PHYSICAL REVIEW E **91**, 022104 (2015)

Families of Fokker-Planck equations and the associated entropic form for a distinct steady-state probability distribution with a known external force field

Somayeh Asgarani*
Department of Physics, Payame Noor University, PO Box 19395-3697, Tehran, Iran
(Received 1 November 2014; published 5 February 2015)

A method of finding entropic form for a given stationary probability distribution and specified potential field is discussed, using the steady-state Fokker-Planck equation. As examples, starting with the Boltzmann and Tsallis distribution and knowing the force field, we obtain the Boltzmann-Gibbs and Tsallis entropies. Also, the associated entropy for the gamma probability distribution is found, which seems to be in the form of the gamma function. Moreover, the related Fokker-Planck equations are given for the Boltzmann, Tsallis, and gamma probability distributions.

DOI: 10.1103/PhysRevE.91.022104 PACS number(s): 02.50.−r, 05.45.−a

I. INTRODUCTION

One of the most important phenomenological equations of nonequilibrium statistical physics is the *linear* Fokker-Planck equation (FPE) [1–6], which expresses the time evolution of the probability distribution related to a given physical system, in the presence of an external potential [7]. This equation describes properly many physical phenomena such as normal diffusion, which is essentially related to the Boltzmann-Gibss (BG) formalism, in the sense that the Boltzmann distribution, usually given by the maximization of the BG entropy under certain constraints, can also be interpreted as the stationary solution of the linear FPE.

However, it is well accepted that some physical phenomena such as anomalous diffusion can not be properly described by linear FPE, including for instance particle transport in disordered media (containing impurities or defects) [8], surface growth [9], and diffusion of micelles in salted water [10]. In order to deal with such anomalous systems, modifications in the linear FPE have been carried out. *Nonlinear* FPEs [1,2,4,11–13], which in many cases appear as simple phenomenological generalizations of the standard linear FPE, can be derived by using kinetic transport theory and linear nonequilibrium thermodynamics [11] or directly from a standard master equation, by introducing nonlinear effects on its associated transition probabilities [14–17]. Another approach to describe these systems is the nonextensive statistical mechanics [18]. The powerlike probability distribution that maximizes the entropy proposed by Tsallis [19] often appears as a stationary solution of nonlinear FPEs [1,2,4,11–13].

In Refs. [16,17], the H-theorem has been proved for systems which are described by nonlinear FPEs in the presence of external potential. For that, a relation involving terms of FPEs and general entropic forms has been proposed. Also, it has been shown that at equilibrium, this relation is equivalent to the maximum entropy principle.

In this work, starting with a given steady-state probability distribution and known potential field, we will propose a method to find associated entropic form, which is equivalent to the result of the MaxEnt principle. Also, we can find the terms of the associated FPEs for a distinct probability distribution and potential field. In the other words, starting with a stationary probability distribution and defining the potential field, it is possible to find the time evolution of that probability distribution, namely, the related set of FPEs. In the next section, following [16,17] and using the H-theorem, a relation between the terms of FPEs and its associated entropic forms is given. In Sec. III, by introducing a different method we obtain the related entropic forms for the Boltzmann, Tsallis, and gamma distributions for a known force field. In Sec. IV, the time evolution of a given probability distribution is investigated and in Sec. V, we present a conclusion.

II. A GENERAL NONLINEAR FOKKER-PLANCK EQUATION AND ITS ASSOCIATED ENTROPY

One form of generalized multivariate FPE has been obtained by Frank [11] using kinetic transport theory and linear nonequilibrium thermodynamics. That equation can be expressed in terms of generalized free energy and entropy of the system as

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x} M(P) P \frac{\partial}{\partial x}\frac{\delta F}{\delta P} = \frac{\partial}{\partial x} M(P) P \left[\frac{dU_0(x)}{dx} - Q \frac{\partial}{\partial x}\frac{\delta S}{\delta P}\right], \quad (1)$$

where F, U_0, and S are generalized free energy, generalized internal energy, and generalized entropy, respectively, and M is regarded as a mobility or friction matrix [20–22].

Another approach to find generalized FPE has been done by Schwämmle and co-workers [16,17] which expresses the FPE as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial\{A(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\}, \quad (2)$$

where $A(x)$ is the external force associated with a potential $\phi(x)\,[A(x) = -d\phi(x)/dx, \phi(x) = -\int_{-\infty}^{x} A(x')dx']$ and the functionals $\Psi[P(x,t)]$ and $\Omega[P(x,t)]$ are supposed to be both positive finite quantities, integrable as well as differentiable (at least once) with respect to the probability distribution $P(x,t)$. At the stationary state, namely when the dependence on time

*sasgarani@pnu.ac.ir

disappears, the generalized FPE is reduced to

$$A(x) = \frac{\Omega[P_{eq}]}{\Psi[P_{eq}]}\frac{dP_{eq}(x)}{dx}, \quad (3)$$

where P_{eq} refers to the probability distribution at equilibrium.

The H-theorem for a system that exchanges energy with its surrounding corresponds to a well-defined sign for the time derivative of the free-energy functional, which can be considered as

$$\frac{dF}{dt} \leqslant 0, \quad (4)$$

where F is the free-energy functional, defined as

$$F = U - \frac{1}{\beta}S; \quad U = \int_{-\infty}^{\infty} dx\phi(x)P(x,t) \quad (5)$$

and S is a general entropic form satisfying the following conditions:

$$S = \int_{-\infty}^{\infty} dx g[P(x,t)]; \quad g(0) = 0; \quad g(1) = 0; \quad \frac{d^2 g}{dP^2} \leqslant 0. \quad (6)$$

Using the above definitions, it is possible to find a condition which preserves the H-theorem [Eq. (4)] [16,17]:

$$\frac{d^2 g}{dP^2} = -\beta\frac{\Omega[P]}{\Psi[P]}. \quad (7)$$

It should be noticed that Eq. (7) expresses an important correspondence between whole families of the FPEs, defined in terms of the functionals $\Omega[P]$ and $\Psi[P]$, and their related entropic forms in the presence of an external potential. On the other hand, starting with a given entropic form, one can find the class of FPEs associated to it and vice versa. For example, by imposing $\Omega[P] = Da[P]$ and $\Psi[P] = Pa[P]$ with constant D and arbitrary function $a[P]$, the Boltzmann-Gibbs entropy is given. Also, in [17] it is shown that at equilibrium, Eq. (7) is equivalent to the maximum entropy principle.

III. A METHOD OF FINDING ENTROPIC FORM RELATED TO A DISTINCT PROBABILITY DISTRIBUTION AND A GIVEN POTENTIAL FIELD, USING STEADY-STATE FPE

The probability distribution of a system is usually obtained through the maximization of the entropic form under certain constraints (the so-called MaxEnt principle). Also, this probability distribution can be given as the stationary solution of the FPE. For example, using the MaxEnt principle to find the probability distribution of BG entropy is the same as finding the stationary solution of the linear FPE [1–6]. Also, some other kinds of probability distributions are given as stationary solutions of nonlinear FPEs [1,2,4,11–13]. In both procedures, knowing the functionality of potential (or force) is essential. In the former case, the potential field is entered as an energy constraint $U = \int_{-\infty}^{\infty} dx\phi(x)P(x,t)$, and in the latter, the force field, $A(x) = -d\phi(x)/dx$, appears in the FPE [Eq. (2)].

However, stationary probability distributions are found experimentally in many systems in nature. So, the reverse procedure, namely, finding the entropic form of the system, assuming the knowledge of the probability distribution and of the external potential, may be interesting. In the following, we will propose a method to achieve that aim. In that method, we start with the stationary probability distribution of a system as a function of potential field $\phi(x)$. By differentiating $P(x)$ with respect to x, in the other side of the equation $A(x) = -d\phi(x)/dx$ appears. At this stage, we can compare the obtained equation with the stationary FPE [Eq. (3)] and so the ratio $\frac{\Omega[P]}{\Psi[P]}$ is given. Using Eq. (7) and knowing the ratio $\frac{\Omega}{\Psi}$, the entropic form is obtained and also the related FPE can be written. Following the method, we will find the entropic forms for the BG, Tsallis, and gamma distributions.

A. Boltzmann distribution

It is known that the maximization of the BG entropy under the normalization and energy constraints results in

$$P(x) = \alpha e^{-\lambda\phi(x)}, \quad (8)$$

where $\phi(x)$ is the potential field, α is the normalization constant, and λ is the Lagrange multiplier of the energy constraint. Now, we start with the Boltzmann distribution [Eq. (8)] and will attempt to find the entropic form. By differentiation of the probability distribution, one finds

$$\frac{dP(x)}{dx} = -\alpha\lambda e^{-\lambda\phi(x)}\frac{d\phi(x)}{dx} = \lambda A(x)P(x), \quad (9)$$

where the relation $A(x) = -\frac{d\phi(x)}{dx}$ is used. Now, we attempt to write the above equation as the stationary FPE [Eq. (3)]. Comparing the above equation with Eq. (3), one can write

$$A(x) = \frac{1}{\lambda P}\frac{dP(x)}{dx} \Rightarrow \frac{\Omega[P]}{\Psi[P]} = \frac{1}{\lambda P}. \quad (10)$$

Substituting Eq. (10) in Eq. (7) and then integrating that equation, one obtains

$$\frac{dg}{dP} = -\frac{\beta}{\lambda}\ln P + C \Rightarrow g[P] = -P\ln P, \quad (11)$$

where we have used the condition $g(0) = g(1) = 0$ to eliminate the constant C, and set the Lagrange multiplier $\beta = \lambda$, which shows that the parameter λ is the same as β, defined in Eq. (5). It should be noted that in the definition of free energy [Eq. (5)], $1/\beta$ is used instead of temperature (T) and so the Boltzmann constant (k_B) is set to 1.

B. Tsallis distribution

The above-mentioned method can be used for the Tsallis probability distribution [18,19] $P(x) = \alpha[1 - \lambda(q-1)\phi(x)]^{\frac{1}{q-1}}$, where α is the normalization constant. Following the method, we have

$$A(x) = \frac{1}{\lambda\alpha^{q-1}}P^{q-2}\frac{dP}{dx} \Rightarrow \frac{\Omega[P]}{\Psi[P]} = \frac{1}{\lambda\alpha^{q-1}}P^{q-2} \equiv DP^{q-2} \quad (12)$$

and so the entropic form can be obtained as

$$\frac{d^2 g}{dP^2} = -\beta DP^{q-2} \Rightarrow g[P] = -\beta D\frac{P^q}{q(q-1)} + CP + E, \quad (13)$$

where $g(0) = g(1) = 0$ results in $E = 0$ and $C = \frac{\beta D}{q(q-1)}$. If we consider $D = \frac{q}{\beta}$, the Tsallis entropy can be given as

$$g_q[P] = -\frac{P^q - P}{q-1}. \tag{14}$$

C. Gamma distribution

One of the examples of the gamma distribution is the energy distribution of an ideal gas proposed by Maxwell around 1860 [23]:

$$f(E) = \frac{dN}{dE} = \frac{2\pi N}{(\pi kT)^{3/2}} E^{1/2} e^{-E/kT}, \tag{15}$$

where dN is the number of molecules with energy between E and $E + dE$. The gamma probability distribution, considered as a combination of exponential and power-law distributions, appears in description of many systems [24–29] and has the form

$$P(x) = \alpha x^{d-1} e^{-\lambda x}, \tag{16}$$

where $\alpha = \frac{\lambda^d}{\Gamma(d)}$ is the normalization constant and the gamma function is defined as

$$\Gamma(d) = \int_0^\infty x^{d-1} e^{-x} dx. \tag{17}$$

We assume that the probability distribution is a gamma function in terms of $\phi(x)$, namely,

$$P(x) = \alpha\phi(x)^{d-1} e^{-\lambda\phi(x)}, \quad \phi(x) \geqslant 0 \tag{18}$$

where the condition $\phi(x) \geqslant 0$ satisfies the positivity of the probability distribution. Following the method, one can differentiate from $P(x)$ and so

$$\frac{dP}{dx} = \left[(d-1)\frac{P(x)}{\phi(x)} - \lambda P(x)\right]\frac{d\phi}{dx}$$
$$\Rightarrow \frac{1}{\left(\lambda - \frac{d-1}{\phi}\right)P}\frac{dP}{dx} = A(x). \tag{19}$$

Comparing the above equation with Eq. (3) and then using Eq. (9), we have

$$-\frac{1}{\beta}\frac{d^2 g}{dP^2} = \frac{\Omega[P]}{\Psi[P]} = \frac{1}{\left(\lambda - \frac{d-1}{\phi}\right)P}, \quad \lambda\phi(x) \geqslant d-1 \tag{20}$$

where the condition $\lambda\phi(x) \geqslant d-1$ is set because of the concavity condition of the entropy. It should be noted that the fraction $\frac{\Omega[P]}{\Psi[P]}$ is a function of P; so in Eq. (20), ϕ must be written as a function of P. Using Eq. (18), we can write

$$\phi(P) = \frac{1-d}{\lambda} W[z], \quad \frac{\lambda}{1-d}\left(\frac{P}{\alpha}\right)^{\frac{1}{d-1}}, \tag{21}$$

where $W[z]$ is the Lambert W function [30] and defined in the relation $z = W[z]e^{W[z]}$. Using the above equations, one can write

$$-\frac{1}{\beta}\frac{d^2 g}{dP^2} = \frac{\Omega[P]}{\Psi[P]} = \frac{W[z]}{\lambda P(1+W[z])} = \frac{d-1}{\lambda}\frac{dW}{dP}, \tag{22}$$

where we have used

$$\frac{dW[z]}{dz} = \frac{W[z]}{z(1+W[z])}. \tag{23}$$

Integrating Eq. (22), one obtains

$$\frac{dg}{dP} = \frac{\beta(1-d)}{\lambda} W[z] + C_1 \Rightarrow g(p)$$
$$= \frac{\beta(1-d)}{\lambda}\int_0^P W[z]dP + C_1 P + C_2, \tag{24}$$

where C_1 and C_2 are integration constants. By defining $z = AP^{\frac{1}{d-1}}$, where $A = \frac{\lambda\alpha^{\frac{1}{1-d}}}{1-d}$, for $g(P)$ we have

$$g(P) = \frac{-\beta(1-d)^2}{\lambda A^{d-1}}\int_0^z W[z]z^{d-2}dz + C_1 P + C_2. \tag{25}$$

If we consider $t = W[z]$,

$$\int_0^z W[z]z^{d-2}dz = \int_0^{W[z]} t^{d-1}e^{(d-1)t}dt + \int_0^{W[z]} t^d e^{(d-1)t}dt$$
$$= \frac{1}{(1-d)^d}\Gamma(d,0,(1-d)W[z])$$
$$+ \frac{1}{(1-d)^{d+1}}\Gamma(d+1,0,(1-d)W[z]), \tag{26}$$

where $\Gamma(d,x_1,x_2)$ is the *generalized incomplete gamma function* and defined as the difference of two *incomplete gamma functions*

$$\Gamma(d,x_1,x_2) = \int_{x_1}^{x_2} x^{d-1}e^{-x}dx = \Gamma(d,x_1) - \Gamma(d,x_2) \tag{27}$$

with $\Gamma(d,x) = \int_x^\infty x^{d-1}e^{-x}dx$. Using the definitions $\alpha = \frac{\lambda^d}{\Gamma(d)}$ and $A = \frac{\lambda\alpha^{\frac{1}{1-d}}}{1-d}$ and substituting Eq. (26) in Eq. (25), the entropic form can be given as

$$g(P) = \frac{-\beta}{\Gamma(d)}\{(1-d)\Gamma(d,0,(1-d)W[z(P)])$$
$$+ \Gamma(d+1,0,(1-d)W[z(P)])\} + C_1 P + C_2. \tag{28}$$

For $d \neq 1$, $g(0) = 0$ results in $C_2 = 0$ and C_1 can be obtained by the condition $g(1) = 0$. But, in the case $d = 1$, we have $\lim_{d\to 1}(1-d)W[z] = -\ln\frac{P}{\lambda}$, and so $g(P)$ can be written as

$$g(P) = -\beta\Gamma\left(2,0,-\ln\frac{P}{\lambda}\right) + C_1 P + C_2 = -P\ln P, \tag{29}$$

where $\beta = \lambda$ and the constants $C_1 = -(1+\ln\beta)$ and $C_2 = \beta$ satisfy $g(0) = g(1) = 0$. It is clear that the BG entropy is recovered from Eq. (28) in the limit $d = 1$, as expected.

Other types of gamma entropy have also been introduced in Refs. [31,32], by considering some scaling properties, which recover some known entropies such as the BG and Tsallis entropies as special limits.

IV. TIME EVOLUTION OF A GIVEN PROBABILITY DISTRIBUTION IN THE PRESENCE OF AN EXTERNAL POTENTIAL FIELD

By using the expressed method in the previous section, one can find the ratio $\frac{\Omega}{\Psi}$ for a given probability distribution. It is clear that if both Ω and Ψ are multiplied by a function of P (for example $b[P]$), the ratio $\frac{\Omega}{\Psi}$ does not change. So, there are families of FPEs, corresponding to the same ratio associated with a single stationary probability distribution or a single entropic form. This ambiguity also exists in the Frank FPE [Eq. (1)], where one can choose different functions of $M(P)$. For example, two special cases of Eq. (1) are obtained when the drift term (the first term in the right-hand side of the equation) or the diffusion term (the second term in the right-hand side of the equation) become linear with respect to P. In Refs. [14–16], $\Psi[P]$ is set equal to P which is equivalent to set $M(P)=1$. In this section, we also consider $\Psi[P]=P$; according to the previous section, we can have the ratio $\frac{\Omega}{\Psi}$. So, $\Omega[P]$ and the related FPE are obtained. In the three forenamed cases, we have the following:

The Boltzmann distribution. In that case, according to Eq. (10) and $\Psi[P]=P$, one obtains $\Omega[P]=\frac{1}{\lambda}$ and so the related FPE [Eq. (2)] can be written in the form

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\{A(x)P\}+\frac{1}{\lambda}\frac{\partial^2 P}{\partial x^2}, \tag{30}$$

which is the familiar linear Fokker-Planck equation.

The Tsallis distribution. According to Eq. (12) and $\Psi[P]=P$, we have $\Omega[P]=DP^{q-1}$ [Eq. (12)] and the correspondent FPE is obtained as

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\{A(x)P\}+D\frac{\partial}{\partial x}\left(P^{q-1}\frac{\partial P}{\partial x}\right) \tag{31}$$

or

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\{A(x)P\}+\frac{D}{q}\frac{\partial^2}{\partial x^2}P^q, \tag{32}$$

which becomes the same as the expressed FPE in Ref. [1], only by the substitution $q\to 2-q$ or $q-1\to 1-q$ in the FPE [Eq. (32)] and also in the definition of probability distribution.

The gamma distribution. According to Eq. (22) and $\Psi[P]=P$, we obtain $\Omega[P]=\frac{W[z(P)]}{\lambda\{1+W[z(P)]\}}$ and so the related FPE is obtained as

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\{A(x)P\}+\frac{1}{\lambda}\frac{\partial}{\partial x}\left\{\frac{W[z(P)]}{1+W[z(P)]}\frac{\partial P}{\partial x}\right\} \tag{33}$$

with $z(P)$ defined by Eq. (21).

V. CONCLUSION

In this study, a method is proposed to derive a particular kind of Fokker-Planck equations from given relationships between stationary probability densities and potential functions. These Fokker-Planck equations are nonlinear with respect to their probability densities. The stationary probability densities of those so-called nonlinear Fokker-Planck equations maximize certain entropy functionals under the constraints of a canonical ensemble. Therefore, the entropy functionals are determined by the method as well. This phenomenological approach addresses an interesting problem of how to construct a stochastic process on the basis of information about how a system responds in the stationary case to a potential force. The method can also have engineering applications in the study of noise generators that asked the following question: Given an arbitrary probability density and a linear force (i.e., quadratic potential), how can we construct a Fokker-Planck equation such that the probability density is the stationary probability density of that equation? The Langevin equation of such a Fokker-Planck equation can then be used in engineering applications as noise generator for the desired probability density [33–35]. It would be interesting that a similar question can be addressed from two perspectives (this work and the work by Primak) which demand new attempts to study.

[1] A. R. Plastino and A. Plastino, Non-extensive statistical mechanics and generalized Fokker-Planck equation, Physica A (Amsterdam) **222**, 347 (1995).

[2] C. Tsallis and D. J. Bukman, Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis, Phys. Rev. E **54**, R2197 (1996).

[3] L. Borland, Microscopic dynamics of the nonlinear Fokker-Planck equation: A phenomenological model, Phys. Rev. E **57**, 6634 (1998).

[4] E. K. Lenzi, R. S. Mendes, and C. Tsallis, Crossover in diffusion equation: Anomalous and normal behaviors, Phys. Rev. E **67**, 031104 (2003).

[5] T. D. Frank and A. Daffertshofer, Nonlinear Fokker-Planck equations whose stationary solutions make entropy-like functionals stationary, Physica A (Amsterdam) **272**, 497 (1999).

[6] L. C. Malacarne, R. S. Mendes, I. T. Pedron, and E. K. Lenzi, N-dimensional nonlinear Fokker-Planck equation with time-dependent coefficients, Phys. Rev. E **65**, 052101 (2002).

[7] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1989).

[8] M. Muskat, *The Flow of Homogeneous Fluids through Porous Media* (McGraw-Hill, New York, 1937).

[9] H. Spohn, Surface dynamics below the roughening transition, J. Phys. I **3**, 69 (1993).

[10] J. P. Bouchaud, A. Ott, D. Langevin, and W. Urbach, Anomalous diffusion in elongated micelles and its Levy flight interpretation, J. Phys. II **1**, 1465 (1991).

[11] T. D. Frank, *Nonlinear Fokker-Planck Equations: Fundamentals and Applications* (Springer, Berlin, 2005).

[12] A. R. Plastino L. Borland and C. Tsallis, Information gain within nonextensive thermostatistics, J. Math. Phys. **39**, 6490 (1998).

[13] P. H. Chavanis, Generalized thermodynamics and Fokker-Planck equations: Applications to stellar dynamics and two-dimensional turbulence, Phys. Rev. E **68**, 036108 (2003).

[14] E. M. F. Curado and F. D. Nobre, Derivation of nonlinear Fokker-Planck equations by means of approximations to the master equation, Phys. Rev. E **67**, 021107 (2003).

[15] F. D. Nobre, E. M. F. Curado, and G. Rowlands, A procedure for obtaining general nonlinear Fokker-Planck equations, Physica A (Amsterdam) **334**, 109 (2004).

[16] V. Schwammle, E. M. F. Curado, and F. D. Nobre, A general nonlinear Fokker-Planck equation and its associated entropy, Eur. Phys. J. B **58**, 159 (2007).

[17] V. Schwammle, F. D. Nobre, and E. M. F. Curado, Consequences of H-theorem from nonlinear Fokker-Planck equations, Phys. Rev. E **76**, 041123 (2007).

[18] E. M. Gell-Mann and C. Tsallis, *Nonextensive Entropy- Interdisciplinary Applications* (Oxford University Press, New York, 2004).

[19] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).

[20] H. C. Ottinger, *Stochastic Processes in Polymeric Fluids* (Springer, Berlin, 1996).

[21] M. Bekele, S. Rajesh, G. Ananthakrishna, and N. Kumar, Effect of Landauer's blow torch on the equilibration rate in a bistable potential, Phys. Rev. E **59**, 143 (1999).

[22] J. M. Rubi and A. Perez-Madrid, Mesoscopic non-equilibrium thermodynamics approach to the dynamics of polymers, Physica A (Amsterdam) **298**, 177 (2001).

[23] J. C. Maxwell, Illustrations of the dynamical theory of gases. Part II. On the process of diffusion of two or more kinds of moving particles among one another, Philos. Mag. **20**, 21 (1860).

[24] B. Hnat, S. C. Chapman, G. Rowlands, N. W. Watkins, and W. M. Farrell, Finite size scaling in the solar wind magnetic field energy density as seen by WIND, Geophys. Res. Lett. **29**, 86-1 (2002).

[25] M. Patriarca, A. Chakraborti, and K. Kaski, A statistical model with a standard Gamma distribution, Phys. Rev. E **70**, 016104 (2004).

[26] D. Saa, On an improvement of the Planck radiation energy distribution, arXiv:physics/0603117v3.

[27] G. E. Crooks and C. Jarzynski, On the work distribution for the adiabatic compression of a dilute classical gas, Phys. Rev. E **75**, 021116 (2007).

[28] G. L. Caer, A new family of solvable Pearson-Dirichlet random walks, J. Stat. Phys. **144**, 23 (2011).

[29] O. E. Garcia, Stochastic modeling of intermittent scrape-off layer plasma fluctuations, Phys. Rev. Lett. **108**, 265001 (2012).

[30] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, On the Lambert W function, Adv. Computat. Math. **5**, 329 (1996).

[31] R. Hanel and S. Thurner, A comprehensive classification of complex statistical systems and an ab initio derivation of their entropy and distribution functions, Europhys. Lett. **93**, 20006 (2011).

[32] S. Asgarani, A set of new three-parameter entropies in terms of a generalized incomplete Gamma function, Physica A (Amsterdam) **392**, 1972 (2013).

[33] S. Primak, Generation of compound non-Gaussian random processes with given correlations, Phys. Rev. E **61**, 100 (2000).

[34] S. Primak, V. Lyandres, and V. Kontorovich, Markov models of non-Gaussian exponentially correlated pricesses and their applications, Phys. Rev. E **63**, 061103 (2001).

[35] T. D. Frank, A Langevin approach for the microscopic dynamics of nonlinear Fokker-Planck equations, Physica A (Amsterdam) **301**, 52 (2001).