} \tag{4}$$

where $\mu$ is the dynamic viscosity, $h$ is the external heat transfer coefficient, $k$ is the thermal conductivity, and $T_\infty$ is the temperature of the fluid outside the liquid film at a large distance from the film itself. A sketch of the geometry of the system and a description of the boundary conditions are illustrated in Fig. 1.

The governing equations and boundary conditions can be rewritten in a dimensionless form,

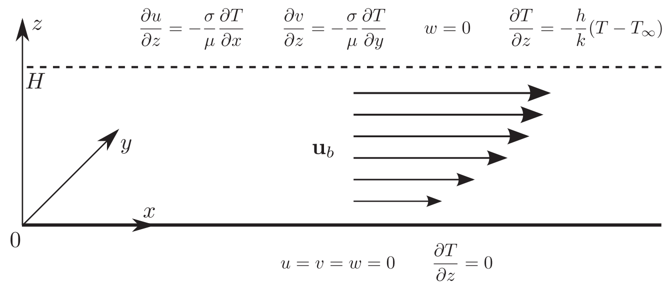


FIG. 1. Sketch of the geometry and the boundary conditions.

namely,

$$\begin{aligned}&\nabla \cdot \mathbf{u} = 0,\\ &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathrm{Pr}(-\nabla P + \nabla^2 \mathbf{u}),\\ &\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T + 2\, \mathfrak{D}_{ij}\, \mathfrak{D}_{ij},\\ &u = v = w = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{for} \quad z = 0,\\ &\frac{\partial u}{\partial z} = -\mathrm{Ma}\frac{\partial T}{\partial x}, \quad \frac{\partial v}{\partial z} = -\mathrm{Ma}\frac{\partial T}{\partial y},\\ &w = 0 \quad \text{for}\ z = 1,\\ &\frac{\partial T}{\partial z} + \mathrm{Bi}T = 0 \quad \text{for} \quad z = 1.\end{aligned} \tag{5}$$

Here Bi is the Biot number, Pr is the Prandtl number, and Ma is the Marangoni number

$$\mathrm{Bi} = \frac{Hh}{k}, \quad \mathrm{Pr} = \frac{\nu}{\alpha}, \quad \mathrm{Ma} = \frac{\sigma H \Delta T}{\alpha \mu}, \tag{6}$$

with $\Delta T = \nu\alpha/H^2 c$. The scaling employed to obtain the dimensionless formulation is

$$\begin{aligned}&\frac{\alpha}{H^2}t \to t, \quad \frac{\mathbf{x}}{H} \to \mathbf{x}, \quad \frac{H}{\alpha}\mathbf{u} \to \mathbf{u}, \quad \frac{T - T_\infty}{\Delta T} \to T,\\ &\frac{H^2}{\mu\alpha}P \to P, \quad \frac{H^2}{\alpha}\,\mathfrak{D}_{i,j} \to \mathfrak{D}_{i,j}, \quad H\nabla \to \nabla,\\ &H^2\nabla^2 \to \nabla^2.\end{aligned} \tag{7}$$

We mention that $\mathrm{Bi} \to \infty$ defines the limiting case of a perfectly isothermal upper boundary, while $\mathrm{Bi} \to 0$ defines a perfectly insulated upper boundary.

## III. BASIC SOLUTION: PARALLEL FLOW

A basic stationary parallel flow in the $(x,y)$ plane inclined an angle $\phi$ with respect to the $x$ axis and driven by a constant pressure gradient is considered. The basic flow is assumed to be dynamically and thermally developed. It must be mentioned that the term $\nabla P$ in Eq. (5) is, rigorously speaking, a dynamic pressure gradient, namely, the difference between the pressure gradient and the gravitational body force. The latter force is uniform and parallel. Thus, the basic parallel flow considered here can be driven either by a constant pressure gradient or, equivalently, by the gravitational body force. In the latter case, the thermally insulated plate is to be considered as inclined to the vertical, thus allowing a falling film flow.

In the basic parallel flow, the stationary velocity and temperature fields depend only on the $z$ coordinate, so the solution of Eq. (5) is given by

$$\begin{aligned} P_b &= Ax\cos\phi + Ay\sin\phi + \text{const}, \\ \mathbf{u}_b &= \frac{Az}{2}(z-2)\hat{\mathbf{n}}, \\ T_b &= \frac{A^2}{12}\left(3 - 6z^2 + 4z^3 - z^4 + \frac{4}{\text{Bi}}\right), \end{aligned} \tag{8}$$

where $\hat{\mathbf{n}} = (\cos\phi, \sin\phi, 0)$ and the subscript $b$ identifies the basic state. The angle $\phi$ can be chosen arbitrarily, so $0 \leqslant \phi \leqslant \pi/2$. The coefficient $A$ can be determined by assuming the Péclet number to be equal to the average value of the basic flow velocity over the film section

$$\text{Pe} = \int_0^1 \mathbf{u}_b \cdot \hat{\mathbf{n}}\, dz \Longrightarrow A = -3\,\text{Pe}. \tag{9}$$

It should be noted that, because of the assumptions just made, the case of an adiabatic upper boundary can here be treated only as a limiting case. In fact, the proper investigation of the double-adiabatic case requires a totally different basic temperature profile: Temperature must grow in the $\hat{\mathbf{n}}$ direction in order to convect downstream the excess heat generated by viscous dissipation. A nonvanishing streamwise temperature gradient allows one to obtain a stationary solution for this special case, which will not be treated here.

Another important aspect to be noted is that the basic solution (8) does not describe any influence of the temperature field on the velocity field. In general, Eq. (5) models such an influence through the Marangoni effect, namely, through the boundary conditions constraining $\partial u/\partial z$ and $\partial v/\partial z$ at $z = 1$. Whenever the temperature field is uniform along $x$ and $y$, actually there is no Marangoni effect. This means that both $\partial u/\partial z$ and $\partial v/\partial z$ vanish at $z = 1$. This condition is that observed with the basic solution (8), where $T_b$ is just a function of $z$. In fact, Eq. (8) does not contain the Marangoni number. The action of the Marangoni effect in producing a thermal influence on the velocity field is expected when the basic solution is perturbed. The perturbed temperature depends on all three coordinates, so in general both $\partial u/\partial z$ and $\partial v/\partial z$ are expected to be nonzero at $z = 1$. Thus, the perturbed fields should produce a Marangoni effect and hence a possible convective thermal instability of the basic flow. There is an evident exception to this argument taking place in the limiting case Bi $\to \infty$ (perfectly isothermal upper boundary). When Bi $\to \infty$, the boundary condition for $T$ at $z = 1$ implied by Eq. (5) is $T = 0$, which means $\partial T/\partial x = 0$ and $\partial T/\partial y = 0$ at $z = 1$ and, as a consequence, no Marangoni effect both in the basic flow and in the perturbed flow. We thus infer that, in the limiting case Bi $\to \infty$, no convective thermal instability of the basic flow is possible. We mention that the behavior described above is a well-known feature arising also in the analysis of the Marangoni-Bénard problem, as pointed out, for instance, by Koschmieder [21].

## IV. LINEAR STABILITY ANALYSIS: HIGHLY VISCOUS FLUIDS

The main goal of the analysis proposed here is understanding more clearly the role played by the viscous dissipation in the onset of thermal convection driven by surface tension. We note that the effect of viscous dissipation is enhanced if the fluid has a very high viscosity. Highly viscous fluids are characterized by high Prandtl numbers, so a sensible assumption is taking Pr $\to \infty$. This assumption allows us to simplify the momentum balance equation by neglecting the contribution of the total derivative with respect to time. The local balance equations (5) may thus be rewritten as

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \nabla^2 \mathbf{u} &= \nabla P, \\ \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T &= \nabla^2 T + 2\,\mathfrak{D}_{i,j}\,\mathfrak{D}_{i,j}. \end{aligned} \tag{10}$$

In order to perform a linear stability analysis we now redefine the velocity, pressure, and temperature fields as composed of a basic stationary state plus an arbitrary small perturbation, namely,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_b(z) + \epsilon\mathbf{U}(\mathbf{x},t), \\ P &= P_b(x,y) + \epsilon\mathfrak{P}(\mathbf{x},t), \\ T &= T_b(z) + \epsilon\Theta(\mathbf{x},t), \end{aligned} \tag{11}$$

where $\epsilon$ is a constant assumed to be small enough to neglect the contributions of the terms of order $\epsilon^2$, i.e., the nonlinear terms in the disturbances. After the substitution of Eq. (11) into Eq. (10), the perturbed equations are linearized and the basic state is subtracted so that we obtain

$$\begin{aligned} &\nabla\cdot\mathbf{U} = 0, \quad \nabla^2\mathbf{U} = \nabla\mathfrak{P}, \\ &\frac{\partial\Theta}{\partial t} + W\frac{dT_b}{dz} + u_b\frac{\partial\Theta}{\partial x} + v_b\frac{\partial\Theta}{\partial y} \\ &\quad = \nabla^2\Theta + 2\frac{du_b}{dz}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + 2\frac{dv_b}{dz}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right). \end{aligned} \tag{12}$$

We are considering a basic flow arbitrarily inclined in the $(x,y)$ plane. In order to carry out a normal mode analysis, based on plane-wave solutions, we can assume invariance of the perturbations in an arbitrarily fixed direction in the $(x,y)$ plane, say, the $y$ axis. We can thus reduce, without any loss of information, the dimension of the problem, setting to zero all the derivatives with respect to $y$. We note that, by considering the $y$ component of the local momentum balance equation (12) and keeping in mind that $\partial\mathfrak{P}/\partial y = 0$, one obtains

$$\nabla^2 V = 0, \quad V(x,0,t) = 0, \quad \frac{\partial V}{\partial z}(x,1,t) = 0. \tag{13}$$

The only possible solution of (13) is $V = 0$. We can now introduce a stream-function formulation for the velocity field components $(U,W)$,

$$U = \frac{\partial\Psi}{\partial z}, \quad W = -\frac{\partial\Psi}{\partial x}, \tag{14}$$

and take the curl of the local momentum balance equation (12). Thus, the set of governing equations and boundary conditions can be written as

$$\begin{aligned}
&\nabla_2^4 \Psi = 0,\\
&\frac{\partial \Theta}{\partial t} - \frac{dT_b}{dz}\frac{\partial \Psi}{\partial x} + u_b \frac{\partial \Theta}{\partial x}\\
&\quad = \nabla_2^2 \Theta + 2\frac{du_b}{dz}\left(\frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Psi}{\partial x^2}\right),\\
&\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial z} = 0, \quad \frac{\partial \Theta}{\partial z} = 0 \quad \text{for} \quad z = 0,\\
&\frac{\partial^2 \Psi}{\partial z^2} = -\mathrm{Ma}\frac{\partial \Theta}{\partial x}, \quad \frac{\partial \Psi}{\partial x} = 0 \quad \text{for} \quad z = 1,\\
&\frac{\partial \Theta}{\partial z} + \mathrm{Bi}\Theta = 0 \quad \text{for} \quad z = 1,
\end{aligned} \tag{15}$$

where

$$\nabla_2^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial z^2}, \quad \nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{16}$$

The disturbances can be expressed as plane waves having the form

$$\begin{Bmatrix} \Psi(x,z,t) \\ \Theta(x,z,t) \end{Bmatrix} = \begin{Bmatrix} i f(z)/\mathrm{Pe}^2 \\ g(z) \end{Bmatrix} e^{i[ax-(\omega+i\lambda)t]}, \tag{17}$$

where $a$ is the wave number, $\omega$ is the angular frequency, and $\lambda$ is the growth rate of the disturbance. When positive, $\lambda$ yields an unstable mode, while $\lambda < 0$ describes a mode damped in time and hence stable. After substituting Eq. (17) into the governing equations (15), we look for the neutrally stable modes, i.e., we set $\lambda = 0$. The neutrally stable modes are in fact the modes that define the threshold between stability and convective instability. The following set of governing equations is obtained:

$$\begin{aligned}
&f'''' - 2a^2 f'' + a^4 f = 0,\\
&g'' - (a^2 + iau_b - i\omega)g\\
&\quad + \frac{2iu_b'}{\mathrm{Pe}^2} f'' + \frac{a}{\mathrm{Pe}^2}(2iau_b' - T_b')f = 0,\\
&f = f' = 0, \quad g' = 0 \quad \text{for} \quad z = 0,\\
&f = 0, \quad f'' = -a\,\mathrm{Ma}\,\mathrm{Pe}^2 g \quad \text{for} \quad z = 1,\\
&\quad g' + \mathrm{Bi}g = 0 \quad \text{for} \quad z = 1,
\end{aligned} \tag{18}$$

where primes denote derivatives with respect to $z$. One may note that the first of Eqs. (18) does not contain $g$ and can be solved analytically to determine $f$ up to an arbitrary multiplicative constant as follows:

$$\begin{aligned}
f(z) = &\frac{\sinh(az)(\sinh a - z\sinh a + az\cosh a)}{2a(a\cosh a - \sinh a)}\\
&- \frac{az\sinh a\cosh(az)}{2a(a\cosh a - \sinh a)}.
\end{aligned} \tag{19}$$

Equation (19) satisfies the boundary conditions $f(0) = f'(0) = 0$ and $f(1) = 0$. In Eq. (19), the arbitrary multiplicative constant is fixed by normalizing $f$ so that $f''(0) = 1$. We note that this normalization breaks the scale invariance of the boundary value problem (18), thus allowing a unique solution for $g$. The evaluation of $g$ is accomplished by solving the second of Eqs. (18), using Eq. (19), and prescribing the boundary conditions $g'(0) = 0$ and $g'(1) + \mathrm{Bi}g(1) = 0$. Finally, the remaining boundary condition, i.e., $f''(1) = -a\,\mathrm{Ma}\,\mathrm{Pe}^2 g(1)$, is used to provide the dispersion relation, which completely characterizes the parametric condition of neutral stability.

## V. LONGITUDINAL ROLLS

The inclination angle $\phi = \pi/2$ defines the longitudinal rolls. These rolls are called longitudinal since the basic flow is parallel to their axes. We start the present investigation with the longitudinal rolls, leaving the analysis of the oblique and transverse rolls for the next section. Due to the principle of exchange of stabilities, the boundary value problem for longitudinal rolls can be solved with $\omega = 0$ and Eq. (18) simplifies to

$$\begin{aligned}
g'' - a^2 g + 3az[3 + (z-3)z]f = 0,&\\
g' = 0 \quad \text{for} \quad z = 0,&\\
g' + \mathrm{Bi}g = 0 \quad \text{for} \quad z = 1,&
\end{aligned} \tag{20}$$

with the function $f$ given by Eq. (19). One may note that Eq. (20) can be solved analytically. The function $g(z)$, with given $(a,\mathrm{Bi})$, will not be written here explicitly for the sake of brevity. The number of governing parameters decreases if we define

$$\Lambda = \mathrm{Ma}\,\mathrm{Pe}^2. \tag{21}$$

In order to obtain a dispersion relation for the neutrally stable modes, we now invoke the boundary condition

$$f''(1) = -a\Lambda g(1). \tag{22}$$

Thus, one obtains the dispersion relation

$$\Lambda = \frac{80}{3}\,\frac{a^4(a - \sinh a\cosh a)(\mathrm{Bi}\cosh a + a\sinh a)}{(9a^4 + 5a^2 - 45)a^2\sinh a + 30\sinh^3 a - 5a^3\cosh a[2a^2 + \cosh(2a) - 4]}. \tag{23}$$

Equation (23) allows one to draw the neutral stability curve $\Lambda(a)$ for a given value of the Biot number. Figure 2 shows some neutral stability curves for different values of Bi. For nonvanishing values of Bi, the curves reported in Fig. 2 have the upward concave shape, with an absolute minimum, typical of thermal instability. The limiting case $\mathrm{Bi} \to 0$ is peculiar, as the neutral stability curve is monotonically increasing. In this case, the absolute minimum of $\Lambda(a)$ is with $a \to 0$. We must keep in mind that the behavior in the limit $\mathrm{Bi} \to 0$ is to be intended just as an asymptotic regime. In fact, as we have already pointed out, a setup with both the lower and the upper wall kept perfectly adiabatic is incompatible with a basic stationary temperature varying only along the $z$ axis. This feature is clearly displayed in Eq. (8), where the

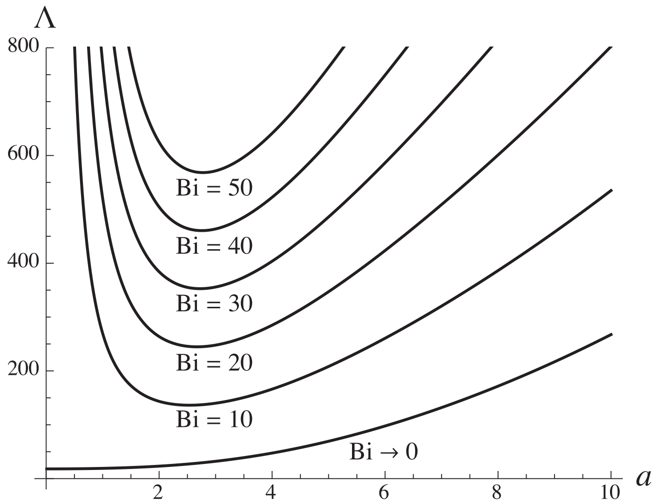


FIG. 2. Longitudinal rolls: neutral stability curves for different values of the Biot number.

expression of $T_b$ becomes singular when $\mathrm{Bi} \to 0$. Figure 2 also shows that the neutral stability curve $\Lambda(a)$ moves upward as Bi increases. As evident also from the right-hand side of Eq. (23), when $\mathrm{Bi} \to \infty$, the neutral stability curve jumps to infinity. The physical meaning of this result is that the basic flow where the lower boundary is thermally insulated and the upper boundary is kept isothermal turns out to be linearly stable versus longitudinal rolls. In order to find the minima of the governing parameter $\Lambda$ as a function of $a$, i.e., the critical value $\Lambda_{cr}$ and the corresponding critical wave number $a_{cr}$, we evaluate the derivative of $\Lambda$ with respect to $a$, by employing Eq. (23), and we set it to zero. Figure 3 shows the critical values of the parameter $\Lambda$ and of the wave number $a$ as functions of Bi. Figure 3 shows in particular the asymptotic behavior of the critical values $a_{cr}$ and $\Lambda_{cr}$ as the Biot number tends to infinity. In this limit, $a_{cr}$ tends to a constant value, namely, $a_{cr} = 2.859\,36$, while $\Lambda_{cr}$ approaches a linearly increasing trend. Thus, by employing Eq. (23), when $\mathrm{Bi} \gg 1$ one obtains the asymptotic expressions

$$a_{cr} = 2.859\,36, \quad \Lambda_{cr} = 30.5750 + 10.7634\,\mathrm{Bi}. \tag{24}$$

On the other hand, in the limit $\mathrm{Bi} \to 0$, one has the power series expansion of Eq. (23) in the neighborhood of $a = 0$ given by

$$\Lambda = \frac{560}{31} + \frac{2632a^2}{2883} + O(a^4). \tag{25}$$

One may question the feasibility of the values of $\Lambda$ as large as $560/31 \cong 18$ in real life systems. An example can be made relative to an engine oil film with $\sigma \cong 4 \times 10^{-5}$ N/mK, $\rho_0 \cong 9 \times 10^2$ kg/m$^3$, $c \cong 2 \times 10^3$ J/kg K, $\alpha \cong 7 \times 10^{-8}$ m$^2$/s, and $H \cong 10^{-2}$ m. On account of Eqs. (6), (7), and (21), one may evaluate $\Lambda = 40U_0^2$, where $U_0$ is the dimensional average velocity of the basic flow. In order to obtain an instability ($\Lambda \cong 18$) one should have $U_0$ not smaller than 0.6 m/s. Such values of $U_0$ are definitely conceivable in practical cases [22]. A review with several examples of applications involving liquid film flows can be found in [23]. As already mentioned in the Introduction, another important point to be made here is that the critical value of the Reynolds number associated with $\Lambda$ can be smaller than the classical critical Reynolds number for a transition to turbulence. In order for this to occur, the Prandtl number must be high enough, as discussed in [17].

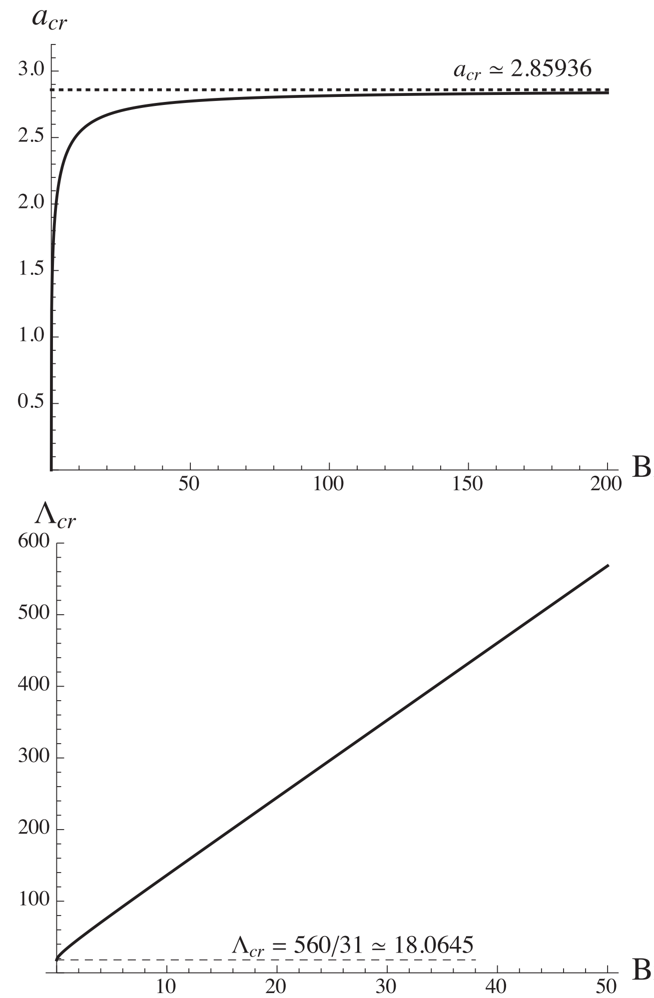


FIG. 3. Critical values, relative to the longitudinal roll case, of the governing parameter $\Lambda$ and the wave number $a$ as functions of Bi.

## VI. OBLIQUE AND TRANSVERSE ROLLS

Oblique rolls are wavelike disturbances with an orientation such that $0 < \phi < \pi/2$, while transverse rolls are obtained when $\phi = 0$. Departure from longitudinal rolls ($\phi = \pi/2$) is a consequence of the governing differential equation not having constant coefficients, so its solution must be sought numerically. In fact, the differential problem to be solved is

$$\begin{aligned} &g'' - (a^2 + iau_b - i\omega)g + \frac{2iu_b'}{\mathrm{Pe}^2} f'' \\ &\quad + \frac{a}{\mathrm{Pe}^2}(2iau_b' - T_b')f = 0, \\ &g' = 0 \quad \text{for} \quad z = 0, \\ &g' + \mathrm{Bi}\,g = 0 \quad \text{for} \quad z = 1, \end{aligned} \tag{26}$$

where the function $f$ is defined by Eq. (19). Problem (26) is complemented by the additional condition given by Eq. (22). Equations (26) are solved numerically by means of a

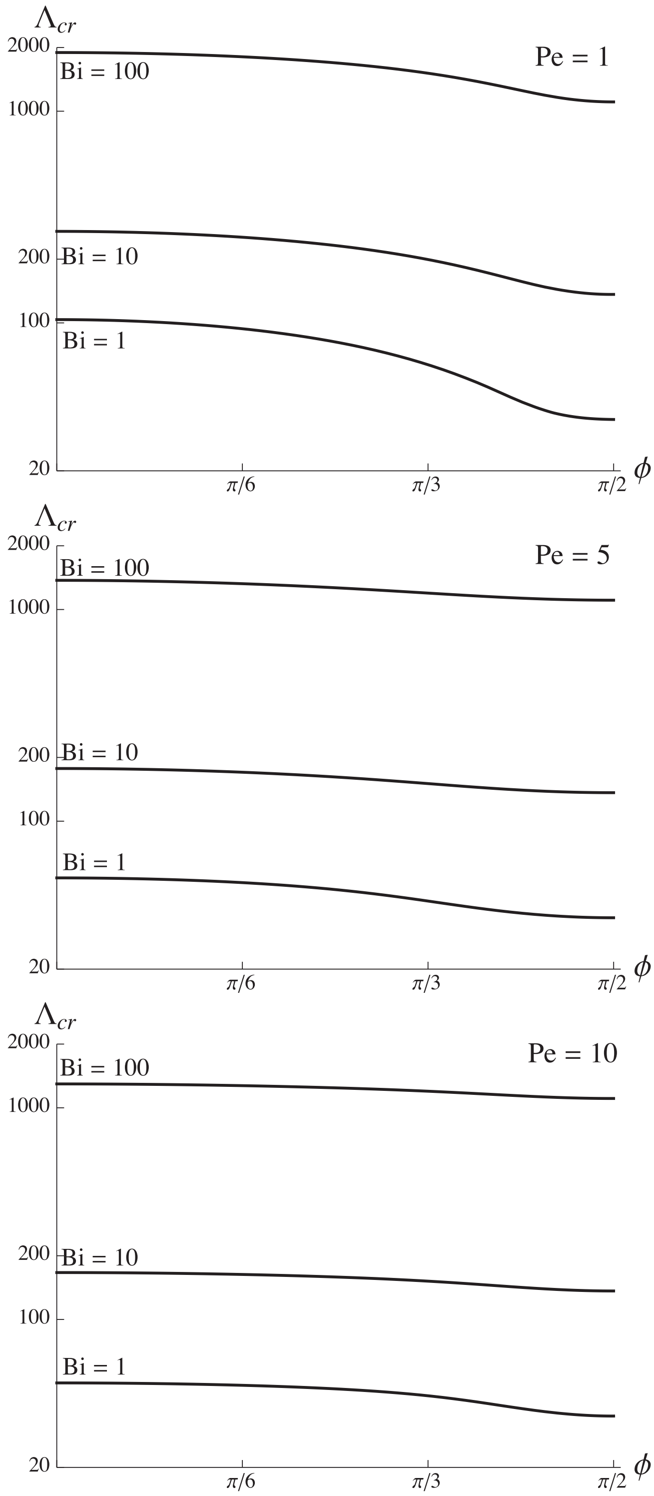


FIG. 4. Critical values of $\Lambda$ as functions of $\phi$ for different values of the Biot number, relative to the cases Pe $= 1$ (top), Pe $= 5$ (middle), and Pe $= 10$ (bottom).

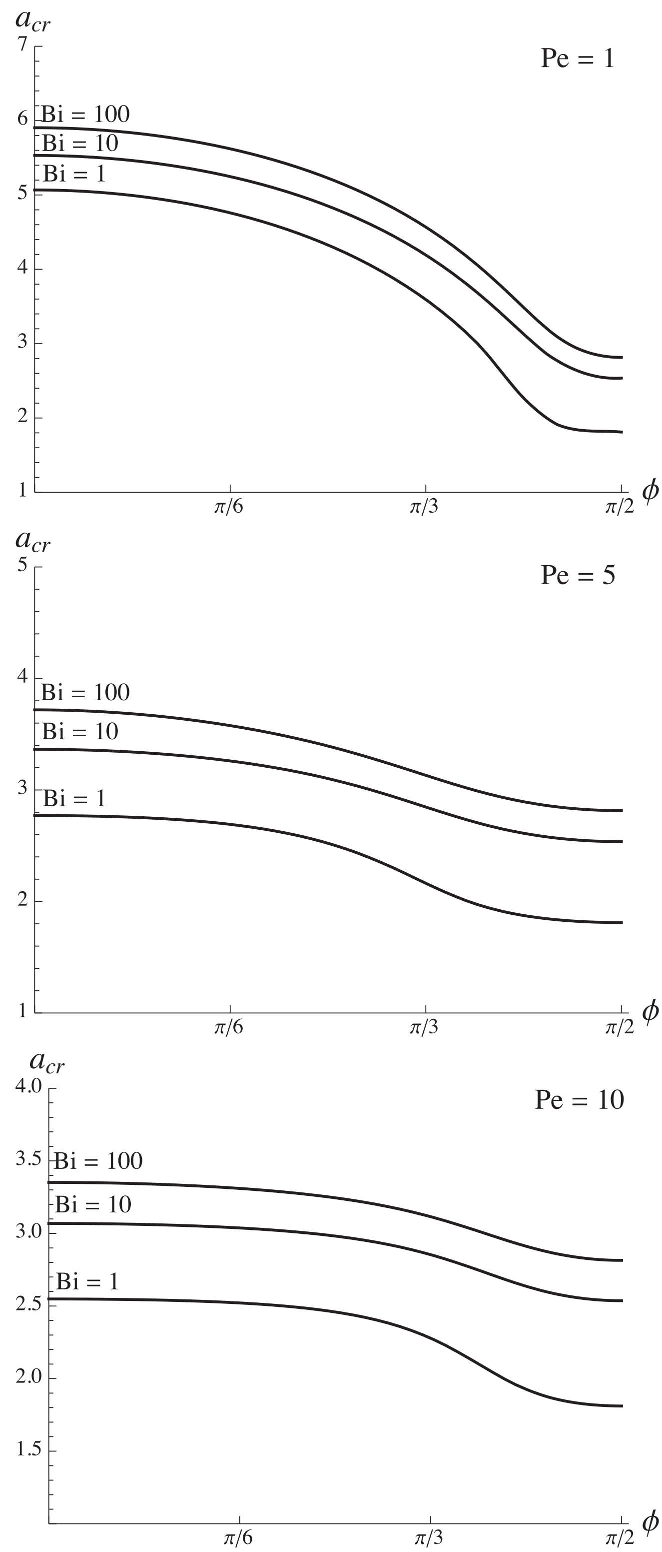


FIG. 5. Critical values of the wave number $a$ as functions of $\phi$ for different values of the Biot number, relative to the cases Pe $= 1$ (top), Pe $= 5$ (middle), and Pe $= 10$ (bottom).

Runge-Kutta method coupled with a shooting method. This numerical procedure is implemented with the software *Mathematica 10* by using the built-in functions `NDSolve` and `FindRoot`.

The investigation of the transition from longitudinal rolls to transverse rolls, obtained by changing $\phi$ from $\pi/2$ to 0, is presented in Figs. 4–6. These figures show the behavior of the critical values of the governing parameters $(\Lambda, a, \omega)$ as functions of the inclination angle $\phi$. Each frame of each figure refers to a different value of the Péclet number and each curve

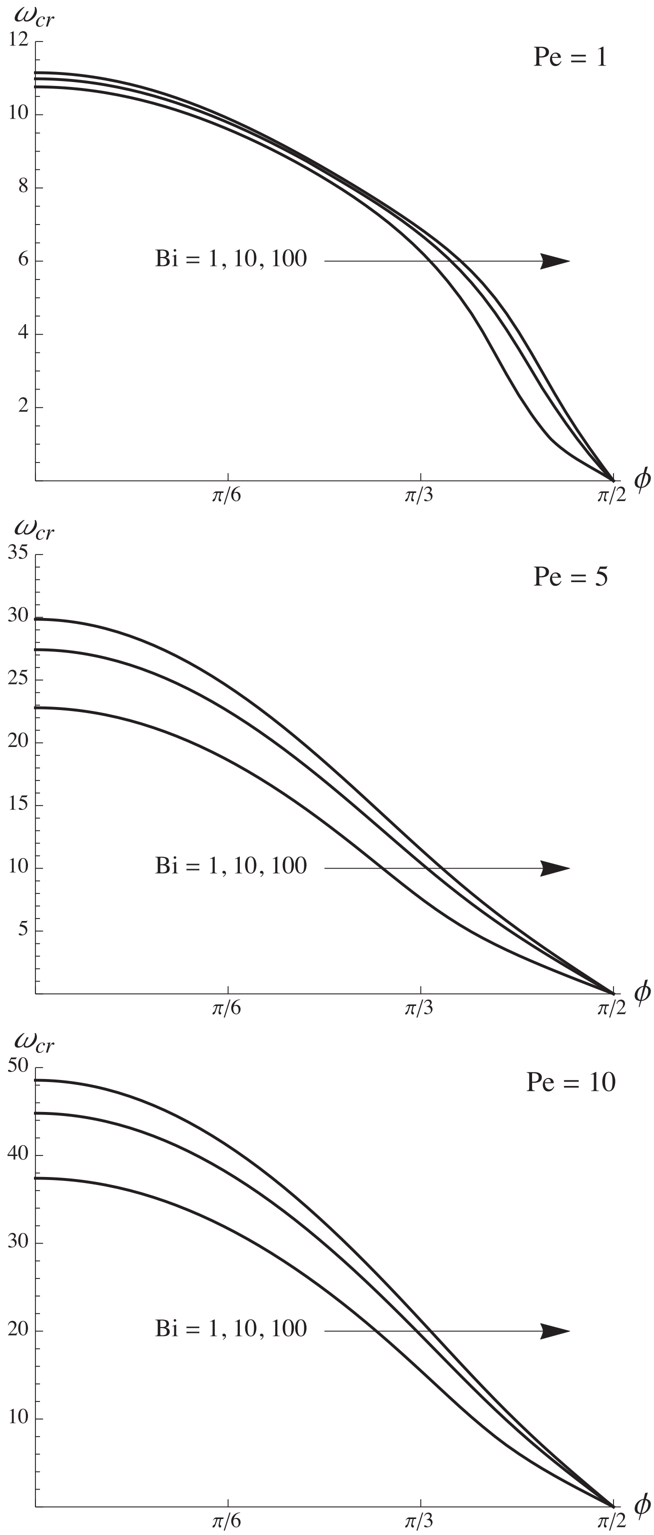


FIG. 6. Critical values of the angular frequency $\omega$ as functions of $\phi$ for different values of the Biot number, relative to the cases Pe = 1 (top), Pe = 5 (middle), and Pe = 10 (bottom).

refers to a different value of the Biot number. The value of each critical parameter presented in these figures decreases monotonically towards the value obtained for longitudinal rolls. Higher Péclet numbers yield lower values of the critical

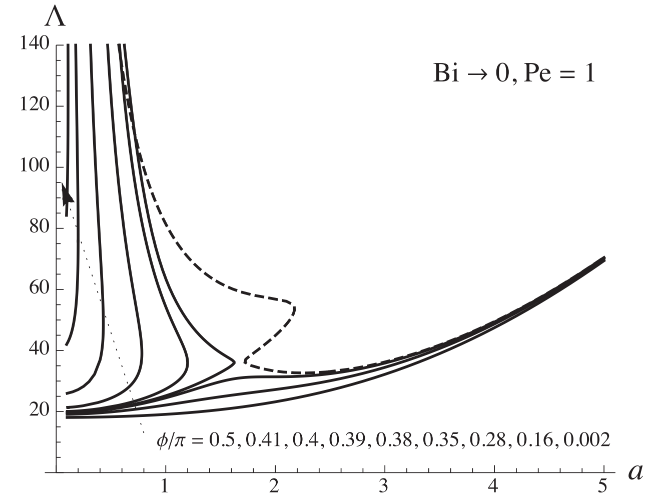


FIG. 7. Neutral stability curves of the oblique rolls as functions of the inclination angle $\phi$ for the limiting case Bi $\to 0$ (adiabatic upper boundary) and Pe = 1. The solid lines refer to the lowest branch. The dashed line is relative to the higher branch with $\phi = 0.39$.

governing parameter $\Lambda_{cr}$ and of the critical wave number $a_{cr}$, while they yield higher values of the critical angular frequency $\omega_{cr}$. Higher values of the Biot number imply higher values of all critical parameters ($\Lambda_{cr}, a_{cr}, \omega_{cr}$). It is evident that the longitudinal rolls turn out to be more unstable with respect to any other possible normal modes. The Péclet number influences the stability of longitudinal rolls only through the parameter $\Lambda = \mathrm{Ma}\,\mathrm{Pe}^2$. On the other hand, Pe appears explicitly in the analysis of oblique and transverse rolls, yielding a destabilizing effect. The influence of an increasing Biot number is stabilizing. This behavior is consistent with what we noted at the end of Sec. III, namely, that the limiting case Bi $\to \infty$ is not subject to any kind of instability.

Figure 7 is relative to the limiting case of an adiabatic top boundary Bi $\to 0$ for a specific value of the Péclet number, namely, Pe = 1. The change of the neutral stability curves versus the inclination angle $\phi$ is illustrated in Fig. 7. The lowest curve is relative to the longitudinal roll case $\phi = \pi/2$. Moving upward from the lowest one, the curves refer to monotonically decreasing values of $\phi$. The continuous lines denote the lowest branch for every prescribed $\phi$. The dashed line corresponds to a higher branch of neutral stability with $\phi = 0.39\pi$. The neutral stability curves depart from the upward concave shape while $\phi$ decreases from $\pi/2$. One observes the detachment of two branches when $\phi$ is within $0.39\pi$ and $0.4\pi$. Then the lowest branch displays a turning point. As pointed out above, the longitudinal rolls turn out to be the most unstable disturbances.

The transverse rolls are defined by the inclination angle $\phi = 0$. No significant simplification of the governing equations can be taken with respect to the general case of oblique rolls. In Fig. 8 we report the critical values of the governing parameter $\Lambda$ as functions of Bi for fixed values of Pe and as functions of Pe for fixed values of Bi. It should be noted that, similarly to the case of longitudinal rolls, the value of $\Lambda_{cr}$ increases almost linearly with Bi. On the other hand, the dependence of $\Lambda_{cr}$ on Pe becomes weaker and weaker as the Péclet number increases.

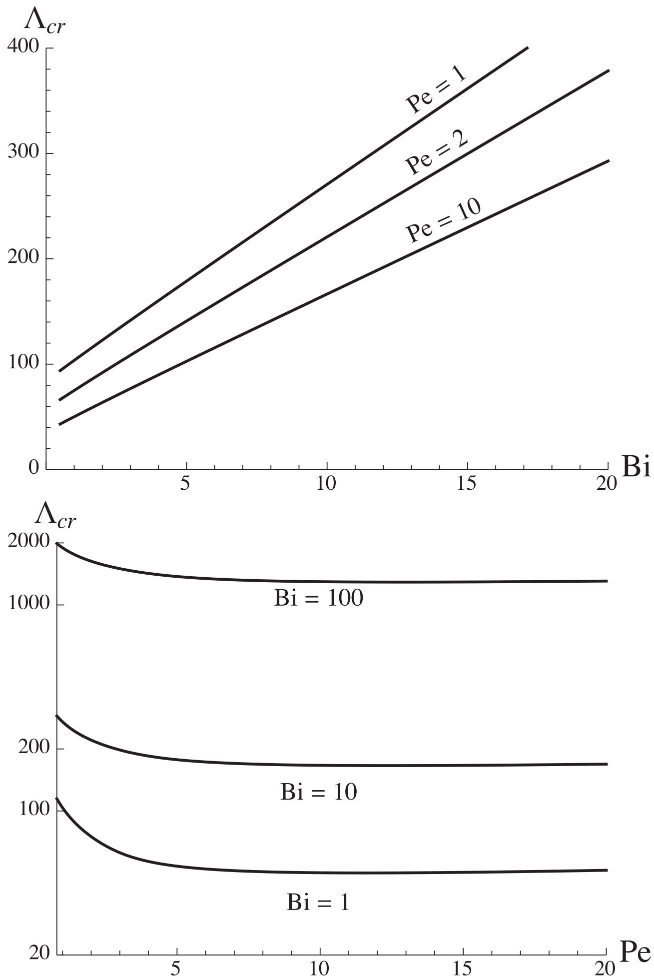


FIG. 8. Critical values of the governing parameter $\Lambda$ for transverse rolls ($\phi = 0$) as functions of Bi for fixed values of Pe (top) and functions of Pe for fixed values of Bi (bottom).

## VII. CONCLUSION

A linear stability analysis of a highly viscous thin liquid film has been carried out. The liquid film flows over a plate and it is internally heated by viscous dissipation. The lower boundary, i.e., the plate, has been assumed to be impermeable and adiabatic, whereas the upper boundary has been modeled as a free nondeformable surface, subject to a Robin temperature boundary condition. The thermoconvective instability investigated here is generated by the interplay between the heating due to viscous dissipation and the temperature-dependent surface tension at the free boundary. A basic fully developed flow inclined arbitrarily and parallel to the plate is imposed such that the temperature and velocity fields depend (nonlinearly) only on the transverse coordinate. The dimensionless governing parameters are the Marangoni number Ma associated with the surface tension, the Biot number Bi relative to the Robin boundary condition at the free surface, and the Péclet number Pe describing the basic flow average velocity. First, the special case of longitudinal rolls is investigated. The eigenvalue problem obtained by employing the normal modes method has been solved analytically in this case. The results have been reported in terms of neutral stability curves $\Lambda(a,\mathrm{Bi})$, where $\Lambda = \mathrm{Ma}\,\mathrm{Pe}^2$. For a given Bi, the absolute minimum of $\Lambda(a,\mathrm{Bi})$ has been determined in order to identify the critical values for the onset of instability. The behavior of the critical values of $\Lambda$ and $a$ has been investigated for different Biot numbers. The oblique and transverse rolls have been studied also by solving numerically the eigenvalue problem for neutral stability. The main results of this analysis are the following.

(i) For every given set of values of the governing parameters, the longitudinal rolls are always more unstable than the oblique and transverse rolls.

(ii) The critical value of $\Lambda$ for the onset of longitudinal rolls is a monotonically increasing function of the Biot number. When $\mathrm{Bi} \to \infty$, the critical value of $\Lambda$ tends to infinity. This finding is consistent with the expected stability of the basic flow when the upper free surface is constrained to be perfectly isothermal.

(iii) The critical value of the wave number for the onset of longitudinal rolls in the case of an isothermal upper boundary, i.e., in the limit $\mathrm{Bi} \to \infty$, tends to the value $a_{cr} = 2.859\,36$. A correlation has been derived for large values of Bi such that $\Lambda_{cr} = 30.5750 + 10.7634\,\mathrm{Bi}$.

(iv) In the limiting case of an adiabatic upper boundary, i.e., when $\mathrm{Bi} \to 0$, the critical values for the onset of longitudinal rolls are $a_{cr} = 0$ and $\Lambda_{cr} = 560/31 \simeq 18.0645$. A series expansion in the neighborhood of $a = 0$ has been performed for small values of Bi, so the asymptotic expression $\Lambda_{cr} = 560/31 + 2632a^2/2883 + O(a^4)$ is obtained.

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