Local search methods based on variable focusing for random K-satisfiability

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(Received 8 July 2014; published 14 January 2015)

We introduce variable focused local search algorithms for satisfiabiliity problems. Usual approaches focus uniformly on unsatisfied clauses. The methods described here work by focusing on random variables in unsatisfied clauses. Variants are considered where variables are selected uniformly and randomly or by introducing a bias towards picking variables participating in several unsatistified clauses. These are studied in the case of the random 3-SAT problem, together with an alternative energy definition, the number of variables in unsatisfied constraints. The variable-based focused Metropolis search (V-FMS) is found to be quite close in performance to the standard clause-based FMS at optimal noise. At infinite noise, instead, the threshold for the linearity of solution times with instance size is improved by picking preferably variables in several UNSAT clauses. Consequences for algorithmic design are discussed.

DOI: 10.1103/PhysRevE.91.013305 PACS number(s): 02.70.−c, 05.40.−a, 89.20.Ff

I. INTRODUCTION

Focused local search algorithms have been found to be fairly efficient in solving random instances of the Boolean satisfiability problem (K-SAT), a famous NP-complete problem for $K \geqslant 3$ [1]. This means that any solution to this problem can be verified quickly, in polynomial time, but that no efficient way to find a solution is known. The simplest, but still very interesting, such algorithm is the one introduced by Papadimitriou in 1991 [2] where variables are flipped randomly, independently, and at a rate proportional to how many unsatisfied clauses they each participate in. Randomness and greediness in focused local search were combined in the Walksat algorithm of Selman, Kautz, and Cohen [3], and several other variants in this direction have been developed and investigated [4–9]. A physics-based outlook on the question of finding solutions follows from considering K-satisfiability as a diluted three-spin Ising model with disorder [10,11]. The variables of SAT formulas are then considered as spins, and the clauses, which consist each of K literals (i.e., variables or their negations), are local energy terms. This glassy model has been analyzed by usual techniques and its phase diagram worked out. However, focused local search does not obey detailed balance since all zero-energy states are left unchanged by this family of algorithms, and equilibrium considerations may therefore not be very relevant for the behavior or design of such algorithms. Indeed, it is well known that the equilibrium phase diagram of random K-SAT does not determine when focused local search works or does not work, and such information, although very important in itself, has not given much of a hint on how to proceed with the development of efficient local algorithms.

In this focused Metropolis search (FMS) and its variants [6–8] have been a main empirical step forward. These algorithms display a linear scaling of the solution times with instance size even in the immediate proximity of the SAT-UNSAT transition [6,8], identified by the ratio $\alpha = M/N$ of the number of clauses M to the number of variables N [12]. Similarly to the Walksat algorithm, focusing on unsatisfied clauses during the search is an important ingredient of this success. It is implemented as in Ref. [2], where first a random choice is made among unsatisfied clauses, and then among the variables participating in the chosen clause.

Thus in the FMS the rate of picking a variable is not uniform among the set of variables participating in unsatisfied clauses, but is biased towards variables participating in many unsatisfied clauses. In this work, we introduce the symmetric approach of uniform focusing on variables in unsatisfied clauses. This is a natural choice since it means a uniform measure on the subset of spins that contribute to the energy. We also consider the dynamics of another quantity that describes the number of variables in unsatisfied clauses.

We investigate the variable-based focused Metropolis search, i.e., V-FMS, which picks variables in unsatisfied clauses uniformly. The idea is to explore the difference of various sampling ideas (clause or variable based). We test in particular the idea of showing that upon using an intuitive rescaling of solution times it performs at least as well as the usual FMS on 3-SAT. We also study an algorithm which has a larger bias than FMS towards selecting variables which participate in many unsatisfied clauses: square-focused Metropolis search (S-FMS) (see also [13]), which picks variables proportionally to the square of the number of unsatisfied clauses they participate in.

These variants are also tried in the large temperature limit. It transpires that there are fundamental differences in the (empirical) phase space as regards the maximum constraint density where solutions are found in linear time. The fact that the kind of sampling is mostly irrelevant for random 3-SAT at optimal noise and a number of other observations should be of importance for choosing further directions in statistical physics-based algorithms: their development and understanding both empirically and theoretically.

The structure of the rest of this work is as follows. Below, we present in pseudolanguage the variable-based focused Metropolis search, i.e., V-FMS as an example of

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variable-based focusing. Then, we demonstrate in Sect. III the differences in the high-temperature behavior. On the other hand we also show that the algorithm can be compared directly to the FMS results utilizing earlier data for FMS published in Ref. [6]. The section also contains several observations about algorithmic behavior. Section IV finishes the paper with conclusions.

II. VARIABLE-BASED FOCUSING

In what follows, the state of the system at a given time is monitored by the number of unsatisfied clauses in the configuration (i.e., assignment of values to the variables), which we use as an energy function. It reaches zero only when a solution is found, and can be used to define an energy landscape [14] (but see also below). We study the variable-focusing case with the behavior of the V-FMS algorithm, presented here in pseudocode:

```
1: S = random assignment of values to the variables
2: while S is not a solution do
3:     V = a variable selected uniformly from those in UNSAT
           clauses
4:     ΔE = change in energy if V is flipped in S
5:     if ΔE ⩽ 0 then
6:         flip V in S
7:     else
8:         flip V in S with probability η^ΔE
9:     end if
10: end while
```

We also study the square-focused Metropolis search (S-FMS). Its only difference from V-FMS is that the variable V in line 3 of the pseudocode is not chosen uniformly from those in UNSAT clauses, but is selected with a probability proportional to the square of the number of unsatisfied clauses this variable is involved in.

We have implemented these algorithms as different search heuristics in a code derived from Walksat version 45 [3,8,15], where we have introduced different arrays to keep track of variables participating in unsatisfied clauses. In this code, η is a positive parameter which has to be adjusted for optimal search. Let us point out that since this process does not obey detailed balance the "energy" E is here just a convenient quantity to describe the dynamics locally, and may not be a good descriptor of the global dynamics. In other words, one may envision other energylike constructions (a global value from a sum of local contributions) that would be more relevant for information about the true distance to the ground state from the viewpoint of algorithmic performance. Instead of E, and given that V-FMS concentrates on variables, it is an alternative and natural idea to consider N_0, the number of variables involved in at least one unsatisfied clause (the cardinality of the support set of the flipping). This quantity also defines an energy landscape which is zero only at solutions, and could in principle have qualitatively different properties. One could use this or other quantities to define the best energy landscapes for the task one is interested in. The S-FMS algorithm suggests that another quantity could be instead used as energy function: this quantity, which we call N_2, is the sum over variables of the square of the number of unsatisfied clauses each variable is involved in.

Note that the corresponding quantity N_1 for clause-focusing (sum over variables of the number of unsatisfied clauses each variable is involved in) is exactly $K \times E$. We can also note that these quantities are, for variables in UNSAT clauses, the first moments of the distribution of the number of unsatisfied clauses they are involved in, with N_0 the "zeroth" moment.

III. ALGORITHMIC PERFORMANCE

A. Role of noise for V-FMS and S-FMS

First we note that the infinite noise limit of the new algorithms is not the usual random version of WalkSAT, studied for instance in Refs. [16–19]. Reference [16] shows that random WalkSAT can solve linearly random 3-SAT instances up to values of the clauses to variables ratio α close to 2.7. Figure 1 shows that variable focusing does worse than that, solving instances linearly up to $\alpha = 2.51 \pm 0.01$. Square focusing improves on the usual clause-focused random algorithm and finds solutions to random 3-SAT instances up to $\alpha = 3.09 \pm 0.01$. This result and similar results obtained in Ref. [13] show that for easy problems, selecting preferably variables in many unsatisfied clauses performs better. Reference [13] also suggests that the behavior of algorithms in the zero noise limit (i.e., greedy algorithms with different types of focusing) would also be different.

Next we test the V-FMS and S-FMS against the known state-of-the-art of the FMS, and we refer in particular to the work of Seitz *et al.* [6]. The important issues here from an empirical perspective are as follows: (i) Are the V-FMS and S-FMS linear for relatively high constraint-densities α? (ii) How does the performance depend on the noise parameter η, i.e., what does the algorithmic landscape look like? Ordinary FMS is biased towards picking trial variables that are present in a greater than average number of unsatisfied clauses. If this bias and the evolution of the set of eligible variables does not matter at optimal noise, then one should get similar performance out of V-FMS and S-FMS.

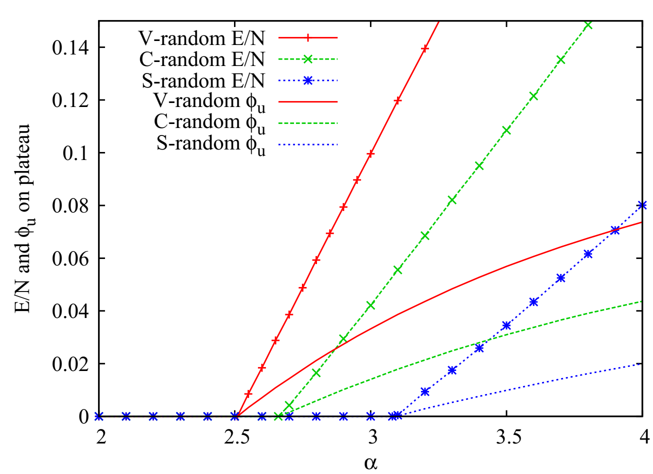

FIG. 1. (Color online) Plateau energy E/N and fraction of unsatisfied clauses $\Phi_u = E/M$ for the random focused algorithms (variable-, clause-, and square-focused random algorithms, respectively V-, C- and S-random), as a function of the ratio of clauses to variables α. Each point is an average over ten simulations with $N = 2 \times 10^5$.

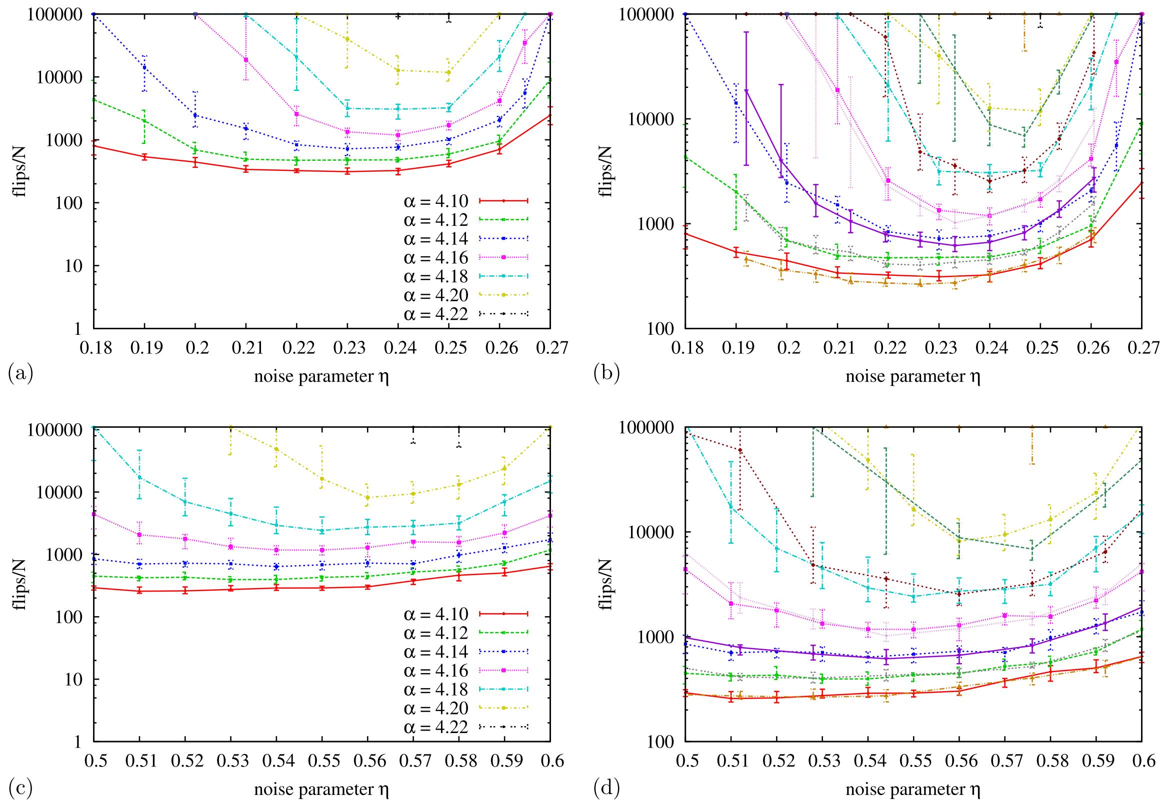

FIG. 2. (Color online) Running times of the local search algorithms. In the left column, running time of the V-FMS (a) and S-FMS (c) algorithms as a function of the noise parameter η, for different values of α (increasing from bottom to top). Each point presents median and quartiles for 20 instances, with $N = 10^5$. The right column presents these results again, with the same symbols, and superimposes the running times of the FMS algorithm taken from [6] for the same values of α, but where the noise parameter η has been rescaled by a factor $0.24/0.35$ (b) and $0.56/0.35$ (d).

Indeed, despite the different sampling rules used for V-FMS, FMS, and S-FMS, Fig. 2 shows evidence supporting the idea that the solution times of the three different algorithms are quite similar, when studied close to their respective optimal noise parameters. The generic feature is known for local search with focusing: the effort to solve problems in the median sense depends on the noise parameter η, or other such parameters of the algorithm used. There is a minimum solution time for an optimal noise parameter, and the minimum gets more marked with the increase of α, as has been reported for the FMS in the past [6]. The right panel of Fig. 2 shows that a simple rescaling of the noise parameter (data from Ref. [6]) suffices to bring the curves of the solution times to fall well on top of each other. This phenomenon is surprising when considering the results at infinite noise of Fig. 1. It implies that at optimal noise, the particular sampling, clause or variable based, is at most of quantitative importance in solving very difficult 3-SAT problems—a clearer difference might well exist of course for other test cases, like $K > 3$ in the SAT class. Let us note that multiplying the noise parameter η by a constant a is equivalent to multiplying the energy E by $\ln \eta a / \ln \eta$ in the pseudocode of Sec. II. We can also observe that the optimal noise is higher in the case of square focusing than for standard FMS, and is even lower for variable focusing, since the two first algorithms choose preferably variables in several unsatisfied clauses, while variables in unsatisfied clauses are picked uniformly with V-FMS. A more "greedy" sampling of variables in unsatisfied clauses, in the sense that variables in many unsatisfied clauses are chosen more often, is compensated by a higher noise in the flipping rates.

Figure 3 affirms the natural expectation that the solution time distribution gets more focused around the typical time value with an increasing instance size, which corresponds to a typically linear running time of the algorithms. The histograms start with increasing N to concentrate, albeit slowly, towards a typical solution time. This concentration of the measure takes here place for an α close to the maximum value up to which the local search methods work (linear algorithms have been reported to perform well at least up to $\alpha = 4.23$ while survey propagation finds solutions for $\alpha = 4.25$).

B. Energy traces

Next we consider the choice of the energy landscape by investigating the combination of N_0 and the energy E. In Fig. 4, we pick a suitable set of parameters including roughly optimal noise values for V-FMS, FMS, and S-FMS. We observe that the three algorithms have close behaviors, but still distinct energy and N_0 traces. This can be related to Fig. 2, where no difference was seen between algorithms on running times after the appropriate rescaling of the noise parameter η is made. The data are presented both in linear and logarithmic time scales to underline the early time behaviors.

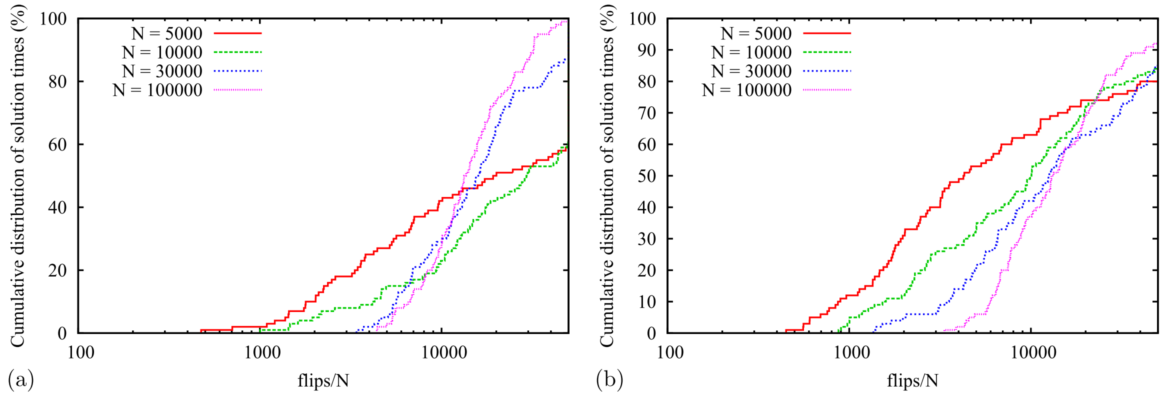

FIG. 3. (Color online) Cumulative distribution (histogram) of running times of V-FMS (a) and S-FMS (b), with $\alpha = 4.2$ and $\eta = 0.25$ (a), respectively $\eta = 0.56$ (b). Each curve is computed from 100 instances.

The main observation is that energy and number N_0 of variables in unsatisfied clauses have a very similar evolution. They are almost proportional during most of the search process evolution, as shown in Figs. 4(c) and 4(d). The proportionality factor increases first and then saturates to a value slightly below $K = 3$, meaning that most variables involved in unsatisfied clauses are involved in only one of them. This of course helps to qualitatively understand why the different sampling does not bring about any major changes in performance (V-FMS compared to FMS and S-FMS). Note that for all three algorithms there is a similar transient from the N_0/E value of a random assignment towards the higher value; already roughly one flip per clause is enough for the transient to get over.

IV. DISCUSSION

We have introduced here algorithms which perform in the region of large α's with an optimized noise parameter in practice as well as the already known focused Metropolis search (FMS). These algorithms have different picking rates for variables in unsatisfied clauses, since focusing is defined differently. They also have different flipping rates, since the optimal noise parameters are different. It can also be checked

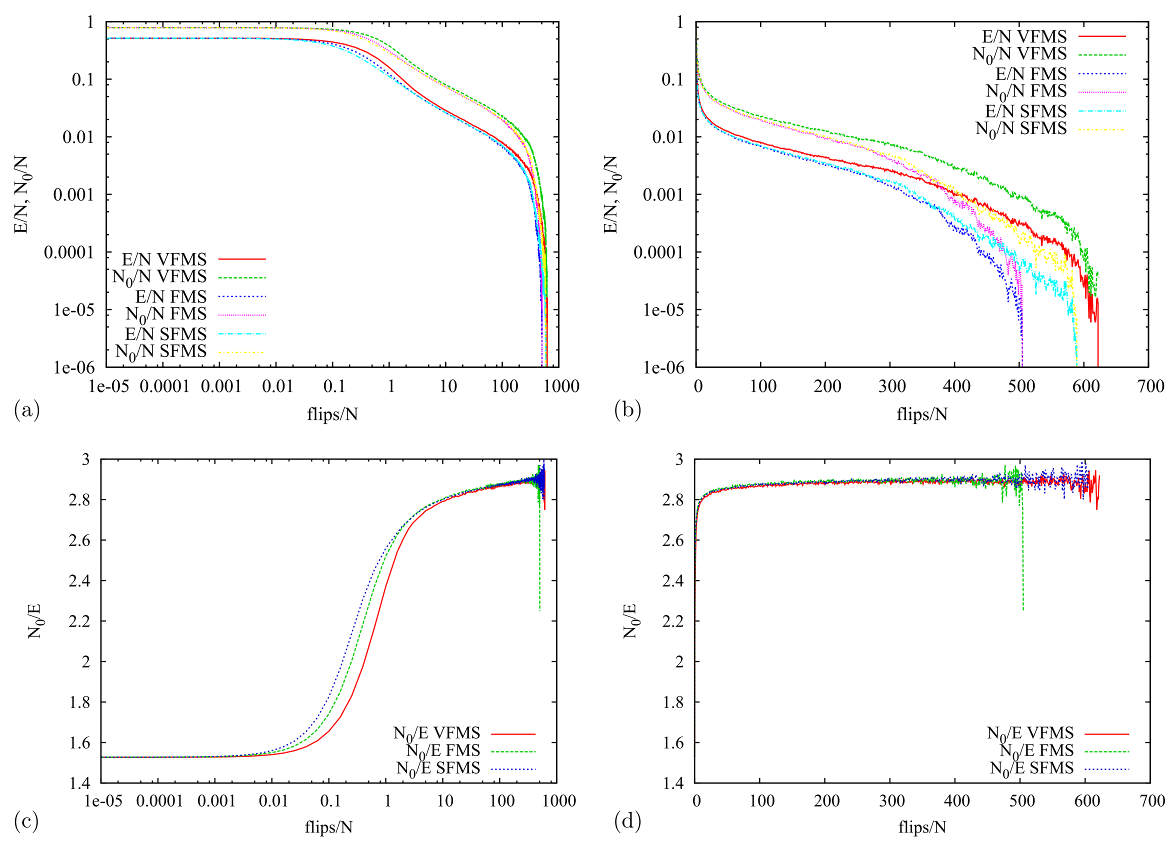

FIG. 4. (Color online) Top panel: average traces, over 20 instances, of the energy E and the number of variables in unsatisfied clauses N_0, for V-FMS ($\eta = 0.23$), FMS ($\eta = 0.33$), and S-FMS ($\eta = 0.53$), with logarithmic (a) and linear (b) abscissa, $\alpha = 4.12$ and $N = 10^5$. Bottom panel: same results, but showing the evolution of N_0/E with logarithmic (c) and linear (d) abscissa.

that the product of the picking and flipping rates is different. Thus it is not trivial that these different algorithms have a very similar performance as the usual FMS with a simple rescaling of parameters.

The variable focusing shows that one has considerable freedom in the choice of focused local search algorithms. This freedom could be used to design successful new algorithms, for instance in the direction of Ref. [20], or to understand the properties of the random K-SAT problem. Indeed, this result can alternatively be seen as a kind of degeneracy of this problem, as these different algorithms seem to make very little difference as far as running times are concerned, if parameters are optimized properly.

We also show that the random (infinite noise) versions of the algorithms we study perform quite differently on random 3-SAT than the standard random WalkSAT. Choosing with increased probability variables in many unsatisfied clauses increases the performance of the random algorithms, so that the square-focused random algorithm solves instances of random 3-SAT linearly close to $\alpha = 3.1$. It is peculiar that in this case the sampling following from variable-based focusing can shift the algorithmic threshold (of the linear time regime) noticeably, whereas in the "real applications" case we mostly concentrate on, this is not the case. A calculation of the dependence of the linearity threshold by a rate equation approach [16,17] might reveal whether the tuning by sampling arises just due to the different flipping rates of variables in the diverse FMS variants due to the different focusing, or if more complicated correlations build up.

The idea of variable focusing arises naturally in theoretical approaches based on describing the algorithm dynamics by a master equation, since the natural time scale for asynchronous updates is 1 over the number of variables [13]. Here this idea is applied only to Metropolis rates, but the concept is general, and it is not clear whether other focused algorithms, like WalkSAT or record-to-record travel, would show more interesting or surprising behaviors than FMS when using other kinds of focusing, such as variable and square focusing, studied here. It is not clear either how these results, even for FMS, translate to higher K, like 4 or 5. This work also questions how "greedy" focusing should be. When variables in unsatisfied clauses are picked with a probability proportional to the power b of the number of unsatisfied clauses they are involved in, only $b = 1$ was studied in the literature, and we study here $b = 0$ (variable focusing) and $b = 2$ (square focusing) for Metropolis rates. What are the results when b is strictly bigger than 2, both at optimal and infinite noise, depending on b? For Metropolis rates, our results show that the optimal noise increases with b and that random (infinite noise) algorithms perform significantly better with increasing b. This suggests that for some finite value of b, infinite noise might be optimal, and that the tuning of the noise parameter could be replaced by the tuning of focusing through b. This would be interesting also because the random algorithms we consider use less information than the algorithms with finite noise: they use only the number of unsatisfied clauses each variable is involved in ("makecount" in WalkSAT [3]), while finite noise algorithms use also the number of clauses when each variable is the only one to satisfy ("breakcount"), in order to compute changes in energy E.

Another point we have raised is that many other quantities than the conventional "energy" E can be considered to design efficient local search, and we have pointed out that one natural such quantity could be N_0, the number of variables in unsatisfied clauses. Higher moments (in our notation $N_2, N_3, \ldots$) of the distribution, for variables in UNSAT clauses, of the number of unsatisfied clauses they are involved in, could also be studied. Some such alternative quantities have already been considered in the past such as "breakcount" (number of clauses which become unsatisfied if a given variable is flipped) and "makecount" (number of clauses which become satisfied) in WalkSAT [3], and the remembered value of a previous energy minimum in focused record-to-record travel [6]; N_0 has however the conceptual advantage that it is simpler and similarly to E can be naturally extended to a global landscape. The tests for 3-SAT reveal that V-FMS and S-FMS are in practice not better or worse than the usual focused Metropolis search. Tests also revealed that E and N_0 are proportional (or nearly so) after an initial transient, and a value is reached which indeed indicates that eligible variables tend to partake only in one unsatisfied clause. Since this is close to the "maximum efficiency" for the ratio of the two quantities, it is an interesting question how to find algorithms that differ in this sense, and if they would be found to be improved over FMS.

The apparent boundary of (typical) linear behavior of focused local search on random K-SAT is hence approximately at the same value of $\alpha \approx 4.23$ with quite different choices of local search, as has been found in the past [6], and also for different kinds of focusing, as found here. Why this is, and if this boundary is in some sense universal, or if other versions of focusing can be designed which perform qualitatively differently, are important challenges for the future.

ACKNOWLEDGMENTS

The authors thank A. Mozeika and S. Seitz for discussions about variable focusing. Professor P. Orponen (Aalto) is acknowledged for discussions. This work was supported by funding from the Center of Excellence program of the Academy of Finland, for the COIN and COMP Centers, as well as the Academy of Finland Distinguished Professor program (E.A.). We acknowledge the computational resources provided by the Aalto Science-IT project.

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