Friction and noise for a probe in a nonequilibrium fluid

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We investigate the fluctuation dynamics of a probe around a deterministic motion induced by interactions with driven particles. The latter constitute the nonequilibrium medium in which the probe is immersed and is modeled as overdamped Langevin particle dynamics driven by nonconservative forces. The expansion that yields the friction and noise expressions for the reduced probe dynamics is based on linear response around a time-dependent nonequilibrium condition of the medium. The result contains an extension of the second fluctuation-dissipation relation between friction and noise for probe motion in a nonequilibrium fluid.

I. THE PROBLEM

Whether the environment of a system is in equilibrium or is driven into a nonequilibrium condition is important for characterizing internal motions and reactions. In other words, the reduced system dynamics will reflect whether the environment is driven or not. For example, the relation between noise and friction on the system need no longer be described via the standard second fluctuation-dissipation (or Einstein) relation, we cannot unambiguously speak about entropy fluxes into the environment in terms of heat as there would be no Clausius relation, and system fluctuations or noise level in general are not simply quantified by a reservoir temperature. The present paper takes up the challenge of characterizing such an effective dynamics for a probe in contact with a nonequilibrium medium. One should have in mind that the medium consists of active elements or driven particles themselves in contact with a big thermal equilibrium reservoir (such as surrounding air or water). Combined, the medium and the heat bath make up the nonequilibrium fluid. The system is called a probe here, but it can also refer in general to a collective or more macroscopic coordinate of the nonequilibrium fluid. The medium (both in contact with the probe and with the heat bath) consists of many (N) particles, which we model via a Langevin dynamics of the overdamped form for positions x_t^j,

$$\dot{x}_t^j = F(x_t^j) - \lambda \partial_j U(x_t^j, X_t) - \sum_{i<j} \partial_j \Phi(x_t^j - x_t^i) + \sqrt{\frac{2}{\beta}} \xi_t^j$$
$$(1.1)$$

with $j = 1, \ldots, N$ labeling the particles and subject to independent standard white noises ξ_t^j. The β denotes the inverse temperature of the background equilibrium reservoir, but the particles are effectively driven by the nonconservative force F. For simplicity, we have not introduced explicitly

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friction and mass parameters, keeping only λ (coupling) and β as parameters. The particle interaction is given through the potential Φ while the potential U depends on the position X_t of the probe, thus representing the backreaction of the probe on each particle. Other and different types of interaction between probe and medium are possible, e.g., in terms of their center-of-mass coordinate with a mean-field-type potential $U(\sum_j x_t^j - X_t)$, not discussed here and inessential for the present level of discussion. Keeping to (1.1), the probe dynamics itself is

$$M\ddot{X}_t + K(X_t, \dot{X}_t) = -\lambda \sum_j \partial_X U(x_t^j, X_t), \qquad (1.2)$$

where we indicate by $K(X_t, \dot{X}_t)$ other aspects of the probe motion in the fluid. Its mass M is obviously also an important parameter for separation of time scales between probe and fluid particle motion.

Equations (1.1) and (1.2) starting at $t = 0$ are the basic evolution equations representing the coupled dynamics of probe and fluid particles. Nonequilibrium works directly on the medium, which is both open to a thermal reservoir and is itself the environment of the probe. The problem of the present paper is to characterize the effective or reduced probe dynamics obtained from "integrating out" the fluid degrees of freedom under the usual assumptions of weak coupling (λ small) for a large environment (N big), which is rapidly relaxing (M large) to every new nonequilibrium steady condition as determined by the instantaneous probe position.

In what follows, we assume that the probe trajectory $[X] = (X_s, 0 \leqslant s \leqslant t)$ deviates only a little, at least for some time span from a deterministic reference trajectory $[Y] = (Y_s, 0 \leqslant s \leqslant t)$ with $Y_0 = X_0$, which is a solution of an averaged out (1.2),

$$M\ddot{Y}_t + K(Y_t, \dot{Y}_t) = -\lambda \sum_j \langle \partial_X U(x^j, Y_t) \rangle^{Y_t}, \qquad (1.3)$$

where on the right-hand side we integrate out the positions x^j of the medium particles using their stationary distribution $\langle \cdot \rangle^{Y_t}$ for the dynamics (1.1) at fixed probe position, i.e.,

evolving with

$$\dot{x}_s^j = F\left(x_s^j\right) - \lambda \partial_j U\left(x_s^j, Y_t\right) - \sum_{i<j} \partial_j \Phi\left(x_s^j - x_s^i\right)$$
$$+ \sqrt{\frac{2}{\beta}} \xi_s^j, \quad s > 0. \tag{1.4}$$

In other words, to find $[Y]$ we apply an infinite time-scale separation between medium and probe motion so that the medium relaxes to its nonequilibrium steady condition at each fixed probe position [1,2]. The trajectory $[Y]$ is then the supposedly unique solution to the probe equation (1.3) for given initial conditions. We use the example of Sec. IV to make that more explicit.

One should think of $[Y]$ as the typical probe motion. The nonequilibrium aspect of the medium is picked up in the effective force on the right-hand side of (1.3), and $[Y]$ will most likely not be constant in time; see also [3–7] for statistical forces from nonequilibrium. That can be imagined as caused by some rotational force F that acts on the medium particles (as illustrated in Sec. IV), or it can be the combined result of strong nonlinearities in the interaction Φ.

When immersing a probe in the fluid, we expect to see also friction and noise on top of the behavior summarized in Eq. (1.3). To understand their origin, we must study the response of the medium particles to the stimulus of X_s fluctuating around trajectory $[Y]$. The reference dynamics for the medium is

$$\dot{x}_s^j = F\left(x_s^j\right) - \lambda \partial_j U\left(x_s^j, Y_s\right) - \sum_{i<j} \partial_j \Phi\left(x_s^j - x_s^i\right)$$
$$+ \sqrt{\frac{2}{\beta}} \xi_s^j, \quad s > 0, \tag{1.5}$$

where we imagine Y_s to be slowly changing. The $[Y]$ here is therefore a quasistatic protocol for the medium particles, which at each moment s are distributed using the stationary expectation $\langle \cdot \rangle^{Y_s}$. Deviations from the real probe trajectory $h_s := X_s - Y_s$ are supposed small for some good amount of time, and hence the real dynamics (1.1) is a perturbation of (1.5). We can, however, find the new distribution of the medium particles by applying linear response theory. It is there that friction appears as the averaged deviation giving the reaction (or indeed, response) of the medium particles to small h_s. In other words, the nonequilibrium fluid back-reacts to the probe motion and makes the friction. Noise is created due to the effectively random effect of the medium particles on the probe. The relation between the noise covariance and the friction kernel is no longer that of the standard Einstein or second fluctuation-dissipation relation. That was already shown in Ref. [8] in the same context as the present paper but for $Y_t \equiv Y_0$ fixed in time. The present paper is thus an extension of [8] for probe motion around (time-dependent) behavior. That requires also an extension of the presently existing results for linear response around nonequilibria. Here we need response theory against a time-dependent background, and the next section will start it, with more general background collected in Appendix. In Sec. IV, we describe the coupled dynamics between fluid

and probe, and we specify friction and noise. Section IV makes things more explicit for driven diffusive particles interacting with a probe in a toroidal trap.

II. LINEAR RESPONSE AROUND TIME-DEPENDENT NONEQUILIBRIA

It suffices here to consider a single overdamped Langevin dynamics $x_s \in \mathbb{R}^d$ (representing a single medium particle) in the presence of rotational forces F, confining potential V, and with a potential U that depends on a time-dependent and deterministic protocol $Y_s \in \mathbb{R}^d$,

$$\dot{\mathbf{x}}_s = \mathbf{F}(\mathbf{x}_s) - \nabla_x V(\mathbf{x}_s) - \nabla_x U(\mathbf{x}_s, \mathbf{Y}_s) + \sqrt{\frac{2}{\beta}} \boldsymbol{\xi}_s. \tag{2.1}$$

Note that for simplicity, we have not considered adding mobilities or position-dependent diffusion constants [and we also set $\lambda = 1$ when compared with (1.1) or with (1.5)]; we just have standard d-dimensional white noise ξ_s with a prefactor including the temperature β^{-1} of a surrounding reservoir in thermal equilibrium. We assume that the system for time-independent or fixed $Y_s \equiv a$ reaches a steady condition that can be described by a smooth density $\rho^a(x)$ on $\mathbb{R}^d$. It obviously requires that the potential V is sufficiently confining, such as by the presence of a harmonic trap or a box with periodic boundary conditions. If, moreover, the protocol is quasistatic with respect to the sufficiently short relaxation time of the medium particle, its position is distributed at each time s by ρ^{Y_s}.

We now perturb for $s > 0$ as

$$\dot{\mathbf{x}}_s = \mathbf{F}(\mathbf{x}_s) - \nabla_x U(\mathbf{x}_s, \mathbf{Y}_s) + \nabla_x h_s \cdot g(\mathbf{x}_s, \mathbf{Y}_s) + \sqrt{\frac{2}{\beta}} \boldsymbol{\xi}_s \tag{2.2}$$

with small but arbitrary time-dependent amplitude h_s and perturbing potential $g(x, Y_s)$. Note that besides the h_s, the time dependence of the perturbation is again through the same protocol Y_s. We could of course imagine other perturbation schemes, but for the present application it suffices to consider the perturbation $Y_s \to Y_s + h_s$, which makes $g(x, Y_s) = -\nabla_Y U(x, Y_s) \in \mathbb{R}^d$.

We assume that we start at time zero from ρ^{Y_0}. The linear response we need here is the difference in expectation, $\delta_g A := \langle A(x_t) \rangle^{[X]} - \langle A(x) \rangle^{Y_t}$, for observable A to first order in h_s, $s \in [0,t]$, where the first expectation is for the perturbed dynamics (2.2) corresponding to protocol $X_t := Y_t + h_t$ and the second expectation $\langle A(x) \rangle^{Y_t} = \langle A(x_t) \rangle^{[Y]}$ is in the quasisteady condition following (2.1). We give the derivation in Appendix, with the following result:

$$\delta_g A = \int_0^t ds \, h_s \cdot R_{gA}(s,t) + O(h^2),$$

with the response coefficient given in terms of connected correlation functions, $\langle w; v \rangle = \langle w v \rangle - \langle w \rangle \langle v \rangle$, in the reference

quasisteady condition of the medium particle,

$$
\begin{aligned}
R_{gA}(s,t) = & \frac{\beta}{2} \frac{d}{ds} \langle g(\mathbf{x}_s, \mathbf{Y}_s) ; A(\mathbf{x}_t) \rangle^{[Y]} \\
& - \frac{\beta}{2} \dot{\mathbf{Y}}_s \cdot \langle \nabla_Y g(\mathbf{x}_s, \mathbf{Y}_s) ; A(\mathbf{x}_t) \rangle^{[Y]} \\
& - \frac{\beta}{2} \langle L_s g(\mathbf{x}_s, \mathbf{Y}_s) ; A(\mathbf{x}_t) \rangle^{[Y]}.
\end{aligned}
\tag{2.3}
$$

Here L_s is the instantaneous backward generator working on a function g of x as

$$
L_s g(\mathbf{x}) := [F(\mathbf{x}) - \nabla_x U(\mathbf{x}, \mathbf{Y}_s)] \nabla_x g(\mathbf{x}) + \frac{1}{\beta} \Delta_x g(\mathbf{x}).
\tag{2.4}
$$

Note that time zero has a special meaning in Eq. (2.3) as marking the beginning of the perturbed evolution. There is another response formula that we need, expressing the change in steady condition when we move $Y_s \to Y_s + a$ to a new protocol that deviates slightly, $||a|| \ll 1$. That corresponds to taking $h_s \equiv a$ in Eq. (2.2) but also allowing sufficient time to reach a new steady condition. As we can only apply (2.3) when the perturbed and the original dynamics have the same initial condition, we have to move the latter to very early times to allow also the dynamics with protocol $Z_s := Y_s + a$ to reach its steady condition. In other words, we obtain from (2.3) the difference between two steady expectations,

$$
\begin{aligned}
& \langle A(x_t) \rangle^{[Y+a]} - \langle A(x_t) \rangle^{[Y]} \\
& = \frac{\beta a}{2} \langle g(x_t, Y_t) ; A(x_t) \rangle^{[Y]} \\
& \quad - \frac{\beta a}{2} \int_{-\infty}^{t} ds\, \dot{Y}_s \cdot \langle \nabla_Y g(x_s, Y_s) ; A(x_t) \rangle^{[Y]} \\
& \quad - \frac{\beta a}{2} \int_{-\infty}^{t} ds\, \langle L_s g(x_s, Y_s) ; A(x_t) \rangle^{[Y]} + O(a^2)
\end{aligned}
\tag{2.5}
$$

where we have used the fact that covariances $\langle g(x_s, Y_s); A(x_t) \rangle^{[Y]} \to 0$ tend to vanish sufficiently fast as $s \downarrow -\infty$.

Formulas (2.3)–(2.5) provide extensions of linear-response theory around time-dependent nonequilibria. In previous work, e.g. [9–11] about nonequilibrium linear response, the nonequilibrium dynamics was not explicitly time-dependent. We need these new formulas (derived in the Appendix), however, for the application described in this paper. Note also that we could have taken observable $A = A_t$ as explicitly time-dependent and not only a function of the state at time t; we will need to apply the formulas to that case.

III. REDUCED DYNAMICS

The purpose of this section is to obtain the reduced dynamics for the probe. The result will be presented in terms of a Langevin equation where noise and friction originate from the interaction of the probe with the nonequilibrium fluid

environment. We must therefore integrate out the particles x_t^j from (1.2).

Denote by

$$
A(x,X) := -\lambda \sum_j \partial_X U(x^j, X)
$$

the force of the medium on the probe. (We continue to use one-dimensional notation for simplicity.) The probe evolution equation (1.2) can be written as the sum of a deterministic and a random contribution,

$$
M\ddot{X}_t + K(X_t, \dot{X}_t) = \langle A(x_t, X_t) \rangle^{[X]} + \eta_t,
\tag{3.1}
$$

defining the noise

$$
\eta_t := A(x_t, X_t) - \langle A(x_t, X_t) \rangle^{[X]}.
\tag{3.2}
$$

We come back later to the nature of that noise. The first term on the right-hand side of (3.1) is an average with respect to the fluid dynamics (1.1) at $t > 0$. To first order in $X_t - Y_t$, that can be considered as the perturbed dynamics,

$$
\begin{aligned}
\dot{x}_t^j = & F(x_t^j) - \sum_{i<j} \partial_j \Phi(x_t^j - x_t^i) - \lambda \partial_j U(x_t^j, Y_t) \\
& + h_t \partial_j A(x_t, Y_t) + \sqrt{\frac{2}{\beta}} \xi_t^j.
\end{aligned}
\tag{3.3}
$$

Remember here that $[Y] = (Y_s, s > 0)$ is the reference probe trajectory, and that the coupling between probe and medium starts at time zero where $Y_0 = X_0$. As (3.3) is of the form (2.2), we rewrite (3.1) as

$$
\begin{aligned}
M\ddot{X}_t + K(X_t, \dot{X}_t) = & \langle A(x_t, X_t) \rangle^{[X]} - \langle A(x_t, X_t) \rangle^{[Y]} \\
& + \langle A(x_t, X_t) \rangle^{[Y]} + \eta_t
\end{aligned}
\tag{3.4}
$$

and we apply the linear-response formula (2.3) to the first line of (3.4). There is a slight catch with respect to Sec. II in that the observable $A = A_t$ is now also explicitly time-dependent, $A_t(x_t) = A(x_t, X_t)$. Nevertheless, the result (2.3) remains valid and translates here into

$$
\langle A_t(x_t) \rangle^{[X]} - \langle A_t(x_t) \rangle^{[Y]} = \int_0^t ds\, (X_s - Y_s) \frac{d}{ds} \mathcal{V}_t(s),
\tag{3.5}
$$

where

$$
\begin{aligned}
\mathcal{V}_t(s) := & \frac{\beta}{2} \langle A(x_s, Y_s) ; A_t(x_t) \rangle^{[Y]} \\
& - \frac{\beta}{2} \int_{-\infty}^{s} du \left[\left\langle \dot{Y}_u \frac{\partial A(x_u, Y_u)}{\partial Y} ; A_t(x_t) \right\rangle^{[Y]} \right] \\
& - \frac{\beta}{2} \int_{-\infty}^{s} du [\langle L_u A(x_u, Y_u) ; A_t(x_t) \rangle^{[Y]}].
\end{aligned}
\tag{3.6}
$$

As in Eq. (2.4), L_s is the backward generator for the unperturbed dynamics,

$$
\begin{aligned}
L_s A\left(x_s, Y_s\right) = & \sum_j F\left(x_s^j\right) \partial_j A\left(x_s, Y_s\right) \\
& - \sum_j \sum_{i<j} \partial_j \Phi\left(x_s^j - x_s^i\right) \partial_j A\left(x_s, Y_s\right) \\
& - \sum_j \lambda \partial_j U\left(x_s^j, Y_s\right) \partial_j A\left(x_s, Y_s\right) \\
& + \beta^{-1} \sum_j \partial_{jj}^2 A\left(x_s, Y_s\right).
\end{aligned}
\tag{3.7}
$$

We integrate (3.5) by parts using $h_0 = 0$,

$$
\begin{aligned}
& \langle A_t(x_t)\rangle^{[X]} - \langle A_t(x_t)\rangle^{[Y]} \\
& = (X_t - Y_t)\mathcal{V}_t(t) - \int_0^t ds\, (\dot{X}_s - \dot{Y}_s)\mathcal{V}_t(s).
\end{aligned}
\tag{3.8}
$$

A. Statistical force

To rewrite the first term on the right-hand side of (3.8), we consider a static perturbation as in Eq. (2.5) with $h_t \equiv h$. Since h is now constant, the perturbed protocol $[Y+h] = (Y_s + h, s > 0)$ is still quasistatic and the difference in steady components is described in Eq. (2.5):

$$
\begin{aligned}
\langle A_t(x_t)\rangle^{[Y+h]} = & \langle A_t(x_t)\rangle^{[Y]} + \frac{\beta h}{2}\langle A(x_t, Y_t); A_t(x_t)\rangle^{[Y]} \\
& - \frac{\beta h}{2}\int_{-\infty}^t du\, \langle \mathcal{K}_u(x_u, Y_u); A_t(x_t)\rangle^{[Y]},
\end{aligned}
\tag{3.9}
$$

where

$$
\mathcal{K}_u\left(x_u, Y_u\right) := \dot{Y}_u \frac{\partial A\left(x_u, Y_u\right)}{\partial Y} + L_u A\left(x_u, Y_u\right).
\tag{3.10}
$$

Comparing with (3.6), we can thus write

$$
\langle A_t\left(x_t\right)\rangle^{[X]} = \langle A_t(x_t)\rangle^{[Y+h_t]} - \int_0^t ds\, (\dot{X}_s - \dot{Y}_s)\mathcal{V}_t(s),
\tag{3.11}
$$

and the temporal boundary term of (3.8) has yielded the zero-order effective force on the probe:

$$
\begin{aligned}
(3.1) &= G(X_t) - \int_0^t ds\, (\dot{X}_s - \dot{Y}_s)\mathcal{V}_t(s), \\
G(X_t) &:= \langle A(x_t, X_t)\rangle^{[Y+h_t]} = \langle A(x, X_t)\rangle^{X_t},
\end{aligned}
\tag{3.12}
$$

where the statistical average corresponds to taking the average over the medium on the right-hand side of (3.8) for infinite

time separation between probe and fluid. That is exactly the reference dynamics (1.3). Note that it could very well be that G is a conservative force, and still the resulting probe dynamics will show fluctuations that betray the nonequilibrium nature of the medium. That is encoded in the relation between friction and noise, as shown in the following subsection.

B. Friction term

The friction appears from the integral term in Eqs. (3.8) and (3.11), with friction kernel

$$
\begin{aligned}
\gamma(t, s) := & \frac{\beta}{2}\langle A(x_s, Y_s); A(x_t, Y_t)\rangle^{[Y]} \\
& - \frac{\beta}{2}\int_{-\infty}^s du\, \left\langle \dot{Y}_u \frac{\partial A(x_u, Y_u)}{\partial Y}; A(x_t, Y_t)\right\rangle^{[Y]} \\
& - \frac{\beta}{2}\int_{-\infty}^s du\, \langle L_u A(x_u, Y_u); A(x_t, Y_t)\rangle^{[Y]},
\end{aligned}
\tag{3.13}
$$

where we have worked to order λ^2 (replacing there X_t with Y_t). Starting from (3.4) and using (3.11), we arrive at the probe effective evolution equation,

$$
M\ddot{X}_t + K(X_t, \dot{X}_t) = G(X_t) - \int_0^t ds\, (\dot{X}_s - \dot{Y}_s)\gamma(t, s) + \eta_t,
\tag{3.14}
$$

where we still need to discuss the noise η_t.

C. Noise

The noise is introduced in Eq. (3.2). Its average is $\langle \eta_t\rangle^{[X]} = 0$, and the two-time correlations are

$$
\langle \eta_s \eta_t\rangle^{[X]} = \langle A(x_s, X_s); A(x_t, X_t)\rangle^{[X_t]}.
$$

Considering the same perturbative regime as above, using the weak coupling (λ small), we relate these average values to those made over the quasisteady condition for protocol $[Y]$:

$$
\langle \eta_s \eta_t\rangle = \langle A(x_s, Y_s); A(x_t, Y_t)\rangle^{[Y]}
\tag{3.15}
$$

to significant order. Note that the noise perceived by the probe need not be Gaussian (we have not required a linear coupling between probe and medium, nor did we specify the role of the interactions in the medium), and we have not insisted on obtaining a Markov (memoryless) limit. That would require a more detailed study of time scales.

D. Second fluctuation-dissipation relation

If we insert (3.15) in Eq. (3.13), we obtain

$$
\gamma(t, s) = \frac{\beta}{2}\langle \eta_s \eta_t\rangle - \frac{\beta}{2}\int_{-\infty}^s du\, \left\langle \dot{Y}_u \frac{\partial A(x_u, Y_u)}{\partial Y}; A(x_t, Y_t)\right\rangle^{[Y]} - \frac{\beta}{2}\int_{-\infty}^s du\, \langle L_u A(x_u, Y_u); A(x_t, Y_t)\rangle^{[Y]},
\tag{3.16}
$$

which is different from the second fluctuation-dissipation relation valid for systems in contact with an equilibrium environment [8,12–15]. The modification to the standard fluctuation-dissipation relation (cf. [9,16]) can be given in terms of

$$
\mathcal{X}_{\mathrm{st}} := \frac{1}{2}\left[1 + \frac{\int_{-\infty}^s du\, \left\langle \dot{Y}_u \frac{\partial A(x_u, Y_u)}{\partial Y} + L_u A(x_u, Y_u); A(x_t, Y_t)\right\rangle^{[Y]}}{\langle A(x_s, Y_s); A(x_t, Y_t)\rangle^{[Y]}}\right].
$$

Both friction kernel and noise are just functions of t-s, and assuming sufficient decay in memory so that $\mathcal{X}_{\text{st}} = \mathcal{X} \exp[-\kappa(t - s)]$ for some large $\kappa > 0$, we could call $\beta_{\text{eff}}^{-1} := [(1 - \mathcal{X})\beta]^{-1}$ an *effective* temperature, as it would restore the Einstein relation

$$\gamma(t,s) = \beta_{\text{eff}} \langle \eta_s \eta_t \rangle,$$

but that is not quite sufficient for a thermodynamic meaning. Moreover, it makes sense to emphasize instead the *difference* between the terms in Eq. (3.16) that make the noise. Following previous work on nonequilibrium linear response [17–19], we speak about an entropic and a frenetic contribution; see also Appendix. The entropic part is purely dissipative and is proportional to the noise correlation; the frenetic component takes into account the changes in dynamical activity due to the perturbation. The latter refers here to the time-symmetric activity of the medium particles and how that changes by a change in the probe position.

IV. EXAMPLE

To understand what we have in mind for the reference protocol Y_t or for the X_t in Eq. (1.1), we consider here a driven diffusion first for one medium particle $x_t \in S^1$ on the circle S^1 of unit length,

$$\dot{x}_t = E - V'(x_t) - \lambda U'(x_t, Y_t) + \sqrt{\frac{2}{\beta}} \xi_t. \quad (4.1)$$

The potential V is periodic in $x \to x + 1$, and $E > 0$ is a constant driving. There is a coupling with the probe at position Y_t through the biperiodic potential $U(x, Y)$. Again, ξ_t denotes standard white noise and $U'(x, Y_t) = \partial_x U(x, Y_t)$. We assume that the probe moves very slowly with respect to the medium. As a result, the particle moves around the circle, reaching at each time a steady angular velocity

$$J_\lambda(t) = \langle E - V'(x) - \lambda U'(x, Y_t) \rangle^{Y_t} \quad (4.2)$$

with expectation under the quasistationary probability density $\rho^{Y_t}(x)$, $x \in S^1$. This current and probability density should be interpreted as follows. One considers a great many of such identical independent medium particles with positions x_t^j suspended in a viscous fluid in the same toroidal trap modeled by S^1 and then ρ^{Y_t} gives the real particle density in the steady regime at probe position Y_t. Similarly, the mass center over all particles actually moves with angular velocity $J_\lambda(t)$ given in Eq. (4.2).

The probe motion itself would naturally also be overdamped, and following (1.3) we put as reference dynamics

$$\dot{Y}_t = \lambda \langle U'(x, Y_t) \rangle^{Y_t}, \quad (4.3)$$

where we have used the action-reaction principle [20]. Comparing with (4.2), we have

$$\dot{Y}_t = E - \langle V'(x) \rangle^{Y_t} - J_\lambda(t)$$
$$= J_0 - J_\lambda(t) + \langle V'(x) \rangle - \langle V'(x) \rangle^{Y_t}, \quad (4.4)$$

where J_0 is the stationary current and $\langle \cdot \rangle$ is the stationary expectation at zero coupling $\lambda = 0$. The reference probe motion (4.3) is thus characterized as follows: (naturally) there

is no motion for $\lambda = 0$; for zero medium driving $E = 0$, we have $J_0 = 0 = J_\lambda(t)$ and there may be various stationary points $Y_t \equiv y$ for which $\langle V'(x) \rangle = \langle V'(x) \rangle^y$; for $V = 0$, we see that the probe will move around the circle at an angular speed $J_0 - J_\lambda(t)$, which is less than the current J_0 of the free medium particles in case the latter are slowed down by the probe $[J_\lambda(t) < J_0]$—obviously, when the coupling U is rotation-invariant, we have $J_\lambda(t) = J_\lambda$ independent of time and the probe just rotates at fixed speed.

The probe will not strictly follow the dynamics (4.4), as there will be fluctuations due to the corpuscular nature of the nonequilibrium fluid. In other words, the probe has a position X_t not exactly equal to Y_t, and therefore the more correct Langevin equation for the medium particles is

$$\dot{x}_t^j = E - V'(x_t^j) - \lambda \, \partial_j U(x_t^j, X_t) + \sqrt{\frac{2}{\beta}} \xi_t^j, \quad (4.5)$$

where X_t moves around Y_t. If $X_t - Y_t$ is small, we can expand the potential U around $X_t = Y_t$ with (4.4) thus representing zero order. As in the previous section, we can use the linear-response formulas (2.3)–(2.5) to study how the probe motion perturbs the fluid dynamics. From there, via (1.2), the effective dynamics of the probe motion would appear. Specific expressions will of course depend on the choices of the potential V and U, but in principle everything needed for friction and noise covariance is computable from two-time correlation functions under the quasistationary dynamics (4.1).

V. SUMMARY AND OUTLOOK

The character of particle motion in nonequilibrium fluids is a subject of increasing interest and is indeed relevant for a great variety of physical contexts ranging from motion in stellar and atmospheric environments to motion on the cytoskeleton of living cells; see, e.g., [3,6,21,22].

Our setup has been as follows:

(i) Driven particles make up the medium, both in contact with a probe and with a thermal equilibrium reservoir. Their dynamics is modeled via a Langevin dynamics satisfying local detailed balance. We assume a reference probe trajectory obtained from an infinite time-scale separation between probe and medium.

(ii) A reduced dynamics for the probe is obtained, in the form of a generalized Langevin equation containing a statistical force, friction, and a noise term. The derivation of the friction kernel is based on linear response for the backreaction of medium particles on the motion of the probe, which is considered to be a small perturbation of the reference trajectory.

There remain many interesting physical and computational points to be studied in the future and to be confronted with controlled experimental realizations.

The reduced probe dynamics is a nonequilibrium one, not satisfying the second fluctuation-dissipation relation. One can try to consider introducing an effective temperature in terms of the ratio between the frenetic and the entropic component of the friction, but it remains to be seen how useful that is for the probe's fluctuation and diffusion behavior. More generally, response theory for this probe motion has not been

physically discussed in the literature so far; the point is that we have no condition here of local detailed balance as there is no calorimetric way to speak about entropy fluxes in the nonequilibrium environment.

Another point of interest is to study the frenetic contribution to the friction; it could very well dominate when the environment is sufficiently far away from equilibrium, and for example giving rise to the possibility of negative friction (as an analog to what can happen in the out-of-equilibrium version of the first fluctuation-dissipation theorem [17]).

But even the statistical force G is largely unexplored for nonequilibrium environments; it can certainly contain a rotational contribution, which is a less studied topic in the general theory of fluctuation-induced or Casimir forces. The simplest example was presented in the preceding section for motion in a toroidal trap, but even there more computational effort must be employed to give precise characterizations of the probe motion.

In this paper, we have considered a Brownian motion in a sea of active particles. There are many possible experimental realizations, and the phenomenology is vast. One of the more straightforward examples is the bacteria dynamics in Ref. [23]. The bacteria can be treated as active particles, and they move in a fluid following some chemical concentration gradient. A practical way to model and simulate active particles is via Janus particles, e.g., in Refs. [24,25]. These kinds of particles are made by two different phases that react differently to an external stimulus, such as an electromagnetic field or a chemical gradient. Yet another well-studied nonequilibrium medium is the granular materials. Probes immersed in granular media have been extensively studied for their induced motion; see, e.g., [26,27] for an example of simulation and experiment. We hope that further quantitative comparison between the theoretical results here and those in the experiments will become available.

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APPENDIX: LINEAR-RESPONSE THEORY

1. Girsanov formula

Consider the perturbed Langevin equation

$$\dot{x}_t = \nu_t(x_t)[F_t(x_t) - \nabla U(x_t)] + \nabla D_t(x_t)$$
$$+ h_t \nu_t(x_t) \nabla V_t(x_t) + \sqrt{2D_t(x_t)}\xi_t, \qquad \text{(A1)}$$

where h_t (nonzero for $t > 0$) is a small time-dependent parameter. The perturbed backward generator is

$$L_t^h = \nu_t(x_t)[F_t(x_t) - \nabla U(x_t)] \cdot \nabla + \nabla[D_t(x_t) \cdot \nabla]$$
$$+ h_t \nu_t(x_t) \nabla V_t(x_t) \cdot \nabla \qquad \text{(A2)}$$

involving a modification of the potential to $U_t^h(x_t) = U(x_t) - h_t V_t(x_t)$. We have assumed that the mobility and the diffusion coefficient have not changed. We can then calculate the density of the perturbed probability $\mathcal{P}_{x_0}^h$ on trajectories ω over a time

span $[0,t]$ with respect to the original one $\mathcal{P}_{x_0}$ both starting from x_0. In other words, there is an excess action $\mathcal{A}$, with formally

$$\mathcal{P}_{x_0}^h(\omega) = \mathcal{P}_{x_0}(\omega) e^{-\mathcal{A}(\omega)}$$

and, with $h_t \nu_t = D_t \varphi_t$,

$$-\mathcal{A}(\omega) = \frac{1}{2}\int_0^t dx_s \circ \varphi_s \nabla V_s(x_s)$$
$$-\frac{1}{2}\int_0^t ds\, \varphi_s \nu_s(x_s) F_s(x_s) \nabla V_s(x_s)$$
$$+\frac{1}{2}\int_0^t ds\, \varphi_s \nu_s(x_s) \nabla U(x_s) \nabla V_s(x_s)$$
$$-\frac{1}{2}\int_0^t ds\, \varphi_s \nabla[D_s(x_s)\nabla V_s(x_s)] + O(h_s^2)$$
$$\text{(A3)}$$

using Stratonovich stochastic integration in the first line. That is called a Girsanov formula, and for diffusion processes it very much resembles standard path integration; see [19]. Note that in Eq. (A3) the backward generator L_s appears. We use the fundamental theorem of calculus to rewrite

$$-\mathcal{A}(\omega) = \frac{1}{2}[\varphi_t V_t(x_t) - \varphi_0 V_0(x_0)]$$
$$-\frac{1}{2}\left[\int_0^t \varphi_s \frac{\partial V_s(x_s)}{\partial s}ds - \int_0^t \frac{d\varphi_s}{ds}V_s(x_s)\,ds\right]$$
$$-\frac{1}{2}\int_0^t ds\, \varphi_s L_s V_s(x_s). \qquad \text{(A4)}$$

Observe that the excess action $\mathcal{A}$ is made from two contributions: the first line is entropic since it describes the excess entropy flux from the system to the environment due to the perturbation. We mean excess because the system is out of equilibrium even without the perturbation; there is an entropy production also in the unperturbed condition assured by the external force $F_t(x_t)$. The second contribution describes the excess in dynamical activity. It takes into account how much the system is inclined to change state.

2. Response formula

The perturbed average values relate to the unperturbed ones via

$$\delta\langle Q_t(x_t)\rangle_\mu^h = \langle Q_t(x_t)\rangle_\mu^h - \langle Q_t(x_t)\rangle_\mu$$
$$\simeq -\langle \mathcal{A}_t(x_t) Q_t(x_t)\rangle_\mu, \qquad \text{(A5)}$$

where $\delta\langle Q_t(x_t)\rangle^h$ is the generalized susceptibility related to the response by

$$\delta\langle Q_t(x_t)\rangle^h \simeq \int_0^t \varphi_s R_{Q,V}(t,s)ds. \qquad \text{(A6)}$$

A simple inspection of (A4) gives

$$R_{Q,V}(t,s) = \frac{1}{2}\frac{d}{ds}\langle V_s(x_s)\,Q_t(x_t)\rangle_\mu - \frac{1}{2}\left\langle\frac{\partial V_s(x_s)}{\partial s}Q_t(x_t)\right\rangle_\mu - \frac{1}{2}\langle L_s V_s(x_s)\,Q_t(x_t)\rangle_\mu. \tag{A7}$$

The first two terms describe the entropic contribution. The last term represents the frenetic part of the response; it depends on more detailed kinetics such as here via the mobility. This kind of relation is a nonequilibrium extension of the (first) fluctuation-dissipation theorem.

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