Generalized Lyapunov exponent as a unified characterization of dynamical instabilities

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The Lyapunov exponent characterizes an exponential growth rate of the difference of nearby orbits. A positive Lyapunov exponent (exponential dynamical instability) is a manifestation of chaos. Here, we propose the Lyapunov pair, which is based on the generalized Lyapunov exponent, as a unified characterization of nonexponential and exponential dynamical instabilities in one-dimensional maps. Chaos is classified into three different types, i.e., superexponential, exponential, and subexponential chaos. Using one-dimensional maps, we demonstrate superexponential and subexponential chaos and quantify the dynamical instabilities by the Lyapunov pair. In subexponential chaos, we show superweak chaos, which means that the growth of the difference of nearby orbits is slower than a stretched exponential growth. The scaling of the growth is analytically studied by a recently developed theory of a continuous accumulation process, which is related to infinite ergodic theory.

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I. INTRODUCTION

Phenomenological laws such as Ohm's law and equations of state are *average laws* because the variables they deal with, such as pressure, temperature, and electrical current, are averaged quantities [1]. Averaging microscopic variables, we can derive the phenomenological laws or equations from the underlying dynamical systems. Chaos plays an important role in such an averaging procedure. In other words, chaos guarantees to change from a deterministic description to a probabilistic one [2]. One of the most useful tools to characterize chaos in dynamical systems is the Lyapunov exponent. Positive Lyapunov exponents imply chaos, which means that nearby orbits separate exponentially with time (exponential dynamical instability).

Chaos plays a central role not only in equilibrium but also in nonequilibrium statistical mechanics [3–5]. In particular, a chaotic hypothesis, which is a stronger hypothesis than a positive Lyapunov exponent, establishes the fluctuation theorem in nonequilibrium stationary states [3]. Moreover, the role of chaos in nonequilibrium nonstationary phenomena such as anomalous diffusions has been studied [6–8], where infinite invariant measure and the Lyapunov exponent play an important role in characterizing transport coefficients such as diffusion coefficients and drift [6,7]. Prominent features in such dynamical systems with infinite invariant measures are *distributional limit theorems*, that is, time-averaged observables do not converge to a constant but become random [9–15]. Recently, infinite densities have become important in physics of anomalous transports [16,17].

Exponential separation of nearby orbits, i.e., exponential dynamical instability, is clearly indicated by the Lyapunov exponent. On the other hand, dynamical systems may show *nonexponential dynamical instabilities* while they have a sensitive dependence on initial conditions. It is well known that the separation of nearby orbits in Pomeau-Manniville maps [18] with infinite invariant measures is characterized as a subexponential growth (subexponential dynamical instability) [19–21]. More precisely, the average of the logarithm of the separation of nearby orbits grows sublinearly with time, which indicates the zero Lyapunov exponent, while the system has a sensitive dependence on initial conditions. Since subexponential dynamical instability implies infinite invariant measure [21], characterization of nonexponential dynamical instability will be important in physics with infinite densities.

Another characterization of dynamical instability is a mixing property. A concept of mixing in dynamical systems with infinite invariant measures was introduced by Krengel and Sucheston [22]. The typical example of such dynamical systems is a Pomeau-Manneville map with an infinite invariant measure [23]. We note that the Lyapunov exponent converges to zero even though there is a mixing property. Recently, indicators characterizing subexponential dynamical instability have been developed [20,21,24,25]. However, to our knowledge, there are no unified quantities to characterize dynamical instabilities such as superexponential and subexponential instabilities. In this paper, we propose the Lyapunov pair as a unified indicator characterizing various types of chaos.

II. DYNAMICAL INSTABILITY IN ONE-DIMENSIONAL MAPS

Dynamical instability is defined by the sensitive dependence on initial points. In particular, the exponential growth of nearby orbits, i.e.,

$$\left|\frac{\Delta x(n)}{\Delta x(0)}\right| \sim e^{\lambda n}, \Delta x(0) \to 0 \quad \text{and} \quad n \to \infty, \tag{1}$$

is characterized by the Lyapunov exponent λ, where $\Delta x(n)$ is the difference between two orbits at time n. Positive exponent $\lambda > 0$ implies the exponential dynamical instability. Let T be a transformation on a one-dimensional interval I; the Lyapunov exponent can be given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |T'(x_k))|, \tag{2}$$

where $x_k = T^k(x_0)$.

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When the separation of nearby orbits grows subexponentially with time, i.e.,

$$\left|\frac{\Delta x(n)}{\Delta x(0)}\right| \sim e^{\lambda_\alpha n^\alpha}, \quad \Delta x(0) \to 0 \quad \text{and} \quad n \to \infty, \tag{3}$$

where $0 < \alpha < 1$, the growth rate λ_α cannot be determined uniquely. This is because there does not exist a sequence $a_n \propto n^\alpha$ such that $\lambda_\alpha(a_n) = \frac{1}{a_n}\sum_{k=0}^{n-1} \ln|T'(x_k)|$ converges to a nontrivial constant as $n \to \infty$ in a conservative, ergodic, measure-preserving transformation [10,21]. In other words, the exponent λ_α remains a random variable even when the time goes to infinity. In a previous study [21], we investigated the generalized Lyapunov exponent to characterize the subexponential dynamical instability.

Here we use the generalized Lyapunov exponent to characterize chaos with nonexponential dynamical instabilities. The generalized Lyapunov exponent is defined by

$$\Lambda_\alpha \equiv \left\langle \frac{1}{n^\alpha L(n)} \sum_{k=0}^{n-1} \ln|T'(x_k)| \right\rangle, \tag{4}$$

where the sequence $L(n)$ is slowly varying at ∞, $\langle . \rangle$ represents the average with respect to an initial ensemble being Riemann integrable and $\langle \ln|T'(x)|\rangle < \infty$ [15,20,21]. We note that dynamical systems with infinite invariant measures shows aging [15,26]. In particular, the generalized Lyapunov exponent depends on the aging ratio, $T_a \equiv t_a/t$, i.e., the ratio between the measurement time t and the time t_a when the system started [15]. Here we set $T_a = 0$ because we do not consider the aging effect. If $L(n)$ is constant, we set $L(n) \equiv 1$. In this definition, dynamical instability can be represented by the average of the logarithm of the separation of nearby orbits:

$$\left\langle \ln \frac{\Delta x(n)}{\Delta x(0)} \right\rangle \sim \Lambda_\alpha n^\alpha L(n), \quad \Delta x(0) \to 0 \quad \text{and} \quad n \to \infty. \tag{5}$$

We call $(n^\alpha L(n), \Lambda_\alpha)$ the *Lyapunov pair* when $0 < \Lambda_\alpha < \infty$ holds. If the average of the logarithm of the separation of nearby orbits cannot be represented by Eq. (5), e.g., $\langle \ln \frac{\Delta x(n)}{\Delta x(0)}\rangle \propto e^n$, we set $\alpha = \infty$. In the case where there does not exist a sequence such that $0 < \Lambda_\alpha < \infty$, we write the Lyapunov pair $(n^\alpha L(n), \infty)$ if $\sum_{k=0}^{n-1} \ln|T'(x_k)|/n^\alpha L(n)$ converges in distribution (does not converges to 0 nor ∞) and the ensemble average diverges. We call the sequence $n^\alpha L(n)$ in the Lyapunov pair the *dynamical instability sequence*.

Using the dynamical instability sequence $n^\alpha L(n)$, we classify a type of chaos into *superexponential chaos*, *exponential chaos,* and *subexponential chaos* if $n^\alpha L(n)/n$ diverges, converges to constant, and 0 as $n \to \infty$, respectively. In other words, a dynamical system with a large α has a high dynamical instability. Because the generalized Lyapunov exponent gives an averaged growth rate when the separation growth of nearby orbits is given by the form (5), a degree of the dynamical instability cannot be quantified by the generalized Lyapunov exponent only. The generalized Lyapunov exponent represents a degree of the dynamical instability when the dynamical instability sequence is fixed. We note that the dynamical instability sequence of ordinary chaos is n, i.e., $\Lambda_\alpha = \lambda$.

III. DIFFERENT TYPES OF CHAOS IN ONE-DIMENSIONAL MAPS

A. Super-exponential chaos

Here we give two examples for superexponential chaos. One example is the infinite Bernoulli scheme:

$$B\left(\frac{1}{2},\frac{1}{4},\frac{1}{16},\frac{1}{16},\ldots,\underbrace{\frac{1}{2^{2^n}},\ldots,\frac{1}{2^{2^n}}}_{2^{2^n-n-1}\,times},\ldots\right). \tag{6}$$

The transformation $T(x)$ is shown in Fig. 1(a). The transformation has uniform invariant measure on $[0,1]$ [27]. By Birkhoff's ergodic theorem [28], the Lyapunov exponent is given by the ensemble average with respect to the uniform invariant measure:

$$\lambda = \int_0^1 dx \log|T'(x)| = \sum_{k=1}^\infty \frac{2^{2^k-k-1}}{2^{2^k}} \log 2^{2^k} = \infty. \tag{7}$$

Let X be the logarithm of the slope of the infinite Bernoulli scheme, then

$$\Pr\left\{X = \log 2^{2^n}\right\} = p_n \equiv 2^{-n-1}. \tag{8}$$

Therefore,

$$\Pr\{X \geqslant x\} = \sum_{k=n}^\infty p_k = 2^{-n} \sim \frac{\log 2}{x} \quad (x \to \infty), \tag{9}$$

where $x = \log 2^{2^n}$. From the generalized central limit theorem [29], $\sum_{k=0}^{n-1} \ln|T'(x_k)|/n\ln n$ converges to a stable distribution with exponent one. Because the mean does not exist (diverge) in the stable distribution, the Lyapunov pair is given as $(n\ln n, \infty)$.

The other example is the ant-lion map [30]. The ant-lion map $T_{AL}: [0,1] \to [0,1]$ is an infinite-modal map defined by

$$T_{AL}(x) = x + Ax\sin(\pi/x), \tag{10}$$

where $A < 1$ [see Fig. 1(b)]. Fixed points are given by $x = 1/n$ $(n = 1,2,\ldots)$. As shown in Fig. 2(b), the Lebesgue measure where orbits go to the origin (black region) is positive, and there are stable periodic orbits in the black region [see also Fig. 2(a)]. Surprisingly, the origin is an attractor, whereas the derivative of the ant-lion map becomes large (greater than one) around the origin. Such a strange phenomenon is called

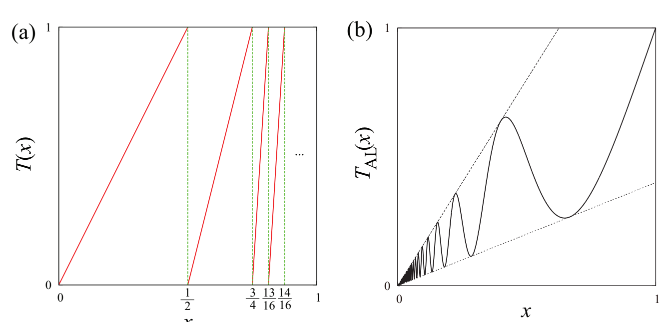

FIG. 1. (Color online) (a) Infinite Bernoulli scheme. Lines represent the transformation. (b) Ant-lion map. The dashed lines represent the envelopes of the map.

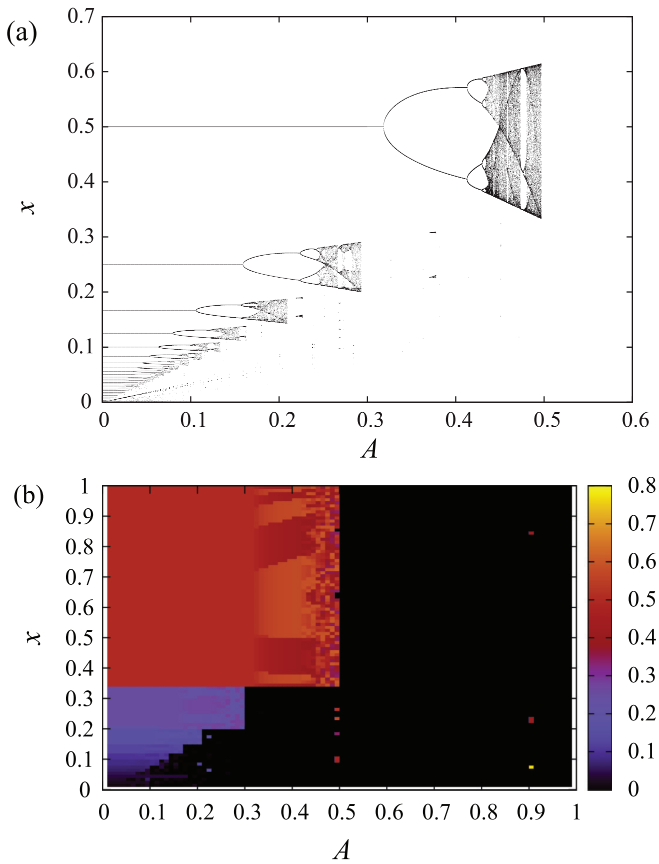

FIG. 2. (Color online) Bifurcation diagram and basins of attraction for the ant-lion map (10). (a) Bifurcation diagram. This diagram is drawn as follows: initial points x_0^i are randomly chosen ($i = 1,\ldots,100$). After 10^4 iterations, we plot the values of $x_{10^4}^i$ ($i = 1,\ldots,100$) for each parameter A. (b) Basins of attraction. Vertical axis represents initial point. The figure is drawn as follows: initial points are chosen as $i/100$ for each parameter $A = i/100$ ($i = 1,\ldots,100$), and then we plot the values of x_{10^3} by their colors.

the ant-lion property [30]. More precisely, orbits which are not stable periodic orbits in the ant-lion map can be represented as $x_n \propto e^{-n\gamma(A)+\sqrt{n}\sigma(A)\xi_n(x_0)}$, where $\gamma(A)$ and $\sigma(A)$ are constants which depend on A, and $\xi_n(x_0)$ represents a correction term which depends on the initial point x_0. These orbits are similar to those generated by a random dynamical system defined by

$$T_R(x) = (1 + A\sin Y)x, \tag{11}$$

where Y is a random variable with uniform density on $[0,2\pi]$. If we assume that the term $\pi/x_n \pmod{2\pi}$ in the ant-lion map is uniformly distributed on $[0,2\pi]$, we have the above random dynamical system. The above assumption is physically reasonable near $x = 0$ because the ant-lion map becomes denser as x closes to the origin [see Fig. 1(b)].

Consider an orbit $z_n = \ln x_n$. Then, we have a biased random walk, i.e., $z_{n+1} = z_n + \ln(1 + A\sin Y_n)$. Because the mean $\langle \ln(1 + A\sin Y_n)\rangle \equiv -\gamma(A)$ is negative, the trajectory z_n shows a drift, i.e., $\langle z_n\rangle \propto -\gamma(A)n$, which implies x_n goes to zero as $n \to \infty$. More precisely, we have $x_n \propto e^{-n\gamma(A)+\sqrt{n}\sigma(A)\xi_n}$ in the random dynamical system [30]. Moreover, the generalized Lyapunov exponent can be obtained by using the random dynamical system [30]. Near the origin we approximate the derivative of the map by

$$T'(x) \sim -\frac{\pi A}{x}\cos(Y). \tag{12}$$

Using trajectories $x_n \sim x_0 e^{-n\gamma(A)+\sqrt{n}\sigma(A)\xi_n}$, we have

$$\left\langle \sum_{k=0}^{n-1} \ln|T'(x_k)| \right\rangle \sim \sum_{k=0}^{n-1} \left\langle \ln\frac{\pi A}{x_k}|\cos(Y)| \right\rangle \tag{13}$$

$$\sim \frac{\gamma(A)}{2} n^2 \quad (n \to \infty). \tag{14}$$

Therefore, the Lyapunov pair is given by $(n^2,\gamma(A)/2)$. Although the origin is an attractor, the ant-lion map has a superexponential dynamical instability. This superexponential dynamical instability validates a randomization of trajectories. In other words, we can use trajectories in the random dynamical system (11) to study statistical properties of the ant-lion map (10) with the aid of its high complexity. This is another evidence of the superexponential chaos. We note that the ant-lion map has infinite invariant measures [30,31].

B. Exponential chaos

When a dynamical system on I has an invariant probability measure m and the function $\ln|T'(x)|$ is an $L^1(m)$ function, i.e., $\int_I |f|\,dm < \infty$, the time average of the function $\ln|T'(x)|$ equals the ensemble average for almost all initial points x_0:

$$\frac{1}{n}\sum_{k=0}^{n-1} \ln|T'(x_k)| \to \int_I \ln|T'(x)|\,dm \quad (n \to \infty). \tag{15}$$

Furthermore, the existence of a probability invariant measure and $\ln|T'(x)| \in L^1(m)$ implies a positive Lyapunov exponent [21]. Therefore, dynamical systems are exponential chaos if and only if the invariant measure m is a probability measure and the function $\ln|T'(x)|$ is an $L^1(m)$ function. In other words, an origin of superexponential chaos in the infinite Bernoulli scheme is a non-$L^1(m)$ property of $\ln|T'(x)|$.

C. Sub-exponential chaos

In a previous paper [21], we show that subexponential instability implies an infinite measure in one-dimensional maps. From infinite ergodic theory [10], if the function $f(x) = \ln|T'(x)|$ is an $L^1(m)$ positive function, one can obtain distributional behavior of the normalized Lyapunov exponent:

$$\frac{1}{a_n}\sum_{k=0}^{n-1} f(x_k) \Rightarrow \left[\int_I f(x)\,dm\right] Y_\alpha \quad (n \to \infty), \tag{16}$$

where a_n is called the *return sequence*, Y_α is a random variable with the normalized Mittag-Leffler distribution of order α [32]. The notation "⇒" means the convergence in distribution. We note that initial points x_0 in the left-hand side of Eq. (16) are random variables. Because the mean of the normalized Mittag-Leffler distribution is one, the generalized Lyapunov exponent is obtained as

$$\Lambda_\alpha = \frac{a_n}{n^\alpha L(n)} \int_0^1 \ln|T'(x)|\rho(x)\,dx. \tag{17}$$

Note that there are no n dependence in the right-hand side (RHS) of Eq. (17). It is also noteworthy that there is at most one infinite invariant measure if T is a conservative, ergodic, nonsingular transformation [10] and that the multiplying constant of the invariant measure m is uniquely determined when the return sequence a_n is specified. In other words, the return sequence a_n is uniquely determined by the choice of an infinite invariant measure. From infinite ergodic theory, the return sequence can be obtained using the *wandering rate* defined by $w_n = m(\cup_{k=0}^{n} T^{-k}B)$, where B is a set with $0 < m(B) < \infty$. In particular, the return sequence is given by

$$a_n \sim \frac{n}{\Gamma(1+\alpha)\Gamma(2-\alpha)w_n}, \tag{18}$$

when w_n is regularly varying at ∞ with index α [10].

Here we consider the map $T_p : [0,1] \to [0,1]$ with $p \geqslant 1$ [33] defined by

$$T_p(x) = x\left[1 + \left(\frac{x}{1+x}\right)^{p-1} - x^{p-1}\right]^{1/(1-p)} \quad (\mathrm{mod}1). \tag{19}$$

The invariant density $\rho_p(x)$ of this map is analytically known as [33]

$$\rho_p(x) = \frac{c}{x^p} + \frac{c}{(1+x)^p}, \tag{20}$$

where c is a multiplicative constant. In what follows, we set $c = 1$ for simplicity. According to the estimation of w_n in Ref. [34], we have

$$w_n \sim \begin{cases} \log n & (p = 1), \\ \frac{p^{1-\alpha}}{p-1} n^{1-\alpha} & (p > 1), \end{cases} \tag{21}$$

where $\alpha = 1/p$. From Eq. (18), the return sequence can be written as

$$a_n \sim \begin{cases} \frac{n}{\log n} & (p = 1), \\ \frac{(p-1)n^{\alpha}}{p^{1-\alpha}\Gamma(1+\alpha)\Gamma(2-\alpha)} & (p > 1). \end{cases} \tag{22}$$

Therefore, the generalized Lyapunov exponent is obtained as

$$\Lambda_\alpha(p) = \frac{p-1}{p^{1-\alpha}\Gamma(1+\alpha)\Gamma(2-\alpha)} \int_0^1 \ln|T_p'(x)|\rho_p(x)\,dx. \tag{23}$$

Figure 3 shows that numerical simulations of the generalized Lyapunov exponents are in good agreement with the theory without fitting.

As an example of subexponential chaos with non-$L^1(m)$ observation function of the Lyapunov exponent, we consider the log-Weibull map [34,35] defined by

$$T_{LW}(x) = \begin{cases} x + x^2 e^{-1/x} & x \in [0,a] \\ \frac{x-a}{1-a} & x \in (a,1] \end{cases}, \tag{24}$$

where a is determined by the equation $a + a^2 e^{-1/a} = 1$ $(0 < a < 1)$. The invariant density has an essential singularity at 0 [34]:

$$\rho(x) = h(x)e^{1/x}/x, \tag{25}$$

where $h(x)$ is continuous and positive on $[0,1]$. The residence time distribution on $(0,a]$ obeys the log-Weibull distribution [35],

$$W(\tau) \sim \exp[-C/\ln(\tau+1)] \quad (\tau \to \infty), \tag{26}$$

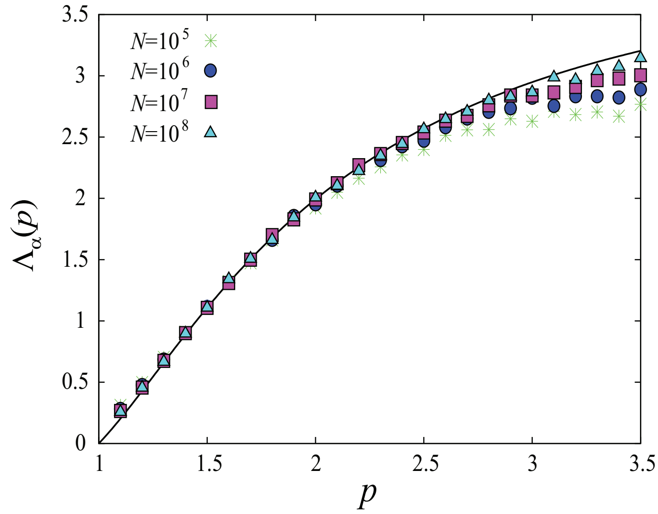

FIG. 3. (Color online) The generalized Lyapunov exponent. Symbols represent the results of numerical simulations for finite lengths N of sums. The solid curve is the theoretical curve (23) without fitting parameter.

where C is a constant. This is why we refer to the map (24) as the log-Weibull map (a logarithmic modification of the Weibull map [36]). We note that $\ln|T_{\mathrm{LW}}'(x)|$ is not an $L^1(m)$ function, i.e., $\int_0^1 \ln|T_{\mathrm{LW}}'(x)|\rho(x)\,dx = \infty$. This class of function is called weak non-L^1 function because of $\ln|T_{\mathrm{LW}}'(0)| = 0$ [37]. Therefore, the distributional limit theorem (16) cannot be applied whereas it is known that the return sequence of the log-Weibull map can be given by $a_n \propto \ln n$ [34]. Instead, another distributional limit theorem will be applied. Although the log-Weibull map does not belong to the maps considered in Ref. [37], a similar distributional limit theorem will hold.

To investigate the scaling of the dynamical instability sequence, we consider the evolution of $\ln|T_{LW}'(x_k)|$ on $[0,a]$ using a continuous approximation. Since a displacement, $x_n - x_{n-1}$, is very small near the fixed point $(x = 0)$, the difference equation (24) can be replaced by the differential equation:

$$\frac{dx}{dt} = x^2 e^{-1/x} \quad (x \leqslant 1). \tag{27}$$

This equation is solved as

$$x(t) = \frac{1}{\ln(e^{1/x_0} - t)} \quad (t < \tau), \tag{28}$$

where x_0 is the initial point and τ satisfies $x(\tau) = 1$, i.e., $\tau = e^{1/x_0} - e$ or $x_0 = 1/\ln(\tau + e)$. When the total residence time (time elapsing from reinjection on $[0,a]$ to escape from it) is given by τ, the partial sum of $\ln|T'(x_k)|$ from time 0 to t during residing on $[0,a]$, i.e., $I(t,\tau) \equiv \sum_{k=0}^{t} \ln|T_{LW}'(x_k)|$ $(t < \tau)$, is approximated by

$$I(t,\tau) \cong \int_0^t \ln|T_{LW}'(x(t'))|\,dt' \cong \ln(\tau+e) - \ln(\tau+e-t), \tag{29}$$

where we approximate the partial sum of $\ln|T'_{LW}(x_k)|$ from time 0 to t as a continuous process [37]. The total increase of the partial sum during residing on $[0,a]$ is given by $I(\tau) \equiv I(\tau,\tau) \cong \ln(\tau+e)-1$. Rigorous discussion on the above approximation has been done in Ref. [37]. Here, we consider a continuous accumulation process [37]. Let $Q(x,t)$ be the probability density function (PDF) that a partial sum is x at time t when the orbit escapes from $[0,a]$, then we have

$$Q(x,t) = \delta(t)\delta(x) + \int_0^\infty dx' \int_0^t dt' \psi(x',t') \times Q(x-x',t-\tau)\, d\tau\, dx', \quad (30)$$

where $\psi(x,\tau) = w(\tau)\delta[x - I(\tau)]$ and $w(\tau)$ is the PDF of the residence time, i.e., $w(\tau) = W'(\tau)$. The conditional PDF of X_t at time t [note that the orbit will be in $(0,a)$] on the condition of $\tau_{N_t+1} = \tau$ (N_t is the number of escapes from $[0,a]$ until time t), denoted by $P(x,t;\tau)$, is given by

$$P(x,t;\tau) = \int_0^x dx' \int_0^t dt' \Psi(x',t';\tau) \times Q(x-x',t-t') + \Psi(x,t;\tau), \quad (31)$$

where $\Psi(x,t;\tau) = \delta[x - I(t,\tau)]\theta(\tau - t)$ and $\theta(x) = 0$ for $x < 0$ and 1 otherwise. It follows that the PDF of X_t at time t reads

$$P(x,t) = \int_0^\infty w(\tau) P(x,t;\tau)\, d\tau. \quad (32)$$

Double Laplace transform with respect to time ($t \to s$) and space ($x \to k$) gives

$$\hat{P}(k,s) \equiv \int_0^\infty dt \int_0^\infty dx e^{-st-kx} P(x,t) = \int_0^\infty \frac{w(\tau)\hat{\Psi}(k,s;\tau)}{1-\hat{\psi}(k,s)}\, d\tau, \quad (33)$$

where

$$\hat{\psi}(k,s) \equiv \int_0^\infty d\tau \int_0^\infty dx e^{-s\tau-kx} \psi(x,\tau) = \int_0^\infty e^{-s\tau} e^{-kI(\tau)} w(\tau)\, d\tau \quad (34)$$

and

$$\hat{\Psi}(k,s;\tau) \equiv \int_0^\infty dt \int_0^\infty dx e^{-st-kx} \Psi(x,t;\tau) = \int_0^\tau e^{-st-kI(t,\tau)}\, dt. \quad (35)$$

Because the Laplace transform of the mean partial sum of $\ln|T(x)|$ denoted by $\hat{H}(s)$, is given by $\hat{H}(s) = -\frac{\partial \hat{P}(k,s)}{\partial k}|_{k=0}$, we have

$$\hat{H}(s) = -\frac{\hat{\psi}'(0,s)}{s[1-\hat{w}(s)]} + \frac{\int_0^\infty dt \left[\int_t^\infty d\tau w(\tau) I(t,\tau)\right] e^{-st}}{1-\hat{w}(s)} \propto -\frac{\hat{\psi}'(0,s)}{s[1-\hat{w}(s)]}, \quad (36)$$

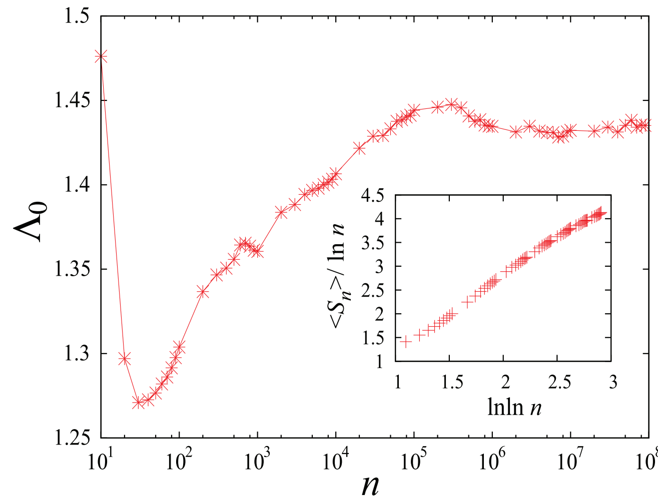

FIG. 4. (Color online) The generalized Lyapunov exponent of the log-Weibull map. The generalized Lyapunov exponent Λ_0 defined by Eq. (40) converges to $A_{LW} \cong 1.43$. Inset figure shows that $\langle S_n \rangle / \ln n$ increases with $\ln\ln n$.

where we used the approximation that the second term has the same order as the first one. Using the asymptotic form of $\hat{\psi}(s)$,

$$\hat{w}(s) \sim W(1/s) \sim \exp\left[-\frac{C}{\ln(1/s)}\right] \quad (s \to 0), \quad (37)$$

and $\int_0^\infty e^{-s\tau} \ln(\tau+e) w(\tau)\, d\tau = O[\ln\ln(1/s)]$ (see the Appendix), we have

$$\hat{H}(s) \sim A_{LW} \frac{\ln(1/s)\ln\ln(1/s)}{s} \quad (s \to 0), \quad (38)$$

where A_{LW} is a constant. The inverse Laplace transform reads

$$\left\langle \sum_{k=0}^{n-1} \ln|T'_{LW}(x_k)| \right\rangle \sim A_{LW} \ln n \ln\ln n \quad (n \to \infty). \quad (39)$$

It follows that the generalized Lyapunov exponent of the log-Weibull map is given by

$$\Lambda_0 \equiv \left\langle \frac{1}{\ln n \ln\ln n} \sum_{k=0}^{n-1} \ln|T'_{LW}(x_k)| \right\rangle \to A_{LW} \quad (40)$$

as $n \to \infty$. We note that the dynamical instability sequence, $\ln n \ln\ln n$, is not the same as the return sequence because of a non-$L^1(m)$ property of $\ln|T(x)|$. Therefore, the Lyapunov pair is given by $(\ln n \ln\ln n, A_{LW})$, where A_{LW} is numerically obtained as $A_{LW} \cong 1.43$. Because the separation of nearby orbits grows slower than a stretched-exponential growth, the dynamical instability is much weaker than that in Pomeau-Manneville map. We call this chaos superweak chaos. Figure 4 shows the generalized Lyapunov exponent converges to a constant.

IV. CONCLUSION

We have proposed the Lyapunov pair as a unified characterization of dynamical instabilities, such as superexponential and subexponential dynamical instabilities. The dynamical instability sequence represents a separation growth of nearby orbits, while the generalized Lyapunov exponent Λ_α characterizes the growth rate of the separation, i.e., $|\Delta x(n)/\Delta x(0)| \sim e^{\Lambda_\alpha n^\alpha L(n)}$. In the log-Weibull map, we show that the dynamical instability sequence is represented as $\ln n \ln\ln n$, which means that the separation growth of nearby orbits is slower than a stretched-exponential as well as a power-law growth (superweak chaos). In deterministic subdiffusion, the mean square displacement grows sublinearly, $\langle x(t)^2\rangle \propto t^\alpha$, whereas the time-averaged mean square displacement grows linearly, $\overline{\delta^2}(\Delta) \sim D_\alpha \Delta$, but D_α remains random [6]. Using the Lyapunov pair, we can characterize the subdiffusive exponent α and the mean of diffusion coefficients through the relation $\langle x(t)^2\rangle \propto t^\alpha L(t)$ and $\langle D_\alpha\rangle \propto \Lambda_\alpha$, where $t^\alpha L(t)$ is the dynamical instability sequence [6]. Therefore, the Lyapunov pair will play an important role in anomalous transports.

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APPENDIX: SCALING OF THE LAPLACE TRANSFORM

We derive the asymptotic form of the Laplace transform of the function $f(\tau) = \ln(\tau + e)w(\tau)$. The asymptotic form of $f(\tau)$ is given by

$$f(\tau) \sim \frac{\ln(\tau + e)}{(\tau+1)[\ln(\tau+1)]^2} \sim \frac{1}{\tau \ln \tau} \quad (\tau \to \infty). \tag{A1}$$

We decompose the integration as follows:

$$\int_0^\infty e^{-s\tau} f(\tau)\, d\tau = \int_0^{\tau^*} e^{-s\tau} f(\tau)\, d\tau + \int_{\tau^*}^{1/s} e^{-s\tau} f(\tau)\, d\tau + \int_{1/s}^{\infty} e^{-s\tau} f(\tau)\, d\tau. \tag{A2}$$

The first term in the RHS of Eq. (A2) can be represented by

$$0 < \int_0^{\tau^*} e^{-s\tau} f(\tau)\, d\tau < \int_0^{\tau^*} f(\tau)\, d\tau < \int_0^{\tau^*} \frac{1}{\tau+1}\, d\tau \tag{A3}$$

for some $\tau^* > 0$. Therefore, the first term is bounded for $s \ll 1$. On the other hand, the second and third terms in the RHS of Eq. (A2) can be estimated by

$$e^{-1}\int_{\tau^*}^{1/s} f(\tau)\, d\tau < \int_{\tau^*}^{1/s} e^{-s\tau} f(\tau)\, d\tau + \int_{1/s}^{\infty} e^{-s\tau} f(\tau)\, d\tau < e^{-s\tau^*}\int_{\tau^*}^{1/s} f(\tau)\, d\tau + f(1/s)\int_{1/s}^{\infty} e^{-s\tau}\, d\tau. \tag{A4}$$

By

$$\int_{\tau^*}^{1/s} f(\tau)\, d\tau \sim \int_{\tau^*}^{1/s} \frac{1}{(\tau+1)\ln(\tau+1)}\, d\tau = [\ln\ln(\tau+1)]_{\tau^*}^{1/s} = \ln\ln(1/s+1) - \ln\ln(\tau^*+1), \tag{A5}$$

we obtain the leading order of the Laplace transform of $f(\tau)$:

$$\int_0^\infty e^{-s\tau} f(\tau)\, d\tau = O\,[\ln\ln(1/s)] \quad (s \to 0). \tag{A6}$$

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$$E(e^{zY_\alpha}) = \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha)^k z^k}{\Gamma(1+k\alpha)},$$

where $E(\cdot)$ is the expectation.

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