Mean-field approximation for the Sznajd model in complex networks

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This paper studies the Sznajd model for opinion formation in a population connected through a general network. A master equation describing the time evolution of opinions is presented and solved in a mean-field approximation. Although quite simple, this approximation allows us to capture the most important features regarding the steady states of the model. When spontaneous opinion changes are included, a discontinuous transition from consensus to polarization can be found as the rate of spontaneous change is increased. In this case we show that a hybrid mean-field approach including interactions between second nearest neighbors is necessary to estimate correctly the critical point of the transition. The analytical prediction of the critical point is also compared with numerical simulations in a wide variety of networks, in particular Barabási-Albert networks, finding reasonable agreement despite the strong approximations involved. The same hybrid approach that made it possible to deal with second-order neighbors could just as well be adapted to treat other problems such as epidemic spreading or predator-prey systems.

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I. INTRODUCTION

The Sznajd model is a very simple opinion propagation model with an outflowing dynamics that has been employed to describe a wide variety of sociophysics situations in the past decade [1]. In politics, for instance, this model is capable of describing some scaling behavior observed in proportional elections in Brazil [2] and other countries [3].

In this work we consider one of the versions of this model in a complex network that we have named the complex Sznajd model (CSM) [4]. In our version each node of the network stands for a voter who is assigned an integer $\sigma(t) \in \{0,1,2,\ldots,N_c\}$ that represents its opinion in time t; in particular, $\sigma = 0$ stands for an undecided voter. So, once a node X is chosen, the dynamics evolves according the following set of rules [5].

(I) If node X already has an opinion [$\sigma_X(t) \neq 0$], another node Y is picked up at random from the set Γ_X of nearest neighbors of X and rule II is applied; otherwise, nothing happens.

(II a) If node Y is undecided, then it adopts the node X opinion with probability $p_X = 1/q_X$, where q_X is the degree of node X.

(II b) If both nodes X and Y have the same opinion, node X tries to convince each of its neighbors with probability $p_X = 1/q_X$ and node Y does the same with its own neighbors, but now with probability $p_Y = 1/q_Y$.

(II c) If X and Y have different opinions, nothing happens.

Besides these conventional rules, we are also interested in studying the role of a spontaneous opinion change in the dynamics. In this case we consider an extra rule so that node X can choose at random any non-null opinion with probability ω or follow the previous rules with the complementary probability $1-\omega$ [5,6]. We finally define a Monte Carlo time (MCT) as a sequence of N applications of those rules, as usual.

II. THE MASTER EQUATION FOR THE MODEL

From these rules it is possible to write down a master equation for the time evolution of opinions. Let us define the node i as the convinced node, one of its next-nearest neighbors $j \in \Gamma_i$ as the convincing node, and $\eta_\sigma(i,t)$ as the probability of i to have opinion σ at time t [7]. It is also important to define the couple able to convince i as the pair (j,k), with k belonging to the next-nearest neighborhood of j, the so-called second-nearest neighborhood of i, denoted here by γ_i.

Since we are interested mainly in the analysis of the steady states of the dynamics and due to the fact that all nodes eventually get some opinion (at least in connected graphs), not becoming undecided again, it is straightforward that $\eta_0^* = 0$ for any state we must consider. Therefore, to compute a nontrivial steady state η_σ^*, it is enough to write down only the master equation corresponding to a given candidate σ and dismiss rule (II a).

In Eq. (1) we present the obtained master equation. It is easy to understand the meaning of term I in this equation as follows: Suppose that a node $j \in \Gamma_i$, chosen in a given Monte Carlo step with probability $1/N$, picks up at random one of its neighbors $k \in \Gamma_j$ with probability $1/q_j$ and, if $\sigma_i \neq \sigma$ and $\sigma_j = \sigma_k = \sigma$ [with probability $P(\sigma_i \neq \sigma, \sigma_j = \sigma_k = \sigma)$], the pair (j,k) may finally convince i, changing its opinion from $\sigma_i \neq \sigma$ to σ, with probability $1/q_j$. The plus token preceding it stresses the fact that this possibility increases the voting intentions for σ:

$$\Delta\eta_\sigma(i) = +\underbrace{\sum_{j\in\Gamma_i}\sum_{k\in\Gamma_j}\frac{1}{N}P(\sigma_i \neq \sigma, \sigma_j = \sigma_k = \sigma)\frac{1}{q_j}\frac{1}{q_j}}_{\mathrm{I}} + \underbrace{\sum_{k\in\gamma_i}\sum_{j\in\Gamma_i\cap\Gamma_k}\frac{1}{N}P(\sigma_i \neq \sigma, \sigma_j = \sigma_k = \sigma)\frac{1}{q_k}\frac{1}{q_j}}_{\mathrm{II}}$$

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$$
\underbrace{- \sum_{\sigma'=1(\sigma'\neq\sigma)}^{N_c} \sum_{j\in\Gamma_i} \sum_{k\in\Gamma_j} \frac{1}{N} P(\sigma_i{=}\sigma,\sigma_j{=}\sigma_k{=}\sigma') \frac{1}{q_j}\frac{1}{q_j}}_{\mathrm{III}}
\underbrace{- \sum_{\sigma'=1(\sigma'\neq\sigma)}^{N_c} \sum_{k\in\gamma_i} \sum_{j\in\Gamma_i\cap\Gamma_k} \frac{1}{N} P(\sigma_i{=}\sigma,\sigma_j{=}\sigma_k{=}\sigma') \frac{1}{q_k}\frac{1}{q_j}}_{\mathrm{IV}} . \qquad (1)
$$

To understand now term II in (1), suppose that a node $k \in \gamma_i$ is chosen in a given Monte Carlo step (probability $\frac{1}{N}$) and then picks up at random one of the mutual neighbors of k itself and i (i.e., a node $j \in \Gamma_i \cap \Gamma_k$) with probability $1/q_k$. Again, if $\sigma_i \neq \sigma$ and $\sigma_j = \sigma_k = \sigma$ [with probability $P(\sigma_i \neq \sigma, \sigma_j = \sigma_k = \sigma)$], the pair (j,k) may convince i with probability $1/q_j$ (see Fig. 1 for further clarification).

Terms III and IV may be understood in the same way. They stand for a decrease in the voting intentions for σ and must be preceded by a minus token. It is also necessary to sum over all possible candidates $\sigma' \neq \sigma$.

We highlight that the terms II and IV in (1) are the main difference between our approach and previous ones [8,9]. It is easy to realize that these terms are related to transition rates with a relatively low frequency and their absence does not affect the qualitative description of the model. Nevertheless, we show that the presence of the terms can greatly improve its quantitative description, allowing us to take into account the correlations present in the underlying network, which is crucial in the case of the Sznajd model due to its second-neighbor interactions.

III. THE MEAN-FIELD APPROACH

To perform the mean-field approximation we assume no correlation of opinion probabilities, i.e., $P(\sigma_i,\sigma_j,\sigma_k) = P(\sigma_i)P(\sigma_j)P(\sigma_k)$, and opinion homogeneity over the network, namely, $P(\sigma_i = \sigma) = \eta_\sigma$, where $\eta_\sigma(t)$ is the probability of any voter to have opinion σ at time t. In addition, we replace all the local variables in Eq. (1) by their mean value over all the network, namely, the mean degree of a node ($q_i \to \langle q\rangle$), the mean degree of a nearest neighbor of a node ($q_j \to \langle q_n\rangle$), the mean degree of a second-nearest neighbor of a node ($q_k \to \langle q_n'\rangle$), and the mean number of second-nearest neighbors of a node ($q_i' \to \langle q'\rangle$). We also introduce a geometric factor $\langle l\rangle$ to express the mean number of mutual neighbors between i and k as the product $\langle l\rangle\langle q_n'\rangle$ [7].

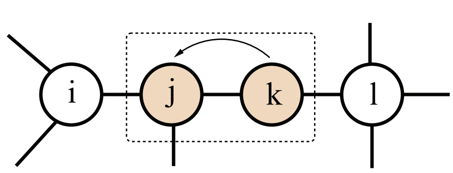

FIG. 1. (Color online) Node k may influence the opinion of i only together with j through the formation of the convincing couple (j,k). If k chooses the neighbor l, for instance, the convincing couple is unable to change the opinion of i, which happens to be out of range.

We may also consider the time increment corresponding to Eq. (1) as $\Delta t = 1/N$ so that, in the thermodynamic limit, the master equation can be reduced to

$$
\frac{\Delta\eta_\sigma}{1/N} \;\to\; \dot{\eta}_\sigma = \frac{1}{\alpha}\eta_\sigma\left((1-\eta_\sigma)\eta_\sigma - \sum_{\sigma'\neq\sigma}\eta_{\sigma'}^2\right), \qquad (2)
$$

where $\alpha \equiv \langle q_n/\rangle(\langle q\rangle + \langle l\rangle\langle q'\rangle)$ is a geometric-dependent parameter and the overdot stands for the time derivative, as usual. As a matter of fact, Eq. (2) constitutes a system of $N_c - 1$ coupled differential equations with fixed points satisfying both the condition $\dot{\eta}_\sigma = 0$ and the normalization constraint in such a way that

$$
\eta_\sigma^*\left(\eta_\sigma^* - \sum_{r=1}^{N_c}(\eta_r^*)^2\right) = 0, \qquad (3)
$$

leading all of its non-null components to share a common value since the quadratic term $Q^*(\{\eta_\sigma^*\}) \equiv \sum_{r=1}^{N_c}(\eta_r^*)^2$ is constant. In this case the fixed points are stable only when $\eta_\sigma^* = 1$ for a specific opinion σ and any other possible configuration is unstable.

The same qualitative result may be obtained if we consider only two possible different probabilities: η_σ, associated with opinion σ, and $\eta_{\sigma'}$, related to all other opinions. In other terms, we consider the approximation $\eta_{\sigma'} = (1-\eta_\sigma)/(N_c - 1)$ and the $(N_c - 1)$-dimensional system is now collapsed in a one-dimensional space, i.e., we choose to analyze the probability of one particular opinion σ and consider all the other opinions sharing the same probability. Within this simplified approach, the system of equations expressed in (2), now projected in a one-dimensional space, can be written as

$$
\dot{\eta}_\sigma = \frac{N_c}{\alpha(N_c-1)}\eta_\sigma(1-\eta_\sigma)\left(\eta_\sigma - \frac{1}{N_c}\right). \qquad (4)
$$

This procedure can be also interpreted as focusing only on two of the expected steady states of the model, namely, an absorbing ferromagneticlike state $\{\eta_\sigma^* = 1, \eta_{\sigma'}^* = 0\}$ and an equiprobable paramagneticlike state $\{\eta_\sigma^* = \eta_{\sigma'}^* = 1/N_c\}$.

The fixed points of (4) are $\eta_\sigma^* = 0,1,1/N_c$, where $\eta_\sigma^* = 0,1$ are stable fixed points and $\eta_\sigma^* = 1/N_c$ is unstable. In this case we may identify three different configurations as steady states of the original system: a stable configuration representing consensus with opinion σ, $\{\eta_\sigma^* = 1, \eta_{\sigma'}^* = 0\}$, as observed numerically, and two configurations representing polarization, $\{\eta_\sigma^* = \eta_{\sigma'}^* = 1/N_c\}$ and $\{\eta_\sigma^* = 0, \eta_{\sigma'}^* = 1/(N_c-1)\}$, not found in numerical simulations, which can be demonstrated as unstable configurations according to calculations by Timpanaro and Prado in [10]. Therefore, in a mean-field approximation, the CSM always evolves to a steady state of consensus as observed in the original Sznajd model [6]:

$$
\begin{aligned}
\dot{\eta}_\sigma = (1-\omega)&\left[\frac{1}{\alpha}\eta_\sigma\left(\eta_\sigma - \sum_{r=1}^{N_c}\eta_r^2\right)\right] \\
&+ \omega\left[(1-\eta_\sigma)\frac{1}{N_c} - \eta_\sigma\left(1 - \frac{1}{N_c}\right)\right]. \qquad (5)
\end{aligned}
$$

We call attention to the fact that, despite the huge contraction performed, the simplified version retains what we consider to be the core of the multidimensional version, the structure of the fixed points (the absence of opinions, consensus, and polarization), which happens also in the study of the system with spontaneous opinion change, as will be seen in the following. For the model with spontaneous opinion change, the master equation is completely analogous to Eq. (2), as we show in Eq. (5). In this case the terms related to the conventional dynamics are preceded by a factor $1-\omega$, whereas the other terms are preceded by a factor ω, which stands for the probability that the chosen voter changes its opinion spontaneously. In particular, $\omega(1-\eta_\sigma)(1/N_c)$ stands for the probability that a voter supporting a candidate other than σ changes its opinion, choosing it among N_c candidates, and $-\omega\eta_\sigma(1-1/N_c)$ stands for the probability that a voter supporting σ changes its mind, choosing another one.

Within the approximation $\eta^*_{\sigma'}=(1-\eta_\sigma)/(N_c-1)$, the fixed points for the model with spontaneous opinion change are

$$\eta^*_\sigma \equiv \overline{\eta} = \frac{1}{N_c},$$
$$\eta^*_\sigma \equiv \eta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha\frac{N_c-1}{N_c}\frac{\omega}{1-\omega}} \tag{6}$$

and their stability depends on ω as follows: η_+ is stable in $\omega \leqslant \omega_t$, whereas $\overline{\eta}$ is unstable in $\omega \leqslant \overline{\omega}$ and η_- is unstable in $\overline{\omega} \leqslant \omega \leqslant \omega_t$, where $\overline{\omega}=[1+\alpha N_c]^{-1}$ and $\omega_t=[1+4\alpha(N_c-1)/N_c]^{-1}$, as shown in Fig. 2. In this case the fixed point can also be interpreted as three different configurations of the system, namely, an ordered configuration representing consensus with opinion σ, $\{\eta^*_\sigma=\eta_+,\eta^*_{\sigma'}=(1-\eta_+)/(N_c-1)\}$, and the disordered configurations $\{\eta^*_\sigma=\eta_-,\eta^*_{\sigma'}=(1-\eta_-)/(N_c-1)\}$ and $\{\eta^*_\sigma=\eta^*_{\sigma'}=1/N_c\}$ representing polarization.

Although the fixed points shown above [Eq. (6)] were obtained through the $\eta_{\sigma'}=(1-\eta_\sigma)/(N_c-1)$ approximation, we highlight that they are also solutions of Eq. (5) (see Fig. 2). It can also be shown that the stability properties of the approximated fixed points $\{\eta^*_\sigma=\eta_+,\eta^*_{\sigma'}=(1-\eta_+)/N_c\}$ and $\{\eta^*_\sigma=\eta^*_{\sigma'}=1/N_c\}$ are compatible with a complete multidimensional mean-field analysis (see the Appendix). In addition, the predicted stability of these solutions for $\overline{\omega}<\omega<\omega_t$ defines an interval of bistability, actually observed in all simulations performed.

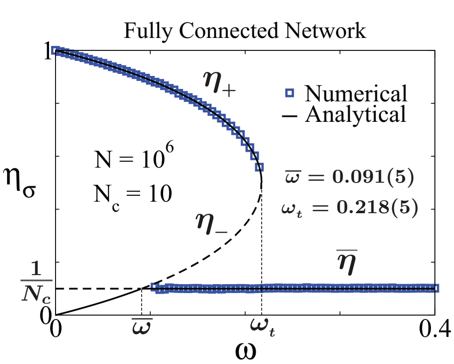

FIG. 2. (Color online) Bifurcation diagram for the CSM. Mean-field results are shown by solid (stable) or dashed (unstable) lines, depending upon their stability, whereas numerical results taken in a fully connected network are represented by symbols. The error bars in the numerical results are smaller than the symbols.

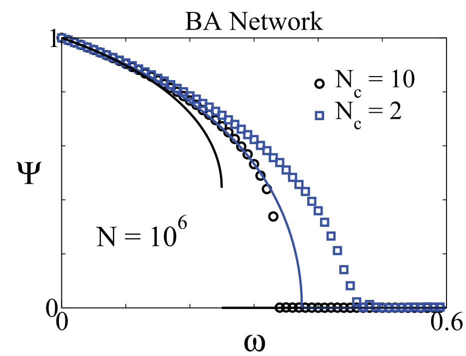

FIG. 3. (Color online) Order parameter for the CSM. Mean-field results are shown by solid lines whereas numerical results taken in a BA network are shown by symbols. The error bars in the numerical results are smaller than the symbols.

As would be expected, we can identify some limitations in the applicability of the assumption of equal sharing of remaining votes; for example, the predicted configuration of polarization $\{\eta^*_\sigma=\eta_-,\eta^*_{\sigma'}=(1-\eta_-)/N_c\}$ in $\omega<\overline{\omega}$ was never found numerically and a complete multidimensional analysis reveals its instability. Nevertheless, the approximated approach is fairly successful once all the others solutions provided by it are also exact solutions of the complete multidimensional system, as shown in the Appendix, with stability analysis failing only in that case.

IV. CRITICAL BEHAVIOR OF THE MODEL

It is also convenient to define an order parameter

$$\Psi = \frac{N_c}{N_c-1}\left(\max_\sigma\{\eta_\sigma\} - \frac{1}{N_c}\right), \tag{7}$$

where ω_t plays the role of a transition temperature between the phases corresponding to consensus and polarization. In other words, the system can undergo a *discontinuous* phase transition, depending on the rate of spontaneous opinion change, as shown in Figs. 3 and 4. In the particular case $N_c=2$ the order parameter is a continuous function on ω, meaning that $\omega_t(N_c=2)$ plays the role of a critical point according the phase diagram shown in Fig. 5.

Considering ensembles with 100 network samples of size $N=10^6$ voters, we performed Monte Carlo simulations in a wide variety of networks to confirm the analytical predictions. The existence of a discontinuous phase transition was verified in all the cases and the numerical transition point shows acceptable agreement with the one predicted analytically in most of them (see Table I).

In the case of Barabási-Albert networks the transition point computed numerically is $\omega_t=0.335(3)$, whereas the mean-field prediction is $\omega_t=0.255(3)$. The simple mean-field value found before by Vannucchi and Prado was $\omega_t=0.099$ [8], in stark contrast with our results, showing that the present terms considered in the master equation are indeed mainly responsible for the improvements in the results (see Table I

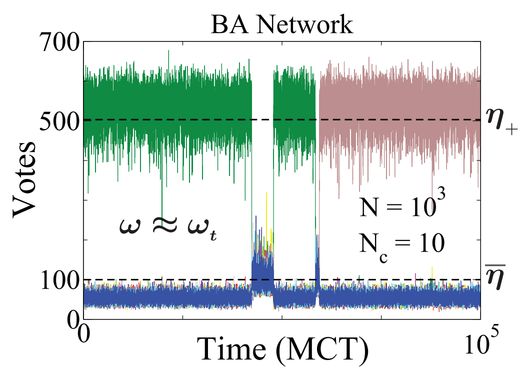

FIG. 4. (Color online) Typical time series for the model in a BA network near its transition point $\omega \approx \omega_t$. In this case the system can alternate between polarization and consensus, even interchanging the majority opinion.

for further comparisons). For $N_c = 2$ the mean-field approach still describes qualitatively well the behavior of the system exhibiting a continuous phase transition with numerical and analytical transition points respectively equal to $\omega_t = 0.485(3)$ and $\omega_t = 0.375(1)$.

It is important to stress that the parameter $\langle l \rangle$ is usually much smaller than unity, showing that the transition rate related to this factor is rare, as mentioned earlier (see Table II). Similar approaches, in previous works, usually omit this transition rate either because the Sznajd model is always analyzed in a fully connected network [9,11] or because of the presumed smallness of the term [8]. However, if we ignore those interactions completely, the transition point found in the mean-field approach does not show acceptable agreement with the one computed numerically.

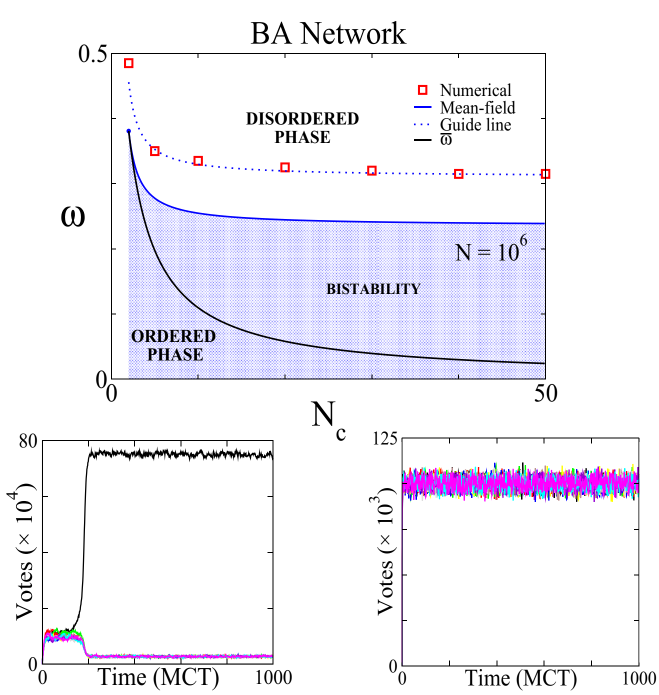

FIG. 5. (Color online) Phase diagram for the CSM in a BA network. Typical time series for each phase are shown in the bottom row of insets: the ordered phase on the left and the disordered phase on the right. The error bars in the numerical results are smaller than the symbols.

TABLE I. Comparison between the mean-field and numerical results of the transition point ω_t for the main networks studied. For complex networks, m is the minimum degree of a node in a Barabási-Albert (BA) network and s is the rewiring probability of an edge in a Watts-Strogatz (WS) network.

Network	Transition point		
($N_c = 10$)	Mean-field results		Numerical
($N = 10^6$)	Simple	Higher order	results
BA ($m = 5$)	0.070(5)	0.255(5)	0.335(5)
WS ($s = 0.01$)	0.22(1)	0.33(1)	0.215(5)
square lattice	0.217	0.328	0.215(5)
cubic lattice	0.217	0.337	0.285(5)
hypercubic lattice ($d = 4$)	0.217	0.342	0.355(5)
hypercubic lattice ($d = 6$)	0.217	0.347	0.355(5)
Bethe lattice ($z = 3$)	0.217	0.316	0.350(5)
fully connected	0.217	0.217	0.218(5)
hypercubic lattice ($d \to \infty$)	0.217	0.357	
Bethe lattice ($z \to \infty$)	0.217	0.357	

Moreover, heuristic arguments on hypercubic lattices allow us to claim other related results such as $\langle l \rangle = 0$ and $\langle l \rangle \langle q' \rangle \propto \langle q \rangle$ in the limit of high dimensions, leading to $\alpha = \frac{1}{2}$ and $\omega_t = [1 + 2(N_c - 1)/N_c]^{-1}$. It is easy to evaluate those parameters on a Bethe lattice and confirm our expectations as we highlight in Table II. We can also compare our expectations with simulations in a fully connected network, but the results are slightly different, mainly because of the complete absence of a second-nearest neighborhood in this case that does not allow any transitions related to terms II and IV in the master equation.

TABLE II. Summary of the geometric-dependent parameters of the model for the main networks studied. For complex networks the results are averages taken over ensembles with 100 network samples of size $N = 10^6$ nodes where m is the minimum degree of a node in BA networks and s is the rewiring probability of an edge in WS networks.

	Geometric parameter				
Network	$\langle q \rangle$	$\langle q_n \rangle$	$\langle q' \rangle$	$\langle l \rangle$	α
BA ($m = 5$)	9.99	37(1)	335(5)	0.1064(1)	0.813(8)
WS ($s = 0.01$)	4.00(1)	4.00(1)	8.00(1)	0.38(1)	0.57(1)
linear lattice	2	2	2	0.50	0.67
square lattice	4	4	8	0.38	0.57
cubic lattice	6	6	18	0.28	0.55
hypercubic lattice	$2d$	$2d$	$2d^2$	$\frac{2d-1}{2d^2}$	$\frac{2d}{4d-1}$
Bethe lattice	z	z	$z(z-1)$	$\frac{1}{z}$	$\frac{z}{2z-1}$
fully connected	$N-1$	$N-1$	0	0	1

TABLE III. Comparison between the transition point of the complex Sznajd model and the threshold percolation q_c of models such as isotropic percolation and the contact process in some of the networks studied. For the contact process we can define the threshold percolation as $q_c = 1/(1+\lambda_c)$, where λ_c is the usual transition point of the model related to its infection rate.

	Transition point			
Network	Site percolation	Bond percolation	Contact process	Complex Sznajd
square lattice	0.407	0.500	0.378	0.215(1)
cubic lattice	0.688	0.706	0.432	0.285(1)
hypercubic lattice ($d = 4$)	0.803	0.840	0.456	0.355(1)
Bethe lattice ($z = 3$)	0.500	0.500		0.350(5)

It is also worth making further comparisons with other opinion formation models well known in the statistical physics literature. For example, the majority-vote model exhibits a similar phase transition between a ferromagnetic phase (ordered) and a paramagnetic phase (disordered), even without any spontaneous opinion change, with the critical parameter given by $\omega_t = 0.135$ in a pair mean-field approximation on square lattices (denoted here by the same notation as that used before for the sake of clarity) [12]. Other models describing spreading diseases and prey-predator biological populations can also be put in the same context of comparisons and display a phase transition between an active state and an absorbing state on square lattices with the pair mean-field critical parameter respectively equal to $\omega_t = 0.379$ [13,14] and $\omega_t = 0.235$ [15], showing results closer to the Sznajd model than the majority-vote model. More comparisons with other important models are made in Table III [16].

We would like to stress that a modified version of the Sznajd model proposed by Timpanaro and Prado in [17] can also exhibit an active phase similar to the spreading disease models aforementioned. In this particular version a scheme of cyclic interacting opinions is imposed on the rules of the model and it is possible to observe the coexistence of different opinions in a stationary state even when $\omega = 0$ (a topological metastable configuration resembling a limit cycle) [17]. In our simulations we observed such an active phase only for a very specific set of networks, even considering the usual rules of the model and in the case of a Bethe lattice with coordination number $z = 3$ a continuous transition from it to polarization takes place at $\omega_t = 0.350(5)$ (see Table I).

V. CONCLUSIONS

In summary, we studied a variation of the Sznajd model on a general network. We proposed a master equation to describe the evolution of opinions in the model and studied its steady states in a mean-field approximation. We also studied the role of a spontaneous opinion change, which changes the opinion of a voter at random in the dynamics. In this case, we found the possibility of a discontinuous phase transition between a state in which a single candidate has the majority of votes (consensus) and another state in which the votes are well distributed among all the candidates (polarization) as the rate of spontaneous opinion change increases. In addition, Monte Carlo simulations in a wide variety of networks were performed and confirm the existence of a discontinuous phase transition showing acceptable agreement with the mean-field results.

Our approach also allows us to take into account the influence of correlations present in the underlying network and to estimate the transition point of the phase transition with much more accuracy than analytical calculations performed by Vannucchi and Prado in previous works [8]. We believe the particular way we introduced the correlations in the mean-field approximation, by numerical evaluation of the parameters network, may be extended to other systems where second-order interactions are important, without all the extra work of a complete pair approximation.

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APPENDIX: DETAILED DISCUSSION OF THE EXACT FIXED POINTS OF THE MASTER EQUATION

After the mean-field approximation, the master equation associated with the CSM can be written as a system of $N_c - 1$ coupled differential equations according to

$$\dot{\vec{\eta}} = \vec{f}(\vec{\eta},\alpha,t), \tag{A1}$$

$$\sum_{r=1}^{N_c} \eta_r = 1, \tag{A2}$$

where α is the parameter related to the underlying network considered and $\vec{f}(\vec{\eta},\alpha,t)$ is a vector function of the opinion probability vector $\vec{\eta}(t)$ with components given by

$$f_\sigma = \frac{1-\omega}{\alpha}\eta_\sigma\left(\eta_\sigma - \sum_{r=1}^{N_c}\eta_r^2\right) + \omega\left[(1-\eta_\sigma)\frac{1}{N_c} - \eta_\sigma\left(1-\frac{1}{N_c}\right)\right], \tag{A3}$$

constrained to the normalization condition (A2). The fixed points are the solutions of $\dot{\vec{\eta}} = \vec{f}(\vec{\eta},\alpha,t) = 0$, which leads to a system of coupled quadratic algebraic equations for the opinion probabilities η_σ^* described by

$$(\eta_\sigma^*)^2 - \left(\frac{\alpha\omega}{1-\omega} + \sum_{r=1}^{N_c}(\eta_r^*)^2\right)\eta_\sigma^* + \frac{\alpha\omega}{1-\omega}\frac{1}{N_c} = 0. \tag{A4}$$

Due to the fact that $Q^*(\vec{\eta}^*) = \sum_{r=1}^{N_c}(\eta_r^*)^2$ is a constant for any configuration of fixed points, we have only two possible values for the components of them, namely,

$$\eta_\sigma^* = \frac{1}{2}\left(\frac{\alpha\omega}{1-\omega} + Q^*\right) \pm \frac{1}{2}\sqrt{\left(\frac{\alpha\omega}{1-\omega} + Q^*\right)^2 - \frac{4\alpha}{N_c}\frac{\omega}{1-\omega}}, \tag{A5}$$

limited by the normalization constraint $\sum_{r=1}^{N_c}\eta_r^*=1$. Interestingly, we stress that the solutions obtained through the assumption of equal sharing of remaining votes, i.e., $\eta_{\sigma'}=(1-\eta_\sigma)/(N_c-1)$, are also fixed points in the multidimensional system. The stability is obtained through the analysis of eigenvalues λ related to

$$\left|\frac{\partial\vec{f}}{\partial\vec{\eta}}-\lambda I\right|=0 \tag{A6}$$

or explicitly

$$\begin{vmatrix} D_1-\lambda & A_{12} & A_{13} & \cdots & A_{1N_c}\\ A_{21} & D_2-\lambda & A_{23} & \cdots & A_{2N_c}\\ A_{31} & A_{32} & D_3-\lambda & \cdots & A_{3N_c}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A_{N_c1} & A_{N_c2} & A_{N_c3} & \cdots & D_{N_c}-\lambda\end{vmatrix}=0, \tag{A7}$$

with

$$D_i=\frac{1-\omega}{\alpha}\left(2\eta_i(1-\eta_i)-\sum_{r=1}^{N_c}\eta_r^2\right)-\omega \tag{A8}$$

and

$$A_{ij}=-2\left(\frac{1-\omega}{\alpha}\right)\eta_i\eta_j. \tag{A9}$$

We may now consider all the particular cases of our interest in the following.

Fixed point $\eta_\sigma^*=\eta_{\sigma'}^*=\bar{\eta}=1/N_c$

Given

$$D=\left(\frac{1-\omega}{\alpha}\right)\frac{N_c-2}{N_c^2}-\omega \tag{A10}$$

and

$$A=-\left(\frac{1-\omega}{\alpha}\right)\frac{2}{N_c^2}, \tag{A11}$$

we have the stability ruled by

$$[D-\lambda+(N_c-1)A](D-\lambda-A)^{N_c-1}=0. \tag{A12}$$

The first determinant factor leads to a negative eigenvalue

$$\lambda=-\left(\frac{1-\omega}{\alpha}\right)\frac{1}{N_c}-\omega<0, \tag{A13}$$

while the second gives us

$$\lambda=\frac{1-(1+\alpha N_c)\omega}{\alpha N_c} \tag{A14}$$

and thus the system is stable if $\omega>\bar{\omega}=(1+\alpha N_c)^{-1}$.

Fixed points $\eta_\sigma^*=\eta_\pm$ and $\eta_{\sigma'}^*=(1-\eta_\pm)/(N_c-1)$

Now we analyze the stability of the other inhomogeneous fixed point, in which one opinion has probability

$$\eta_\sigma^*=\eta_\pm=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{\alpha\omega}{1-\omega}\frac{N_c-1}{N_c}}$$

and the others have $\eta_{\sigma'}^*=(1-\eta_\pm)/(N_c-1)$. We may define

$$D_\sigma=\frac{2(N_c-1)}{N_c}\omega-\frac{1-(N_c-2)\eta_\pm}{\alpha(N_c-1)}(1-\omega), \tag{A15}$$

$$D_{\sigma'}=\left(\frac{2}{N_c(N_c-1)}\right)\omega+\left(\frac{(N_c-3)-(N_c-2)(N_c+1)\eta_\pm}{\alpha(N_c-1)^2}\right)(1-\omega), \tag{A16}$$

$$A_{\sigma\sigma'}=-\left(\frac{2\eta_\pm}{\alpha}\right)\left(\frac{1-\eta_\pm}{N_c-1}\right)(1-\omega)=-\frac{2\omega}{N_c}, \tag{A17}$$

and

$$A_{\sigma'\sigma'}=\frac{2}{N_c(N_c-1)}\omega+\frac{2(\eta_\pm-1)}{\alpha(N_c-1)^2}(1-\omega). \tag{A18}$$

Therefore, the stability is now obtained by solving

$$[(D_\sigma-\lambda)(D_{\sigma'}-\lambda)-(N_c-1)A_{\sigma\sigma'}^2+(N_c-2)(D_\sigma-\lambda)A_{\sigma'\sigma'}](D_{\sigma'}-\lambda-A_{\sigma'\sigma'})^{N_c-2}=0, \tag{A19}$$

resulting in a quadratic equation for λ with solutions

$$\lambda_\pm=\omega-\left(\frac{1-\omega}{\alpha}\right)\left(\frac{1}{N_c-1}-\frac{N_c-2}{N_c-1}\eta_\pm\right)\pm\omega. \tag{A20}$$

It can be seen that $\lambda_-<0$ always and $\lambda_+<0$ if

$$\frac{4\alpha\omega}{1-\omega}(N_c-1)-N_c<\pm(N_c-2)\sqrt{1-\frac{4\alpha\omega}{1-\omega}\frac{N_c-1}{N_c}}. \tag{A21}$$

Of course, since $\eta_\pm$ are real, $[4\alpha\omega/(1-\omega)](N_c-1)\leqslant N_c$ and the above inequality is satisfied for η_+. In the case of η_-, $\lambda_+<0$ implies

$$N_c\left(1-\frac{4\alpha\omega}{1-\omega}\frac{N_c-1}{N_c}\right)>(N_c-2)\sqrt{1-\frac{4\alpha\omega}{1-\omega}\frac{N_c-1}{N_c}}, \tag{A22}$$

now with both sides being positive. We then obtain

$$\frac{4(N_c-1)}{N_c^2}[1-(1+\alpha N_c)\omega]>0 \tag{A23}$$

and $\lambda_+<0$ for η_- if $\omega<\bar{\omega}=(1+\alpha N_c)^{-1}$. Related to the second factor of Eq. (A19), we still have

$$\lambda=\left(\frac{1-\omega}{\alpha}\right)\left(\frac{1-N_c\eta_\pm}{N_c-1}\right). \tag{A24}$$

This last result, earlier hidden by the $\eta_{\sigma'}=(1-\eta_\sigma)/(N_c-1)$ approximation, further implies that $\eta_\pm$ are stable only if they are greater than $1/N_c$, a condition not fulfilled by η_- when $\omega<\bar{\omega}=(1+\alpha N_c)^{-1}$, explaining therefore its instability.

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