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Elliptical vortex solutions, integrable Ermakov structure, and Lax pair formulation of the compressible Euler equations

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The 2+1-dimensional compressible Euler equations are investigated here. A power-type elliptic vortex ansatz is introduced and thereby reduction obtains to an eight-dimensional nonlinear dynamical system. The latter is shown to have an underlying integral Ermakov-Ray-Reid structure of Hamiltonian type. It is of interest to notice that such an integrable Ermakov structure exists not only in the density representations but also in the velocity components. A class of typical elliptical vortex solutions termed *pulsrodons* corresponding to warm-core eddy theory is isolated and its behavior is simulated. In addition, a Lax pair formulation is constructed and the connection with stationary nonlinear cubic Schrödinger equations is established.

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I. INTRODUCTION

The Navier-Stokes equations describing fluid motion are fundamental equations in hydromechanics, which were first introduced by Navier in 1821 and developed by Stokes in 1845. In many cases, the viscosity of the fluids is quite small and may be neglected, which reduces the Navier-Stokes equations to the Euler equations. Hence, one can regard the Euler equations as the limit cases of the Navier-Stokes equations for a large Reynolds number [1]. In addition to the fundamental application in fluids, the Euler equations are also found in many other physical fields such as plasmas, condensed matters, astrophysics, oceanic and atmospheric dynamics, etc. [2–6]. Due to their extensive applications, much literature has been devoted toward seeking exact solutions of the Euler equations [7–11], especially for the investigation of their integral structures. For example, vortex structures underlying the 2D Euler equations were investigated by Kirchhoff [12], while the Hamiltonian structures were discussed by Arnold [13]. Extensive studies on the symplectic structures underlying the 2D Euler model were carried out by Marsden and Ratiu *et al.* [14]. Lax pair formulations were investigated by Friedlander and Vishik in the Lagrangian coordinates [15] and subsequently by Li in the Eulerian coordinates [16,17]. Important contributions were also made by Childress and Lou [18] with regard to the Lax pairs of the Euler equations.

As an important integrable characteristic of a differential equation, the Ermakov-Ray-Reid systems have their origin in the work of Ermakov [19] and were developed by Ray and Reid [20,21]. The main theoretical interest in the systems centers around their admittance of a distinctive integral of motion, namely the Ermakov-Lewis-Ray-Reid invariant together with a concomitant nonlinear superposition principle (e.g., see Refs. [22–25]). In terms of physical applications, Ermakov systems are found in nonlinear optics [26–29], hydrodynamics [30], quantum mechanics [31], elasticity [32], cosmology [33], molecular structures [34], partial differential equations of mathematical physics such as the Riccati equations and nonlinear Schrödinger equations, etc. [35,36].

It is known that the rotating shallow water equation is an approximation to the Euler equations [37]. The former is shown by Rogers and An to admit an underlying integrable structure of Ermakov-Ray-Reid type and thereby exact vortex solutions are constructed [30]. Importantly, such vortex solutions prove to be useful in tidal oscillations, warm-core rings, and other upper-ocean phenomena *et al.* [38–40]. This naturally motivates us to consider whether the Euler equations, like the rotating shallow water system, have an underlying integral structure of Ermakov-Ray-Reid type. If so, what kind of exact solutions can be constructed, accordingly? Can the solutions obtained be applied to explain or predict any phenomenon in physical areas mentioned above? With these questions in mind, we expand the investigations on integral Ermakov structures for the (2+1)-dimensional compressible Euler equations.

The plan of this paper is as follows. In Sec. II, a power-type elliptic vortex ansatz is introduced, and thereby the compressible Euler equations are reduced to a set of nonlinear dynamical system that generalizes what has been given in Refs. [30,38]. Time-modulated physical variables are introduced and the dynamical system is reducible to a form amenable to exact solutions. In Sec. III, it is shown that the nonlinear dynamical system, remarkably, admits an underlying integrable Ermakov structure, which also takes a Hamiltonian form. Interestingly, it is noticed that such an integrable Ermakov structure exists not only in the density quantities, but also in the velocity components, at least in a special reduction. In Sec. IV, a class of typical *pulsrodon* solutions with a breather-type free boundary oscillation is isolated and its behaviors are exhibited. In Sec. V,

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a Lax pair formulation for the compressible Euler equations is constructed and its connection with stationary nonlinear Schrödinger equations is given. Finally, a short conclusion is attached.

II. A POWER-TYPE ELLIPTIC VORTEX ANSATZ OF THE EULER EQUATIONS

The 2+1-dimensional compressible Euler equations considered here take the following form:

$$\begin{aligned}\rho_t + \nabla\cdot(\rho\mathbf{U}) &= 0\\ \rho[\mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} + f\,\mathbf{k}\times\mathbf{U}] + \nabla p &= \mathbf{0},\end{aligned} \tag{2.1}$$

where f is the Coriolis force and $\mathbf{k}$ denotes a unit orthogonal basis of the Eulerian coordinates. While ρ, $\mathbf{U}$, and p stands for the fluid density, velocity and pressure, respectively. And the pressure p is given by

$$p = K\rho^{\gamma}, \tag{2.2}$$

with $\gamma > 0$, and K is a time-dependent function to be determined.

A. A power-type elliptic vortex ansatz

Exact solutions and integrable structure of the compressible Euler Eqs. (2.1) are now sought via a power-type elliptic vortex ansatz

$$\begin{aligned}&\rho = [\mathbf{x}^T\mathbf{E}(t)\mathbf{x} + d(t)]^{\frac{1}{\gamma-1}}, \quad \gamma\neq 1,\\ &\mathbf{U} = \mathbf{L}(t)\mathbf{x} + \mathbf{H}(t),\end{aligned} \quad \mathbf{x} = \begin{bmatrix} x-\bar{q}(t)\\ y-\bar{p}(t)\end{bmatrix}, \tag{2.3}$$

where $\mathbf{L}(t)$ and $\mathbf{E}(t)$ are 2×2 matrices depending only on time, and $\mathbf{E}(t)$ is symmetric and positive definite, namely

$$\mathbf{L} = \begin{bmatrix} u_1(t) & u_2(t)\\ v_1(t) & v_2(t)\end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} a(t) & b(t)\\ b(t) & c(t)\end{bmatrix},$$

$$\text{and}\quad \mathbf{H} = \begin{bmatrix}\dot{\bar{q}}(t)\\ \dot{\bar{p}}(t)\end{bmatrix}. \tag{2.4}$$

This solution ansatz represents a swirling, rotating fluid mass in the shape of a horizontally truncated ellipsoid. Its upper boundary is the flat, horizontal ellipse at $\rho(x,y,t) = 0$. What needs to be pointed out is that the precluded case $\gamma = 1$ may be readily treated via the exponential-type vortex ansatz given by Rogers and An. Readers may refer to Ref. [41] for the details. Therefore, without loss of generality, we shall proceed with the case $\gamma \neq 1$ in the present work.

Substitution of (2.3) into the continuity Eq. $(2.1)_1$ yields

$$\begin{pmatrix}\dot a\\ \dot b\\ \dot c\end{pmatrix} + \begin{pmatrix} 2u_1 + (\gamma-1)(u_1+v_2) & 2v_1 & 0\\ u_2 & \gamma(u_1+v_2) & v_1\\ 0 & 2u_2 & 2v_2 + (\gamma-1)(u_1+v_2)\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \mathbf{0}, \tag{2.5}$$

together with

$$\dot d + d(\gamma-1)(u_1+v_2) = 0\,. \tag{2.6}$$

Insertion of Eq. (2.3) into the momentum Eq. $(2.1)_2$ produces

$$\begin{pmatrix}\dot u_1\\ \dot u_2\\ \dot v_1\\ \dot v_2\end{pmatrix} + \begin{pmatrix}\mathbf{L}^T & -f\mathbf{I}\\ f\mathbf{I} & \mathbf{L}^T\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ v_1\\ v_2\end{pmatrix} + \frac{2\gamma K}{\gamma-1}\begin{pmatrix}a\\ b\\ b\\ c\end{pmatrix} = \mathbf{0}, \tag{2.7}$$

augmented by the linear auxiliary equations

$$\ddot{\bar q} - f\,\dot{\bar p} = 0\,, \qquad \ddot{\bar p} + f\,\dot{\bar q} = 0\,. \tag{2.8}$$

At this stage, the spacial structure of the orignal Euler equations has been removed. Hence, the solution of the Euler equations is encoded in the seven-dimensional time-dependent nonlinear system Eqs. (2.5) and (2.7). Once the solution of the latter is known, the quantities d, $\bar p$, and $\bar q$ can be easily obtained via the relation Eqs. (2.6) and (2.8).

Some relations that are key to the subsequent development are obtained. These may be established by appealing to the system Eqs. (2.5)–(2.7) and are now recorded in the following theorem.

Theorem Let

$$R = v_1 - u_2 + f\,, \quad \Delta = ac - b^2\,,$$

$$M = a\,(u_2 - f/2) + b\,(v_2 - u_1) - c\,(v_1 + f/2)\,,$$

$$\begin{aligned}Q = &-a\left(u_2^2+v_2^2\right) + 2b\left(u_1u_2+v_1v_2\right) - c\left(u_1^2+v_1^2\right)\\ &+ \frac{4\gamma K}{(\gamma-1)^2}\Delta - \frac{4\gamma\Delta}{(\gamma-1)^2}\exp\left[\int(\gamma-1)(u_1+v_2)dt\right]\\ &\times\int\dot K\exp\left[\int(1-\gamma)(u_1+v_2)dt\right]dt,\end{aligned} \tag{2.9}$$

then we get the following relations

$$\boxed{\begin{aligned}\dot d &= -(\gamma-1)(u_1+v_2)d\\ \dot R &= -(u_1+v_2)R\\ \dot\Delta &= -2\gamma(u_1+v_2)\Delta\\ \dot M &= -(\gamma+1)(u_1+v_2)M\\ \dot Q &= -(\gamma+1)(u_1+v_2)Q\end{aligned}}\,. \tag{2.10}$$

Interestingly, the latter two relations constitute a generalization of those derived in the context of shallow water theory [30,38]. Moreover, appropriate combinations of the quantities d, R, Δ, M, and Q lead to the important physical invariants of the Euler equations, such as the energy conservation, the potential vorticity and angular momentum conservation, etc.

B. Associated reductions via canonical variables

In the sequel, it proves convenient to proceed in terms of new variables as previously employed in a hydrodynamics context in Refs. [30,38], namely

$$\begin{aligned} G &= u_1 + v_2, \quad G_R = \tfrac{1}{2}(v_1 - u_2), \\ G_S &= \tfrac{1}{2}(v_1 + u_2), \quad G_N = \tfrac{1}{2}(u_1 - v_2), \\ B &= a + c, \quad B_S = b, \quad B_N = \tfrac{1}{2}(a - c). \end{aligned} \tag{2.11}$$

Thus, G and G_R represent the modified versions of the divergence and spin of the velocity field, while G_S and G_N represent the modified shear and normal deformation rates. Then, the system Eqs. (2.5)–(2.7) adopts the form

$$\begin{aligned} \dot{B} + \gamma BG + 4(B_N G_N + B_S G_S) &= 0, \\ \dot{B}_S + \gamma B_S G + BG_S - 2B_N G_R &= 0, \\ \dot{B}_N + \gamma B_N G + BG_N + 2B_S G_R &= 0, \\ \dot{G} + \frac{1}{2}G^2 + 2\left(G_N^2 + G_S^2 - G_R^2\right) - 2fG_R + \frac{2\gamma K}{\gamma - 1}B &= 0, \\ \dot{G}_R + GG_R + \frac{1}{2}fG &= 0, \\ \dot{G}_N + GG_N - fG_S + \frac{2\gamma K}{\gamma - 1}B_N &= 0, \\ \dot{G}_S + GG_S + fG_N + \frac{2\gamma K}{\gamma - 1}B_S &= 0, \end{aligned} \tag{2.12}$$

together with

$$\dot{d} + d(\gamma - 1)G = 0. \tag{2.13}$$

The form of Eq. $(2.12)_4$ suggests introducing a function Ω via

$$G = \frac{2\dot{\Omega}}{\Omega}, \tag{2.14}$$

so that Eqs. $(2.12)_5$ and (2.13) show that

$$G_R = \frac{c_0}{\Omega^2} - \frac{f}{2}, \quad d = c_1 \Omega^{2(1-\gamma)}, \tag{2.15}$$

where c_0 and c_1 are integration constants.

New Ω-modulated variables involving the gas parameter γ are now introduced according to

$$\begin{aligned} &\bar{B} = B\Omega^{2\gamma}, \quad \bar{B}_S = B_S\Omega^{2\gamma}, \quad \bar{B}_N = B_N\Omega^{2\gamma}, \\ &\bar{G}_S = G_S\Omega^2, \quad \bar{G}_N = G_N\Omega^2, \end{aligned} \tag{2.16}$$

whence the nonlinear system Eq. (2.12) reduces to

$$\begin{aligned} \dot{\bar{B}} + \frac{4(\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S)}{\Omega^2} &= 0, \\ \dot{\bar{B}}_S + f\bar{B}_N + \frac{\bar{B}\bar{G}_S - 2c_0\bar{B}_N}{\Omega^2} &= 0, \\ \dot{\bar{B}}_N - f\bar{B}_S + \frac{\bar{B}\bar{G}_N + 2c_0\bar{B}_S}{\Omega^2} &= 0, \\ \dot{\bar{G}}_S + f\bar{G}_N + \frac{2\gamma}{\gamma - 1}\frac{K\bar{B}_S}{\Omega^{2(\gamma-1)}} &= 0, \\ \dot{\bar{G}}_N - f\bar{G}_S + \frac{2\gamma}{\gamma - 1}\frac{K\bar{B}_N}{\Omega^{2(\gamma-1)}} &= 0, \end{aligned} \tag{2.17}$$

together with the second-order nonlinear differential equation

$$\Omega^3\ddot{\Omega} + \frac{1}{4}f^2\Omega^4 + \bar{G}_N^2 + \bar{G}_S^2 - c_0^2 + \frac{\gamma}{\gamma - 1}\frac{K\bar{B}}{\Omega^{2(\gamma-2)}} = 0. \tag{2.18}$$

It is the seven-dimensional nonlinear dynamical system Eqs. (2.17)–(2.18) that will be analyzed in details to construct explicit solutions of the 2+1-dimensional compressible Euler equations.

C. The constraints and first integrals

It is seen that combination of Eqs. $(2.17)_2$ and $(2.17)_3$ together with use of Eq. $(2.17)_1$ produces an integral of motion

$$\bar{B}_S^2 + \bar{B}_N^2 - \tfrac{1}{4}\bar{B}^2 = c_2. \tag{2.19}$$

While, in a similar way, combination of Eqs. $(2.17)_4$ and $(2.17)_5$ delivers

$$\bar{G}_S^2 + \bar{G}_N^2 - \frac{\gamma}{\gamma - 1}\int K\dot{\bar{B}}\Omega^{2(2-\gamma)}dt = 0. \tag{2.20}$$

Observation shows that when

$$\bar{B} = \text{const} \quad \text{or} \quad K = k_1'\Omega^{2(\gamma-2)}, \quad \bar{B} \neq \text{const}, \tag{2.21}$$

a second integral of motion is obtained. The former case leads to the *pulsrodon* solutions in warm-core eddy theory, which is discussed in Sec. IV. While, the latter gives rise to the integrable Hamiltonian Ermakov structure of the compressible Euler equations. For convenience, we first proceed with the latter case, wherein $\bar{B} \neq$ const and the modulation variable K adopts

$$K = k_1'\Omega^{2(\gamma-2)}, \tag{2.22}$$

so that the second integral of motion is

$$\bar{G}_S^2 + \bar{G}_N^2 - k_1\bar{B} = c_3. \tag{2.23}$$

with arbitrary constant c_3 and $k_1 = \frac{\gamma k_1'}{\gamma - 1}$. Remarkably, in this case, the above two integral variants coincide with that have been obtained in the f-plane elliptic warm-core eddy analysis in Refs. [30,38], which is shown to be integrable in the sense of Liouville [42]. However, in the current context, these first integrals exist for the arbitrary γ. Moreover, the nonlinear dynamical system Eq. (2.17) is shown to admit another "hidden" integrals of motion,

$$\begin{aligned} &2(\bar{B}_N\bar{G}_S - \bar{B}_S\bar{G}_N) - c_0\bar{B} = c_4, \\ &\frac{4c_2k_1}{\Omega^2} - \frac{1}{2}\dot{\bar{B}}G\Omega^2 + 2G_R(c_4 + c_0\bar{B}) \\ &\quad - \frac{\bar{B}}{\Omega^2}\left[c_3 + k_1\bar{B} + \Omega^4\left(G_R^2 + \frac{G^2}{4}\right)\right] = c_5. \end{aligned} \tag{2.24}$$

These can be considered as an avatar of the last two relations of Eq. (2.10) and may be readily validated with the aid of the computation software *Maple*.

In summary, under the constraint of Eq. (2.22), the nonlinear dynamical system Eqs. (2.17)–(2.18) admits four integrals of motions, namely

$$\begin{aligned}&\bar{B}_S^2+\bar{B}_N^2-\frac{1}{4}\bar{B}^2=c_2,\quad \bar{G}_S^2+\bar{G}_N^2-k_1\bar{B}=c_3,\\&2(\bar{B}_N\bar{G}_S-\bar{B}_S\bar{G}_N)-c_0\bar{B}=c_4,\\&\frac{4c_2k_1}{\Omega^2}-\frac{1}{2}\dot{\bar{B}}G\Omega^2+2G_R(c_4+c_0\bar{B})\\&\quad-\frac{\bar{B}}{\Omega^2}\left[c_3+k_1\bar{B}+\Omega^4\left(G_R^2+\frac{G^2}{4}\right)\right]=c_5.\end{aligned}\tag{2.25}$$

These invariants prove to be important in the construction of exact solutions and integrable Ermakov structure of the 2+1-dimensional Euler equations. Their relevance will be analyzed in the following section.

III. HAMILTONIAN ERMAKOV STRUCTURE AND INTEGRABILITY

Here, it is demonstrated that the nonlinear dynamical system Eqs. (2.17) and (2.18) has a remarkable underlying integrable structure of the Ermakov-Ray-Reid type, which turns out to also be Hamiltonian. Further investigations show that such Hamiltonian Ermakov systems not only exist in the density quantities but also in the velocity components, at least in a particular reduction.

A. A Hamiltonian Ermakov system in the density quantities

In order to show that the dynamical system Eqs. (2.17) and (2.18) admits Ermakov-Ray-Reid system of Hamiltonian type, we now turn back to reconsider the integral of motions Eq. (2.25). It is noted that the identity

$$\begin{aligned}&(\bar{B}_N\bar{G}_N+\bar{B}_S\bar{G}_S)^2+(\bar{B}_N\bar{G}_S-\bar{B}_S\bar{G}_N)^2\\&\quad=\left(\bar{B}_S^2+\bar{B}_N^2\right)\left(\bar{G}_S^2+\bar{G}_N^2\right)\end{aligned}\tag{3.1}$$

holds. Thus, on the one hand, combination of the relation Eqs. $(2.17)_1$ and (2.25) leads to

$$\dot{\bar{B}}^2=\frac{4}{\Omega^4}[(c_4+c_0\bar{B})^2-(c_3+k_1\bar{B})(\bar{B}^2+4c_2)],\tag{3.2}$$

whereas the differential Eq. $(2.17)_{4,5}$ yields

$$\arctan(\bar{G}_N/\bar{G}_S)=ft-\int\frac{c_4+c_0\bar{B}}{\Omega^2}dt,\tag{3.3}$$

under the constraint condition Eq. (2.22). It reminds us to consider the nonlinear Eq. (2.18), namely

$$\Omega^3\ddot{\Omega}+\tfrac{1}{4}f^2\Omega^4+c_3-c_0^2+2k_1\bar{B}=0.\tag{3.4}$$

The latter, as it stands, is intractable unless $\bar{B}=\lambda+\mu\Omega^3+\nu\Omega^4$ $(\lambda,\mu,\nu\in R)$ when it reduces to the Steen-Ermakov-Pinney equation [19,43,44]. However, in general, it is shown that the nonlinear dynamical system Eqs. (2.17) and (2.18) is encoded in the coupled pair of differential Eqs. (3.2) and (3.4), namely

$$\begin{aligned}&\ddot{\Omega}+\frac{1}{4}f^2\Omega=\frac{1}{\Omega^3}(c_0^2-c_3-2k_1\bar{B}),\\&\ddot{\bar{B}}+\frac{2\dot{\Omega}}{\Omega}\dot{\bar{B}}-\frac{\bar{B}}{\bar{B}^2+4c_2}\dot{\bar{B}}^2\\&\quad=\frac{2}{\Omega^4}\left[k_1(\bar{B}^2+4c_2)+\frac{2(c_4+c_0\bar{B})(c_4\bar{B}-4c_0c_4)}{(\bar{B}^2+4c_2)}\right].\end{aligned}\tag{3.5}$$

Accordingly, once the functions Ω and $\bar{B}$ are known, the remaining quantities $\bar{B}_S$, $\bar{B}_N$, $\bar{G}_S$, $\bar{G}_N$ may be readily obtained via Eqs. (2.25) and (3.3).

Interestingly, it turns out that the coupled Eqs. (3.5) may be reformulated as a Ermakov-Ray-Reid system. Indeed, for instance, in the case of $c_1>0$ and $\bar{B}^2\geqslant -4c_2>0>\bar{B}$, then, at any fixed time t, the semiaxes of the elliptic curves of constant density $\rho=$ const are given by

$$\begin{aligned}\Phi&=\sqrt{\frac{2d(t)}{-\sqrt{(a-c)^2+4b^2}-(a+c)}}=\Omega\sqrt{\frac{c_1}{-2c_2}(-\bar{B}+\sqrt{\bar{B}^2+4c_2})},\\\Psi&=\sqrt{\frac{2d(t)}{\sqrt{(a-c)^2+4b^2}-(a+c)}}=\Omega\sqrt{\frac{c_1}{-2c_2}(-\bar{B}-\sqrt{\bar{B}^2+4c_2})},\end{aligned}\tag{3.6}$$

as proposed by Rogers and An in Ref. [30] in the hydrodynamic shallow water theory. Then, in these variables, the system Eq. (3.5) adopts the form of Ermakov-Ray-Reid type:

$$\boxed{\begin{aligned}\ddot{\Phi}+\frac{1}{4}f^2\Phi&=\frac{1}{\Phi^2\Psi}F(\Psi/\Phi)=\frac{1}{\Phi^2\Psi}\left[J(\Phi/\Psi)-\frac{\Phi}{\Psi}J'(\Phi/\Psi)-\frac{2k_1c_1^2}{\sqrt{-c_2}}\right]\\\ddot{\Psi}+\frac{1}{4}f^2\Psi&=\frac{1}{\Phi\Psi^2}G(\Phi/\Psi)=\frac{1}{\Phi\Psi^2}\left[J(\Phi/\Psi)+\frac{\Phi}{\Psi}J'(\Phi/\Psi)-\frac{2k_1c_1^2}{\sqrt{-c_2}}\right]\end{aligned}},\tag{3.7}$$

with

$$J(\omega)=J(\omega^{-1})=\frac{c_1^2}{2c_2^2}\left[(c_4+2c_0\sqrt{-c_2})^2\frac{\omega}{(\omega+1)^2}+(c_4-2c_0\sqrt{-c_2})^2\frac{\omega}{(\omega-1)^2}\right].\tag{3.8}$$

Moreover, deep inspection shows that the associated Ermakov-Ray-Reid system Eq. (3.7) has an additional property of adopting a Hamiltonian form

$$\frac{d}{dt}\frac{\partial\mathcal{H}}{\partial\dot{\Phi}}=-\frac{\partial\mathcal{H}}{\partial\Phi},\quad \frac{d}{dt}\frac{\partial\mathcal{H}}{\partial\dot{\Psi}}=-\frac{\partial\mathcal{H}}{\partial\Psi},\tag{3.9}$$

with the Hamiltonian invariant

$$\mathcal{H}=\frac{1}{2}(\dot{\Phi}^2+\dot{\Psi}^2)+\frac{1}{8}f^2(\Phi^2+\Psi^2)+\frac{1}{\Phi\Psi}J(\Phi/\Psi)-\frac{1}{\Phi\Psi}\frac{2k_1c_1^2}{\sqrt{-c_2}}+\frac{c_1}{2c_2}(c_5+c_4f). \tag{3.10}$$

Accordingly, the two integrals of motion, namely, the Ray-Ray invariant and Hamiltonian invariant of the system Eq. (3.7), allow their completely integrable and analytical solutions may be explicitly obtained via the procedure described in Ref. [30].

B. A Hamiltonian Ermakov system in the velocity quantities

It has been established that the semiaxes $\{\Phi,\Psi\}$ of the time-modulated ellipse associated with the density representation Eq. (2.3) are governed by an integrable Ermakov-Ray-Reid system, albeit of some complexity. But now, we would like to show that such a Ermakov-Ray-Reid system is also associated with the velocity components, at least, in a particular reduction.

Thus, our attention here is restricted to the irrotational motions with $\mathbf{L}$ and $\mathbf{E}$ in Eq. (2.3) given by

$$\mathbf{L}=\begin{bmatrix}\dot{\alpha}(t)/\alpha(t) & 0\\ 0 & \dot{\beta}(t)/\beta(t)\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}a(t) & 0\\ 0 & c(t)\end{bmatrix}. \tag{3.11}$$

It is observed that this kind of solution is indeed a specialization of Eq. (2.4). Insertion of Eq. (3.11) into the continuity Eq. (2.1) produces

$$\begin{aligned}&\frac{\dot{a}}{a}+(\gamma-1)\frac{\dot{\beta}}{\beta}+(\gamma+1)\frac{\dot{\alpha}}{\alpha}=0\,,\\ &\frac{\dot{c}}{c}+(\gamma-1)\frac{\dot{\alpha}}{\alpha}+(\gamma+1)\frac{\dot{\beta}}{\beta}=0\,,\\ &\frac{\dot{d}}{d}+(\gamma-1)\left(\frac{\dot{\alpha}}{\alpha}+\frac{\dot{\beta}}{\beta}\right)=0\,,\end{aligned} \tag{3.12}$$

whence

$$\begin{aligned}&a=C_I\alpha^{-2}(\alpha\beta)^{1-\gamma},\ \ c=C_{II}\beta^{-2}(\alpha\beta)^{1-\gamma},\\ &d=C_{III}(\alpha\beta)^{1-\gamma}.\end{aligned} \tag{3.13}$$

Substitution of Eq. (3.11) into the momentum Eq. (2.1) now gives

$$\ddot{\alpha}+\frac{2\gamma K}{\gamma-1}a\alpha=0,\quad \ddot{\beta}+\frac{2\gamma K}{\gamma-1}c\beta=0. \tag{3.14}$$

On use of the first two relations of Eq. (3.13), the system Eq. (3.14) is reducible to the following coupled equations:

$$\ddot{\alpha}=\frac{1}{\alpha^2\beta}\frac{-2\gamma K}{(\gamma-1)(\alpha\beta)^{\gamma-2}},\quad \ddot{\beta}=\frac{1}{\alpha\beta^2}\frac{-2\gamma K}{(\gamma-1)(\alpha\beta)^{\gamma-2}}. \tag{3.15}$$

This becomes a Ermakov-Ray-Reid system if the modulation K adopts the form

$$K=\frac{\gamma-1}{\gamma}(\alpha\beta)^{\gamma-2}F\left(\frac{\alpha}{\beta}\right), \tag{3.16}$$

namely,

$$\ddot{\alpha}=-\frac{2}{\alpha^2\beta}F(\alpha/\beta),\quad \ddot{\beta}=-\frac{2}{\alpha\beta^2}F(\alpha/\beta). \tag{3.17}$$

This system constitutes a corresponding Hamiltonian if we additionally require $F(\alpha/\beta)$ to be

$$F(\alpha/\beta)=-c_1^*/2=\text{const}. \tag{3.18}$$

Hence the Hamiltonian Ermakov system is now written down as

$$\ddot{\alpha}=\frac{c_1^*}{\alpha^2\beta}\qquad \ddot{\beta}=\frac{c_1^*}{\alpha\beta^2}, \tag{3.19}$$

with the Hamiltonian invariant

$$H=\frac{1}{2}(\dot{\alpha}^2+\dot{\beta}^2)+\frac{c_1^*}{\alpha\beta}. \tag{3.20}$$

Now, the remaining thing is to consider the constraint condition Eq. (3.16). According to the analysis in the above section, we have

$$\text{tr}\mathbf{L}=u_1+v_2=\frac{2\dot{\Omega}}{\Omega}=\frac{\dot{\alpha}}{\alpha}+\frac{\dot{\beta}}{\beta}, \tag{3.21}$$

whence

$$\alpha\beta=c_2^*\Omega^2. \tag{3.22}$$

Therefore, substitution of Eqs. (3.22) and (3.18) into Eq. (3.16) yields

$$K=\Omega^{2(\gamma-2)}\frac{(1-\gamma)c_1^*}{2\gamma c_2^{*2-\gamma}}. \tag{3.23}$$

This is nothing, remarkably, but the constraint condition of Eq. (2.22).

It has been shown that the reduction Eq. (3.19) to a Hamiltonian Ermakov system involves the velocity component parameters, in contrast to the reduction Eq. (3.7), which involves only the density parameters. The existence of integral Hamiltonian Ermakov systems not only in the density quantities but also in the velocity emphasizes the importance of studying the 2+1-dimensional compressible Euler equations.

IV. THE PULSRODON SOLUTIONS AND NUMERICAL SIMULATIONS

The preceding analysis has been completed with $\bar{B}\neq$ const and the modulated variable K described in Eq. (2.22). In the sequel, we shall discuss the other case $\bar{B}=$ const, which leads to a subclass of typical solutions that corresponds to pulsating elliptical eddies. These are termed as *pulsrodons* since they combine the pulsating characteristic of the circular pulson with the more general elliptic geometry of the rodon.

It is observed that in the particular case $\bar{B}=$ const, the second integral of motion is recorded by

$$\bar{G}_S^2+\bar{G}_N^2=\tilde{C}^2. \tag{4.1}$$

This expression inspires us to introduce a parametrization via

$$\bar{G}_S=\tilde{C}\cos\eta,\quad \bar{G}_N=\tilde{C}\sin\eta, \tag{4.2}$$

with $\eta=\eta(t)$ to be determined. It is seen from Eq. $(2.17)_1$ that the nonlinear dynamical system Eqs. (2.17) and (2.18) is constrained by

$$\bar{B}_S\bar{G}_S + \bar{B}_N\bar{G}_N = 0, \quad (4.3)$$

which may be conveniently parametrized as

$$\bar{B}_S = \alpha\bar{G}_N = \alpha\tilde{C}\sin\eta, \quad \bar{B}_N = -\alpha\bar{G}_S = -\alpha\tilde{C}\cos\eta, \quad (4.4)$$

with $\alpha = \alpha(t)$ to be determined.

Combinations of Eqs. $(2.17)_2$ and $(2.17)_3$ on use of Eqs. (4.2) and (4.4) yield that

$$\dot{\eta} - f + \frac{\bar{B} + 2c_0\alpha}{\alpha\Omega^2} = 0, \quad \dot{\alpha} = 0. \quad (4.5)$$

Similarly, Eqs. $(2.17)_{4,5}$ are reducible to a single condition:

$$\dot{\eta} - f - \frac{2\alpha\gamma}{\gamma - 1}\frac{K}{\Omega^{2(\gamma-1)}} = 0. \quad (4.6)$$

Comparison of these two relations shows that the variable $\bar{B}$ is given by

$$\bar{B} = -2c_0\alpha - \frac{2\alpha^2\gamma}{\gamma - 1}\frac{K}{\Omega^{2(\gamma-1)}}. \quad (4.7)$$

Interestingly, in the present case $\bar{B} = \text{const}$ again requires the modulation K to be of the type of Eq. (2.22). Hence, the constant $\bar{B}$ is determined by the relation

$$\bar{B} = -2c_0\alpha - 2\alpha^2k_1; \quad (4.8)$$

accordingly, η is given by

$$\eta = ft + 2\alpha k_1\int\frac{1}{\Omega^2}dt. \quad (4.9)$$

Moreover, the system Eq. (2.18) is now reducible to an equation of Steen-Ermakov-Pinney type

$$\ddot{\Omega} + \frac{1}{4}f^2\Omega = \frac{\delta_0}{\Omega^3}, \quad \delta_0 = c_0^2 - \tilde{C}^2 + 2k_1c_0\alpha + 2\alpha^2k_1^2. \quad (4.10)$$

The latter originated in the work of Steen [43] and arises in a wide range of areas of physical importance, most notably in nonlinear optics, quantum mechanics (see, e.g., Refs. [45–47]). Another version of this equation has appeared recently in a study of pulsons by Sutyrin [48]. It is distinguished by its admittance of a nonlinear superposition principle, which was derived by the Lie group method [49]. Thus, the general solution of Eq. (4.10) is given by

$$\Omega = \sqrt{\delta_1\Omega_1^2 + 2\delta_2\Omega_1\Omega_2 + \delta_3\Omega_2^2}, \quad (4.11)$$

where Ω_1, Ω_2 are linearly independent solutions of the associated linear oscillator equation

$$\ddot{\Omega} + \tfrac{1}{4}f^2\Omega = 0,$$

with unit Wronskian, namely $W(\Omega_1,\Omega_2) = \Omega_1\dot{\Omega}_2 - \Omega_2\dot{\Omega}_1 = 1$ and the constants δ_i are constrained by the relation

$$\delta_1\delta_3 - \delta_2^2 = \delta_0. \quad (4.12)$$

If we choose Ω_1 and Ω_2 as

$$\Omega_1 = \cos\frac{f}{2}t, \quad \Omega_2 = \frac{2}{f}\sin\frac{f}{2}t, \quad (4.13)$$

then the general solution of the Steen-Ermakov-Pinney equation is determined by

$$\Omega = \sqrt{\delta_4\cos(ft + \theta) + \delta_5}, \quad (4.14)$$

where the constants δ_4 and δ_5 are related by

$$f^2(\delta_4^2 - \delta_5^2) + 4\delta_0 = 0, \quad \theta = \arctan\frac{2\delta_2}{f(\delta_5 - \delta_1)}. \quad (4.15)$$

The reality constraints associated with the relations Eqs. (4.14) and (4.15) require that

$$\delta_5 > \delta_4 \geqslant 0, \quad \tilde{C}^2 < \alpha^2k_1^2 + (c_0 + \alpha k_1)^2, \quad (4.16)$$

without loss of generality.

Thus, in the present case of $\bar{B} = \text{const}$, a subclass of analytical solutions of the 2+1-dimensional compressible Euler equations is now obtained, where the velocity components are given by

$$u_1 = \frac{\dot{\Omega}}{\Omega} + \frac{\tilde{C}}{\Omega^2}\sin\eta, \quad v_1 = \frac{\tilde{C}}{\Omega^2}\cos\eta + \frac{c_0}{\Omega^2} - \frac{1}{2}f, \quad (4.17)$$

$$u_2 = \frac{\tilde{C}}{\Omega^2}\cos\eta - \frac{c_0}{\Omega^2} + \frac{1}{2}f, \quad v_2 = \frac{\dot{\Omega}}{\Omega} - \frac{\tilde{C}}{\Omega^2}\sin\eta,$$

and the density components are

$$a = \frac{-1}{\Omega^{2\gamma}}(2\alpha c_0 + 2k_1\alpha^2 + \alpha\tilde{C}\cos\eta), \quad b = \frac{1}{\Omega^{2\gamma}}\alpha\tilde{C}\sin\eta, \quad (4.18)$$

$$c = \frac{-1}{\Omega^{2\gamma}}(2\alpha c_0 + 2k_1\alpha^2 - \alpha\tilde{C}\cos\eta), \quad d = \frac{c_1}{\Omega^{2(\gamma-1)}}.$$

Remarkably, these subclass solutions are analogous to the *pulsrodons* in the f-plane elliptic warm-core eddy theory [38] and if $\gamma = 2$, they are nothing but the solutions derived in Ref. [38].

Below, the exact solutions for the moving shoreline $\rho = 0$ are used to exhibit typical eddy boundary evolution. Figure 1 shows the time evolution of a small eccentricity elliptical eddy. From the figure, one can see that the clockwise rotation of the elliptical mode is successive but irregular, being faster when the eddy is expanded (wider rim) and slower when the eddy is contracted (smaller rim). A plausible explanation is as follows: for a given eccentricity, the larger the eddy, the greater the radius of curvature at the extremities compared to the radius

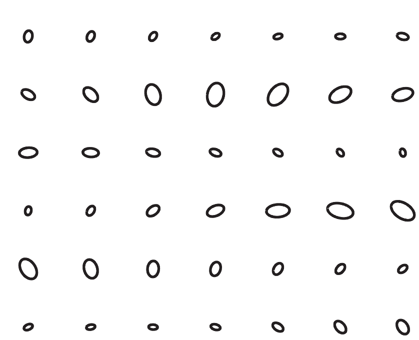

FIG. 1. The temporal evolution of a small eccentricity elliptical eddy.

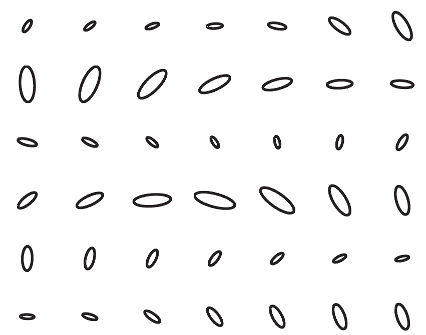

FIG. 2. The temporal evolution of a large eccentricity elliptical eddy.

of inertia, and the lesser the inertial tendency for a particle to overshoot the rim's curve at its point of maximum curvature. In Fig. 2, the eccentricity of the eddy is increased and the same behavior is displayed. Interestingly, this evolution of the upper free surface for such a typical pulsrodon coincides with an oscillating "breather-type" motion.

V. A LAX PAIR FORMULATION

It is now shown that for the arbitrary γ, the nonlinear dynamical system Eqs. (2.5)–(2.7) arising from the power-type elliptic vortex ansatz admits an associated Lax pair representation. For this purpose, we reformulate the nonlinear dynamical system Eqs. (2.5)–(2.7) into a compact matrix form,

$$\begin{gathered}\dot{\mathbf{E}}+\mathbf{E}\mathbf{L}+\mathbf{L}^T\mathbf{E}+(\gamma-1)\mathbf{E}\,\mathrm{tr}\mathbf{L}=\mathbf{0},\\ \dot{\mathbf{L}}+\mathbf{L}^2+f\,\mathbf{M}\mathbf{L}+\frac{2\gamma K}{\gamma-1}\mathbf{E}=\mathbf{0},\quad \gamma\neq 1,\end{gathered}\tag{5.1}$$

together with the linear system

$$\dot{d}+d(\gamma-1)\mathrm{tr}\mathbf{L}=0,\quad \dot{\mathbf{H}}+f\,\mathbf{M}\mathbf{H}=\mathbf{0},\tag{5.2}$$

where $\mathbf{M}$ is the Pauli matrix:

$$\mathbf{M}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.\tag{5.3}$$

In the sequel, it proves convenient to proceed with the gauge transformation,

$$\tilde{\mathbf{L}}=\mathbf{D}\mathbf{L}\mathbf{D}^{-1}+\tfrac{1}{2}f\mathbf{M},\quad \tilde{\mathbf{E}}=\mathbf{D}\mathbf{E}\mathbf{D}^{-1},\tag{5.4}$$

with

$$\mathbf{D}=\exp\left(\tfrac{1}{2}\mathbf{M}ft\right).\tag{5.5}$$

Thus, the matrix system Eq. (5.1) may be readily reducible to

$$\begin{gathered}\dot{\tilde{\mathbf{E}}}+\tilde{\mathbf{E}}\tilde{\mathbf{L}}+\tilde{\mathbf{L}}^T\tilde{\mathbf{E}}+(\gamma-1)\tilde{\mathbf{E}}\,\mathrm{tr}\tilde{\mathbf{L}}=\mathbf{0},\\ \dot{\tilde{\mathbf{L}}}+\tilde{\mathbf{L}}^2+\frac{1}{4}f^2\mathbf{I}+\frac{2\gamma K}{\gamma-1}\tilde{\mathbf{E}}=\mathbf{0}.\end{gathered}\tag{5.6}$$

On use of the relation

$$\mathbf{M}\mathbf{P}\mathbf{M}=\mathbf{P}^T-(\mathrm{tr}\mathbf{P})\mathbf{I},\quad \forall\,\mathbf{P},\tag{5.7}$$

together with the Cayley-Hamilton identity,

$$\tilde{\mathbf{L}}^2-(\mathrm{tr}\tilde{\mathbf{L}})\tilde{\mathbf{L}}+(\det\tilde{\mathbf{L}})\mathbf{I}=\mathbf{0},\tag{5.8}$$

then the system Eq. (5.6) becomes

$$\begin{gathered}\dot{\tilde{\mathbf{Q}}}+\gamma(\mathrm{tr}\tilde{\mathbf{L}})\tilde{\mathbf{Q}}+[\tilde{\mathbf{Q}},\tilde{\mathbf{L}}]=\mathbf{0},\\ \dot{\tilde{\mathbf{L}}}+(\mathrm{tr}\tilde{\mathbf{L}})\tilde{\mathbf{L}}-(\det\tilde{\mathbf{L}})\mathbf{I}+\frac{1}{4}f^2\mathbf{I}+\frac{2\gamma K}{\gamma-1}\tilde{\mathbf{E}}=\mathbf{0},\end{gathered}\tag{5.9}$$

where $\tilde{\mathbf{Q}}$ is a trace-free matrix expressed via

$$\tilde{\mathbf{Q}}=\mathbf{M}\tilde{\mathbf{E}}.\tag{5.10}$$

Since $\mathrm{tr}\mathbf{L}=\mathrm{tr}\tilde{\mathbf{L}}=2\dot{\Omega}/\Omega$, the form of Eq. (5.9) suggests introducing new Ω-modulated matrices via

$$\bar{\mathbf{L}}=\tilde{\mathbf{L}}\Omega^2,\quad \bar{\mathbf{E}}=\tilde{\mathbf{E}}\Omega^{2\gamma},\quad \bar{\mathbf{Q}}=\tilde{\mathbf{Q}}\Omega^{2\gamma},\tag{5.11}$$

so that the system Eq. (5.9) is now reduced to

$$\begin{gathered}\dot{\bar{\mathbf{Q}}}+\Omega^{-2}\,[\bar{\mathbf{Q}},\bar{\mathbf{L}}]=\mathbf{0},\\ \dot{\bar{\mathbf{L}}}-\Omega^{-2}(\det\bar{\mathbf{L}})\mathbf{I}+\frac{1}{4}f^2\Omega^2\mathbf{I}+\frac{2\gamma K}{\gamma-1}\bar{\mathbf{E}}\Omega^{2-2\gamma}=\mathbf{0}.\end{gathered}\tag{5.12}$$

It is observed that Eq. $(5.12)_1$ can be reformulated in terms of two trace-free matrices $\bar{\mathbf{Q}}$ and $\bar{\mathbf{L}}^*$, namely

$$\dot{\bar{\mathbf{Q}}}+\Omega^{-2}\,[\bar{\mathbf{Q}},\bar{\mathbf{L}}^*]=\mathbf{0},\tag{5.13}$$

with $\bar{\mathbf{L}}^*=\bar{\mathbf{L}}-\frac{1}{2}(\mathrm{tr}\bar{\mathbf{L}})\mathbf{I}$ denoting the trace-free part $\bar{\mathbf{L}}$. While Eq. $(5.12)_2$ may be divided into two parts, namely the trace-free part,

$$\dot{\bar{\mathbf{L}}}^*+\frac{\gamma K}{\gamma-1}\Omega^{2-2\gamma}[\bar{\mathbf{Q}},\mathbf{M}]=\mathbf{0},\tag{5.14}$$

and the trace part,

$$\begin{gathered}\mathrm{tr}\dot{\bar{\mathbf{L}}}-2\Omega^{-2}\det\bar{\mathbf{L}}^*-\frac{1}{2}\Omega^{-2}(\mathrm{tr}\bar{\mathbf{L}})^2\\ +\frac{1}{2}f^2\Omega^2+\frac{2\gamma K}{\gamma-1}(\mathrm{tr}\bar{\mathbf{E}})\Omega^{2-2\gamma}=0.\end{gathered}\tag{5.15}$$

Here, it is noted that the matrix system Eq. (5.14) and the scalar Eq. (5.15) are coupled via the relation

$$\dot{d}+d(\gamma-1)\mathrm{tr}\bar{\mathbf{L}}=0,\tag{5.16}$$

where the modulated variable K is given by Eq. (2.22):

$$K=\frac{\gamma-1}{\gamma}k_1\Omega^{2\gamma-4},\quad k_1 \text{ is constant.}\tag{5.17}$$

Without loss of generality, we scale the modulation K to be

$$K=\frac{\gamma-1}{\gamma}\Omega^{2\gamma-4},\tag{5.18}$$

so that Eq. (5.14) becomes

$$\dot{\bar{\mathbf{L}}}^*+\Omega^{-2}[\bar{\mathbf{Q}}\,,\mathbf{M}]=\mathbf{0}.\tag{5.19}$$

Now, a new time variable τ is introduced via

$$d\tau=\Omega^{-2}dt,\tag{5.20}$$

so that the Eqs. (5.13) and (5.20) are reducible to

$$\bar{\mathbf{Q}}'+[\bar{\mathbf{Q}},\bar{\mathbf{L}}^*]=\mathbf{0},\quad \bar{\mathbf{L}}^{*'}+[\bar{\mathbf{Q}}\,,\mathbf{M}]=\mathbf{0},\tag{5.21}$$

where the prime denotes $d/d\tau$. It is remarkable to notice that the matrix system Eq. (5.21) is nothing but the constitutes compatibility condition

$$\mathcal{M}'(\lambda) + [\mathcal{M}(\lambda)\ ,\ \mathcal{L}(\lambda)] = 0, \tag{5.22}$$

for the Lax pair

$$\Psi' = \mathcal{L}(\lambda)\Psi, \quad \mu\Psi = \mathcal{M}(\lambda)\Psi, \tag{5.23}$$

where

$$\mathcal{L}(\lambda) = \bar{\mathbf{L}}^* + \lambda\mathbf{M}, \quad \mathcal{M}(\lambda) = \bar{\mathbf{Q}} + \lambda\bar{\mathbf{L}}^* + \lambda^2\mathbf{M}. \tag{5.24}$$

In the terminology of soliton theory [50], $\mathcal{L}$ and $\mathcal{M}$ are termed Lax matrices for the nonlinear matrix system Eq. (5.21). Analogous results have been obtained in the context of nonisothermal rotating gas clouds and magnetogasdynamics in Refs. [51,52]. As in their work, there is an interesting Steen-Ermakov-Pinney connection. Indeed, on setting

$$\Sigma = \Omega^{-1}, \tag{5.25}$$

then the scalar relation Eq. (5.15) is readily shown to reduce to a Steen-Ermakov-type equation, namely

$$\Sigma'' + (\det\bar{\mathbf{L}}^* - \mathrm{tr}\bar{\mathbf{E}})\Sigma = \frac{f^2}{4\Sigma^3}. \tag{5.26}$$

Results of Refs. [51,52] related to the Lax pairs for the gas cloud and magnetogasdynamic system carry over *mutatis mutandis* to the Lax pair Eq. (5.23) obtained in the present work on the compressible Euler equations.

In the sequel, we would like to show the linear system Eq. (5.23) is gauge equivalent to the standard Lax pair for the stationary reduction of the integrable cubic nonlinear Schrödinger equation. Following the procedure analogous to that set down in Ref. [51], we parametrize the matrices $\bar{\mathbf{L}}^*$ and $\bar{\mathbf{E}}$ into the form of

$$\bar{\mathbf{L}}^* = \begin{pmatrix} \psi & \varphi - \bar{G} \\ \varphi + \bar{G} & -\psi \end{pmatrix}, \quad \bar{\mathbf{E}} = \begin{pmatrix} \frac{\bar{B}}{2} + \bar{\tau} & \varphi \\ \varphi & \frac{\bar{B}}{2} - \bar{\tau} \end{pmatrix}, \tag{5.27}$$

whence the matrix system Eq. (5.21) is reducible to

$$\begin{aligned} \bar{\sigma}' &= -\bar{B}\varphi + 2\bar{G}\bar{\tau}, \quad \varphi' = -2\bar{\sigma}, \\ \bar{\tau}' &= -\bar{B}\phi - 2\bar{G}\bar{\sigma}, \quad \psi' = -2\bar{\tau}, \\ \bar{B}' &= -4(\bar{\sigma}\varphi + \bar{\tau}\psi), \quad \bar{G}' = 0. \end{aligned} \tag{5.28}$$

It is noticed that the system Eq. (5.28) admits the first integrals

$$\begin{aligned} \bar{G} = c_0, \quad \bar{\sigma}^2 + \bar{\tau}^2 - \frac{\bar{B}^2}{4} = c_2, \\ \varphi^2 + \psi^2 - \bar{B} = c_3, \quad 2(\bar{\tau}\varphi - \bar{\sigma}\psi) - c_0\bar{B} = c_4, \end{aligned} \tag{5.29}$$

which coincides with relations that set down in Eq. (2.25). Therefore, the dynamical system may be readily reformulated as

$$\begin{aligned} \varphi'' - 2(\varphi^2 + \psi^2 - c_3)\varphi - 2c_0\psi' = 0, \\ \psi'' - 2(\varphi^2 + \psi^2 - c_3)\psi + 2c_0\varphi' = 0. \end{aligned} \tag{5.30}$$

The form of this coupled system suggests introducing the complex variable via

$$V = \varphi + i\psi, \tag{5.31}$$

so that the pair of second-order differential equations are combined to produce the single complex-valued differential equation

$$ic_0V' + c_3V = -\tfrac{1}{2}V'' + |V|^2V. \tag{5.32}$$

The latter is nothing but the stationary reduction

$$V(x,t) = V(x + c_0t)e^{-ic_3t} \tag{5.33}$$

of the integrable (defocusing) nonlinear Schrödinger equation

$$iV_t = -\tfrac{1}{2}V_{xx} + |V|^2V. \tag{5.34}$$

Interestingly, such a stationary Schrödinger equation is shown to admit Weierstrass function solutions [51] and dressed dark soliton solutions [53].

VI. CONCLUSIONS

It has been shown in this paper that the 2+1-dimensional compressible Euler equations admit an underlying integral Ermakov structure and a Lax pair representation via the power-type elliptic vortex reduction. The existence of such integral structures fully demonstrates the importance of investigation on the Euler equations. Moreover, typical *pulsrodon*-type solutions corresponding to the elliptic warm-core eddy theory are constructed and their behaviors are analyzed. However, there are still many interesting and challenging problems that need further consideration:

(1) It is known that *pulsrodons* are very important in oceanography, atmospheric dynamics, and other physical fields. Whether the *pulsrodons* obtained in this paper can be applied to explain or predict some physical phenomena deserves deep investigation.

(2) Since the Euler equations are the limit cases of the Navier-Stokes (NS) equations for a large Reynolds number, it is conjectured that such integral Ermakov structures must exist in the 2+1-dimensional NS equations. However, the involvement of viscosity terms in the NS equations renders the method proposed invalid. How to improve the elliptic vortex ansatz for the 2+1-dimensional NS model needs intensive investigations.

(3) In light of the limitations inherent in the elliptic vortex reductions, it would be of interest to investigate whether alternative approaches exist, which lead to the integrable Ermakov-Ray-Reid structure for the 3+1-dimensional Euler equations and 3+1-dimensional NS equations.

Because of the importance of the Euler equations, NS equations and the Ermakov-Ray-Reid systems, as well as their extensive physical applications, the models and all the problems mentioned above are worthy of further investigations.

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