Optimal pinning controllability of complex networks: Dependence on network structure

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(Received 6 November 2013; revised manuscript received 8 July 2014; published 5 January 2015)

Controlling networked structures has many applications in science and engineering. In this paper, we consider the problem of pinning control (pinning the dynamics into the reference state), and optimally placing the driver nodes, i.e., the nodes to which the control signal is fed. Considering the local controllability concept, a metric based on the eigenvalues of the Laplacian matrix is taken into account as a measure of controllability. We show that the proposed optimal placement strategy considerably outperforms heuristic methods including choosing hub nodes with high degree or betweenness centrality as drivers. We also study properties of optimal drivers in terms of various centrality measures including degree, betweenness, closeness, and clustering coefficient. The profile of these centrality values depends on the network structure. For homogeneous networks such as random small-world networks, the optimal driver nodes have almost the mean centrality value of the population (much lower than the centrality value of hub nodes), whereas the centrality value of optimal drivers in heterogeneous networks such as scale-free ones is much higher than the average and close to that of hub nodes. However, as the degree of heterogeneity decreases in such networks, the profile of centrality approaches the population mean.

DOI: 10.1103/PhysRevE.91.012803 PACS number(s): 89.75.Hc, 89.75.Fb

I. INTRODUCTION

Network science has witnessed tremendous progress over the last two decades [1,2]. Large-scale systems are often modeled as networked structures comprising a number of nodes that are connected through the edges; examples include the Internet, World Wide Web, power grids, neurons in the brain, and financial systems [3]. A branch of research related to network science is the study of how dynamical phenomena evolve over networked structures [4]. For example, if dynamical systems interact over a network and some conditions are met, some kind of collective behavior (synchronization or consensus) emerges [5,6]. Synchronization has many applications in science and engineering, and many real-world systems are desired to have high synchronization properties [7]. It largely depends on the network structure and some topologies are more synchronizable than others. In general, it has been shown that highly synchronizable networks belong to a class of homogeneous networks with a small number of loops [7–9].

In some applications, dynamical networks (i.e., dynamical systems interacting over a networked structure) are required to be controlled. Controlling dynamical networks has many potential applications. An example is brain networks in which controlling specific regions can be beneficial in minimizing the risks of neuronal disorders for which synchronization plays an important role, e.g., Alzheimer's disease [10] and schizophrenia [11]. Considering the concept of structural controllability, the minimum number of driving nodes have been obtained, i.e., the minimum number of nodes controlling which will allow to control the whole dynamics of the network [12,13]. Cornelius *et al.* [14] extended the seminal work of Liu *et al.* [12] to nonlinear networks. They introduced a method to systematically design compensatory perturbations in order to tune the nodes into a desired state. However, [12–14] are based on structural controllability and do not consider controlling the dynamics to a specific state. Some other works instead considered the problem of pinning control, that is, how one can pin the dynamics of all nodes to a specific state by controlling a number of preselected nodes [15,16]. Usually, a small fraction of the nodes are considered as driver nodes, and the input is fed into them.

The controllability problem through pinning control is often studied through similar techniques available for local stability of the synchronization manifold [17,18]. The master stability function [19] that is a prime choice for many synchronization studies can be easily extended to study controllability of dynamical networks [17,18]. Wu studied the relationship between the effectiveness of pinning control and graph topology [20], and showed that if the number of driver nodes does not grow as fast as the other nodes, higher control gain will be needed in order to have effective pinning control. Li *et al.* considered cost of synchronization and studied its relations with pinning control [21]. They found that one can achieve lower cost by controlling nodes with small degrees. For example, in a starlike network, controlling all noncentral nodes with small feedback gain is better than controlling the central node [21]. However, one still needs to control many nodes, and if the number of drivers matters, the strategy proposed by Li *et al.* [21] is far from being optimal.

Turci and Macau studied the performance of pinning control in accordance with the type of driver nodes [22]. They showed that in order to pin the network to the desired state, one only needs to control a limited number of the nodes. Indeed, their simulation results showed that as the number of driver nodes increases more than a certain amount, the performance of the pinning control (i.e., time to synchronize) does not improve significantly. They also showed in networks with a single driver node, choosing a hub node is likely to result in good performance. However, they did not use any optimization technique to find the most influential driver node. Porfiri and

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Fiorilli proposed node-to-node pinning control [23]. They considered a specific cost for the pinning control in terms of the control gain, location of the driver nodes, and network structural properties. With this cost function, they showed that the optimal control is obtained when all nodes are pinned with the same gain. The node-to-node control strategy periodically switches the location of driver nodes and spans all the nodes. However, this might not be possible in some applications where switching the location of driver nodes is costly. It was shown that considering a specific dynamics in the pinned edges can improve the performance of pinning control [24]. Pinning control has also been extended to discrete-time networks through stochastic pinning strategy [25].

In this paper, we consider the local stability of the pinning synchronization, i.e., the method based on the master stability function approach [18,20–22,26]. This leads to a simple metric quantifying controllability of the network. This controllability measure is based on the eigenvalues of the Laplacian matrix and the information on driver nodes (i.e., the location of the driver nodes and the feedback control gain). Here we develop an optimization technique in order to find the best driver nodes. Having a network with N nodes, the evolutionary optimization technique (differential evolution) considers the controllability measure as the cost function and determines m ($m \ll N$) driver nodes whose controlling can pin the state of all nodes to the desired state.

We compare the performance of the proposed optimal placement strategy for driver nodes with heuristic methods in model networks including scale-free, pure random, and small-world networks. These network structures have heterogeneous (scale-free network) or almost homogeneous (random and small-world networks) degree distribution. Our results show that the proposed optimal placement strategy outperforms heuristic methods in both heterogeneous and homogeneous networks. We also investigate properties of optimally placed driver nodes in terms of different centrality measures including degree, betweenness, closeness, and clustering coefficient. The results show that homogeneous and heterogeneous networks are significantly different in properties of the optimal driver nodes. While driver nodes have close centrality value to hub nodes in heterogonous networks, they have almost the same centrality value of the mean population in homogeneous networks.

II. CONTROLLABILITY OF DIFFUSIVELY COUPLED DYNAMICAL SYSTEMS

We consider identical dynamical systems coupled over undirected networks. Let us consider an undirected and unweighted network with N nodes. A dynamical system sits on each node of the connection graph, and the equations of the motion read

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} l_{ij} H \boldsymbol{x}_j; \quad i = 1,2,\ldots,N, \quad (1)$$

where $\boldsymbol{x}_i \in \mathbb{R}^d$ are d-dimensional state vectors; $F : \mathbb{R}^d \to \mathbb{R}^d$ defines the individual systems' dynamical equation. These dynamical systems are coupled via a unified coupling strength σ and coupling matrix described by binary adjacency matrix $A = (a_{ij})$. The entries of A are 1, if there is a link between the corresponding nodes, or 0, if there is no link. $L = (l_{ij})$ is called a Laplacian matrix, which is a symmetric matrix with vanishing row sums and positive diagonal entries; $l_{ij} = -a_{ji}$ for all pairs of (i,j) and $i \neq j$, and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for all i. The nonzero elements of $d \times d$ projection matrix H determines the coupled elements of the oscillators.

The aim of pinning control is to pin the nodes into a specific dynamics. Let us consider a time-varying reference state (virtual leader to which the dynamical systems should be pinned) as

$$\frac{d\boldsymbol{s}(t)}{dt} = F(\boldsymbol{s}(t)). \quad (2)$$

In order to pin the dynamical network to the reference dynamics $s(t)$, one should design the state feedback controllers as

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} l_{ij} H \boldsymbol{x}_j - \sigma \beta_i k_i (\boldsymbol{s} - \boldsymbol{x}_i); \quad i = 1,2,\ldots,N, \quad (3)$$

where k_i is the feedback control gain, which is considered to be the same for all nodes in this work. β_i determines whether a node should receive the pinning control signal; if a node is receiving the control signal, i.e., it is a driver node, $\beta_i = 1$; otherwise $\beta_i = 0$. In pinning control, driver nodes have the role of pinning other nodes into the desired state $s(t)$, that is, $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$.

One should discriminate between global and local synchronization to the reference state. The dynamical network described by Eq. (3) synchronizes globally to the reference state $s(t)$, if starting from any initial condition, we have

$$\|\boldsymbol{x}_i(t) - \boldsymbol{s}(t)\| \xrightarrow[t\to\infty]{} 0 \; \forall i, = 1,\ldots,N, \quad (4)$$

and synchronizes locally to the reference state, if there exists an $\varepsilon > 0$ such that for any solution with

$$\|\boldsymbol{x}_i(0) - \boldsymbol{s}(t)\| < \varepsilon, \quad (5)$$

we have the relation (4).

Whether or not the network synchronizes to the reference state depends mainly on four causes: (i) the dynamics of the individual systems, expressed by $F(\cdot)$ in Eqs. (1)–(3); (ii) the network structure, represented by the connection graph described by A or L; (iii) the type and strength of the interaction between the individual dynamical systems; and (iv) the driver nodes and feedback control gain. In Eq. (3), σ and H are related to (iii) and β and k are related to (iv). Indeed, the main problem that should be studied in the above configuration is the stability of the solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$. Similar to the approach proposed by the master stability function formalism [19], local stability of the synchronized solution can be evaluated in terms of N independent blocks in the parameters $a = \sigma\lambda_i (i = 1,\ldots,N)$ as

$$\frac{\boldsymbol{\zeta}_i}{dt} = [DF(s) - aDH(s)]\,\boldsymbol{\zeta}_i; \quad i = 1,2,\ldots,N, \quad (6)$$

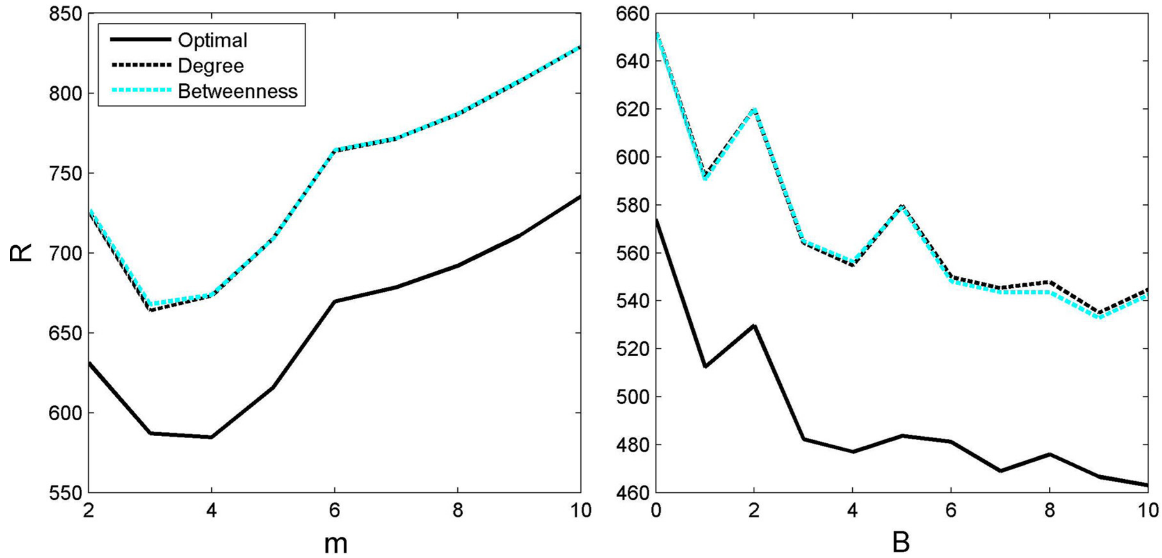

FIG. 1. (Color online) The eigenratio R indicating the controllability (the smaller R is, the more controllable the network) in networks with optimized position for driver nodes (Optimal), those with high degree nodes as drivers (Degree), and those with high betweenness nodes as drivers (Betweenness). The networks are scale free with $N = 500$ and $B = 0$ (left), and $m = 3$ (right). B [Eq. (11)] controls heterogeneity of the network (i.e., the higher the B, the less the heterogeneity), and the connection probability of newly added nodes depends on B. The average degree of the network is almost $2m$ (see text for details). There are $N_d = 5$ driver nodes in the networks. Data show averages over 20 realizations.

where D stands for Jacobian and λ_i's are the eigenvalues of the following matrix:

$$C = \{c_{ij}\} = \begin{pmatrix} l_{11} + k_1\beta_1 & l_{12} & \dots & l_{1N} \\ l_{21} & l_{11} + k_1\beta_2 & \dots & l_{2N} \\ \vdots & \vdots & \dots & \vdots \\ l_{N1} & l_{N2} & \dots & l_{NN} + k_N\beta_N \end{pmatrix}. \tag{7}$$

Indeed, C is the Laplacian matrix L with the information on the driver nodes added to the diagonal entries, i.e., a diagonal element is different only when the corresponding node is among driver nodes. Since L is symmetric, C will also be a symmetric matrix, and thus its eigenvalues are real. Let us denote the eigenvalues of C as $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_N$. When all Lyapunov exponents of Eq. (6) are negative, the synchronized solution is locally stable. The largest Lyapunov exponent of the variational equation (6) $\Lambda(a)$, called master

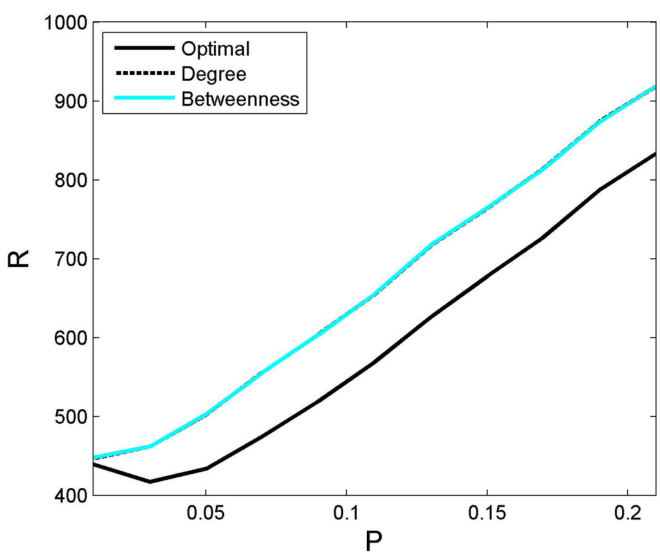

FIG. 2. (Color online) R as a function of connection probability P in Erdos-Renyi networks with $N = 500$. Designations are as in Fig. 1. Note that cyan (gray) and black-dashed lines are largely overlapped in this figure.

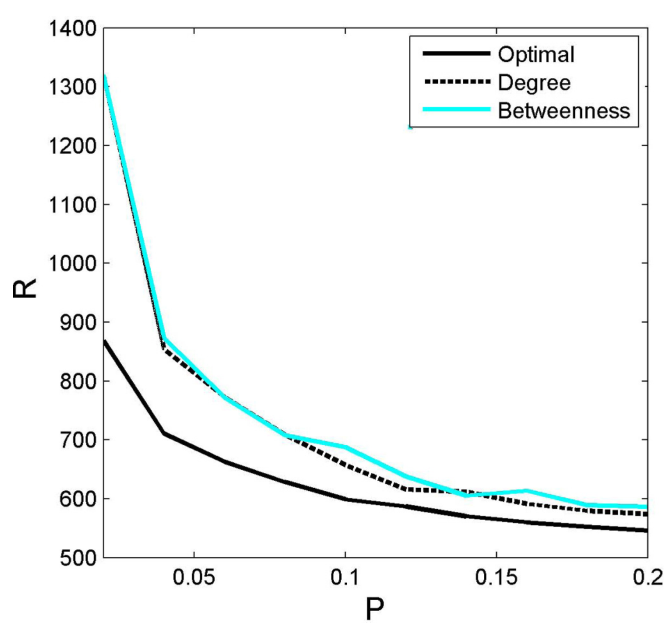

FIG. 3. (Color online) R as a function of rewiring probability P in Watts-Strogatz networks with $N = 500$ and $m = 3$ (right). Designations are as Fig. 1.

stability function, accounts for the linear stability of the synchronization solution; i.e., if $\Lambda(a) < 0$, the synchronized state is stable. In other words, in order for the dynamical network (3) to be synchronized to the reference state, the coupling strength, feedback control gain, and configuration of driver nodes must be in a way such that $\Lambda(a) < 0$. The master stability function is independent of a particular choice of dynamical system and coupling configuration, i.e., independent of H and $F(\cdot)$ in Eqs. (1) and (3).

For many systems, the master stability function is only negative within an interval (a_1,a_2), and hence, the network synchronizes to the reference state in such an interval. Requiring all coupling strengths to lie within this interval, i.e., $a_1 < \sigma\lambda_1 \leqslant \cdots \leqslant \sigma\lambda_N < a_2$, one concludes that if the network locally asymptotically synchronizes to the reference state, we have

$$R = \frac{\lambda_N}{\lambda_1} < \frac{a_2}{a_1} \tag{8}$$

for the corresponding graph.

The right-hand side of (8) depends on the dynamics of the individuals and coupling configuration, while its left-hand side depends on the connection graph, configuration of the driver nodes, and feedback control gain. In this work, we assume that the feedback control gains are all the same, and thus R in Eq. (8) depends on the connection graph and the position of driver nodes. Based on this configuration, one can argue that the larger the range of stabilizing parameters, the better controllable the network. Therefore, it relates the controllability to the eigenratio $R = \lambda_N/\lambda_1$, and concludes that the smaller R, the better its controllability. Considering the stabilizing coupling strength within an interval (σ_1,σ_2), the master stability function formalism requires

$$a_1 < \sigma_1\lambda_2; \quad \sigma_2\lambda_N < a_2. \tag{9}$$

Since a_1 and a_2 are fixed for any dynamical system and coupling configuration, to extend the interval of stabilizing parameter (σ_1,σ_2), and consequently enhance the controllability of the network, one should make the eigenratio R as

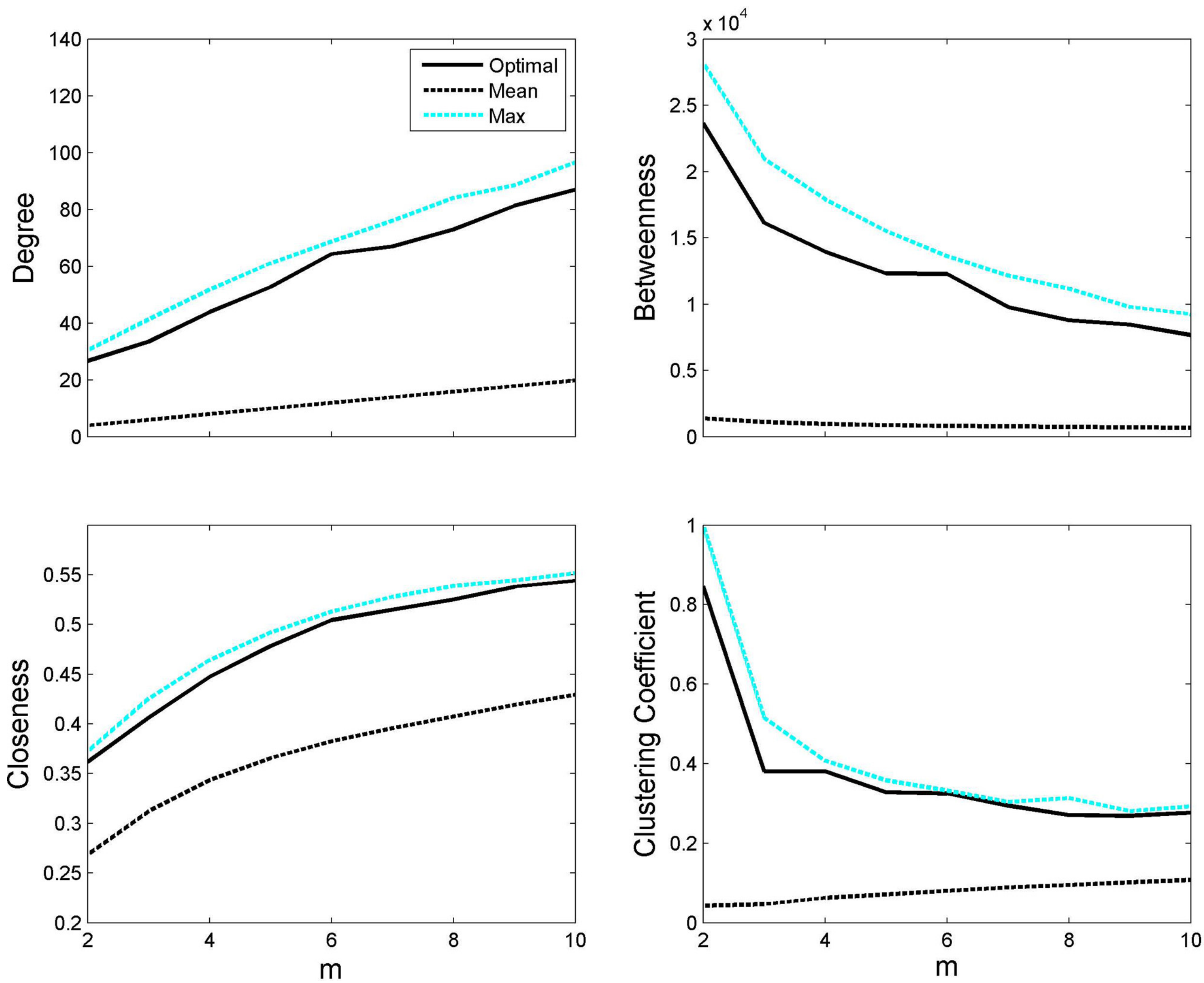

FIG. 4. (Color online) Various centrality properties of driver nodes (degree, betweenness, closeness, and clustering coefficient) as a function of m in scale-free networks with $N = 500$ and $B = 0$. Graphs show mean centrality values of five driver nodes with optimized position in the network (Optimal), mean centrality of the population (Mean), and mean centrality value of five nodes with the highest centrality. Data show averages over 20 realizations.

small as possible. In some systems, a_2 is infinite, and thus, the controllability of the networks is solely determined by λ_1; the higher λ_1, the more controllable the network, i.e., the lower is the stabilizing value of σ above which the synchronized reference solution is stable. In this work, we consider only undirected networks, resulting in symmetric L and C. For directed networks, these matrices are asymmetric, and thus, the eigenvalues are complex. In such cases, one should also consider the imaginary part of the eigenvalues, while the real part is the main indicator of controllability.

III. OPTIMALLY PLACING THE DRIVER NODES

Considering R as an indicator of network controllability, it depends not only on the network topology, but also on the number of driver nodes and their position in the network. Here, we assume that all feedback gains are the same for driver nodes. We fix the number of driver nodes and use an optimization method in order to find the best possible position for these nodes. Often, driver nodes are determined based on their centrality scores. For example, it has been shown that pinning nodes with high degree or betweenness centrality is effective in synchronizing the dynamics into the reference state [17,27,28]. Considering the structural controllability concept [29], it has been shown that the driver nodes are among those with lower than average degree [12]. However, the concept we consider for the controllability is different from that of structural controllability and our results cannot be compared with those obtained in [12]. Let us fix the number of driver nodes as $N_d (N_d \ll N)$. In heuristic methods, first the nodes are sorted based on their degree or betweenness centrality. Then N_d nodes with the highest degree or betweenness are considered as driver nodes. If two nodes have the same degree or betweenness centrality, the driver nodes are selected randomly from them.

In this work, we formulate the problem of finding the best possible N_d driver nodes as an optimization problem with specific cost function. We then use an evolutionary algorithm for the optimization process. Our goal is to minimize the eigenratio R as described by Eq. (8). We assume that all feedback gains are equal, i.e., $k_i = k_j$ for all i and j. Thus, the only free parameters for the optimization process are the position of the driver nodes, i.e., β_i's, which are either 1 or 0,

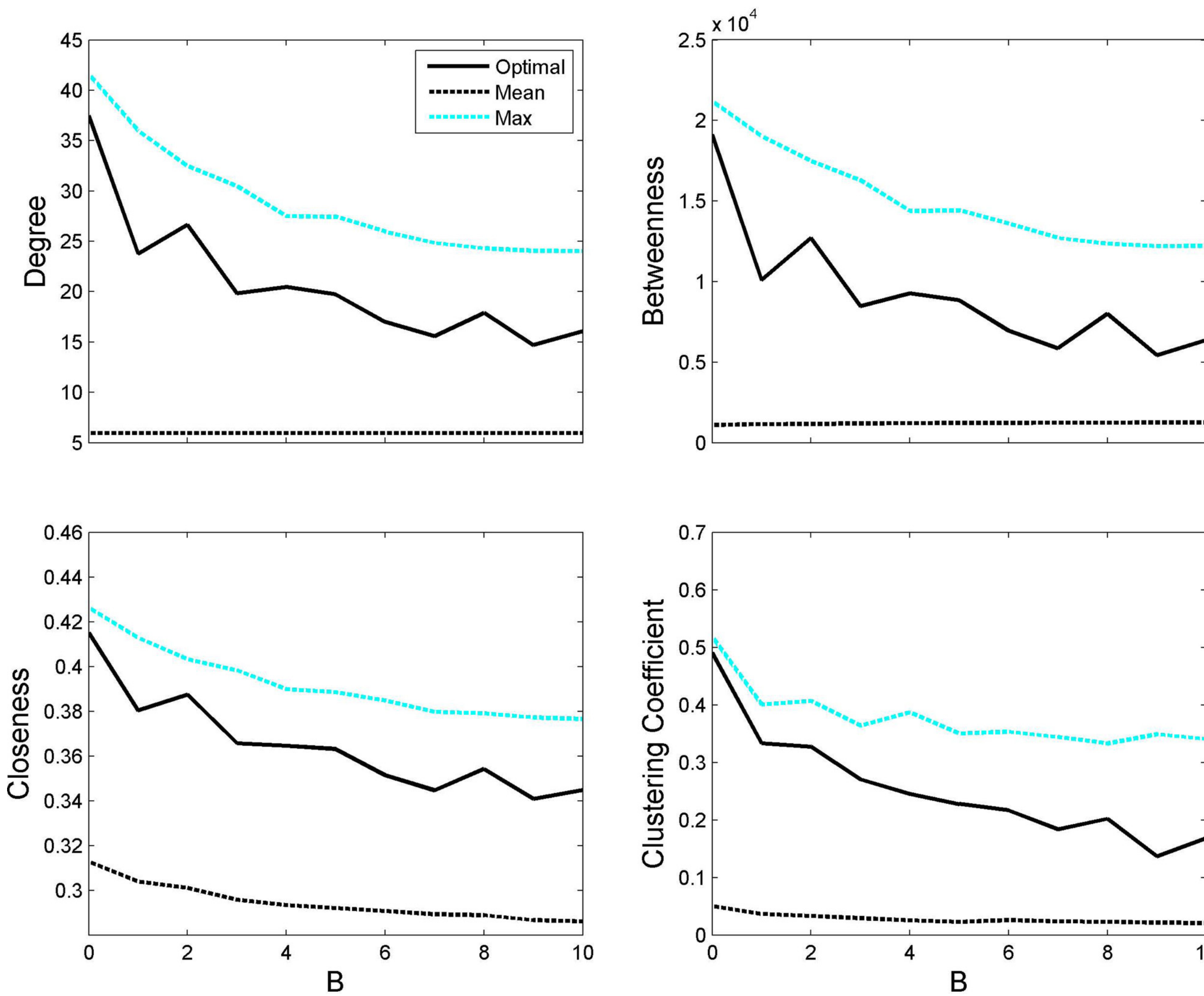

FIG. 5. (Color online) Various centrality properties of driver nodes (degree, betweenness, closeness, and clustering coefficient) as a function of B in scale-free networks with $N = 500$ and $m = 3$. Designations are as in Fig. 4.

1 when node i is a driver node, and 0 otherwise. Furthermore, the total number of driver nodes is N_d. Thus, the optimization problem is

$$\min_{\beta_i} R(\boldsymbol{\beta}) = \frac{\lambda_N(\boldsymbol{\beta})}{\lambda_1(\boldsymbol{\beta})},$$
$$\text{subject to} \sum_{j=i}^{N} \beta_j = N_d \quad \text{and} \quad \beta_j = \{0,1\}, \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ are the parameters that should be optimized.

Since the above optimization problem is not convex, we use an evolutionary algorithm in order to obtain the optimal parameters. In this work, we use differential evolution (DE), which has been shown to be a powerful technique for various optimization processes [30,31]. DE is an evolutionary algorithm designed to solve optimization problems and has the ability to handle nondifferentiable, nonlinear, and multimodal cost functions. Effectiveness of this optimization technique is originated from the fact that DE performs mutation based on the distribution of the solutions in the current population; in spite of blind operators that exist in traditional evolutionary methods [31].

In the DE optimization algorithm, each candidate solution is called a chromosome and there is a population of them in the solution space. DE can handle all types of binary, integer, and continuous chromosomes. Crossover operators such as binomial and exponential crossover are used to combine the chromosomes [31]. If the building blocks are important in chromosomes, binomial crossover is often used; otherwise, exponential crossover is favored. The optimization algorithm finds a solution for the minimization problem as expressed in (10). Here, we use binary representation for chromosomes and fix the population size as the size of the network N.

IV. RESULTS AND DISCUSSION

A. Model networks

We apply the proposed optimization method to find driver nodes in a number of model networks, including preferential attachment scale-free, random, and small-world networks. We construct scale-free networks using an algorithm proposed in [32], which itself is a generalization of the original preferential attachment growing procedure introduced by Barabasi and Albert [33]. Namely, starting with a network of $m + 1$ all-to-all connected nodes, at each step, a new node is added

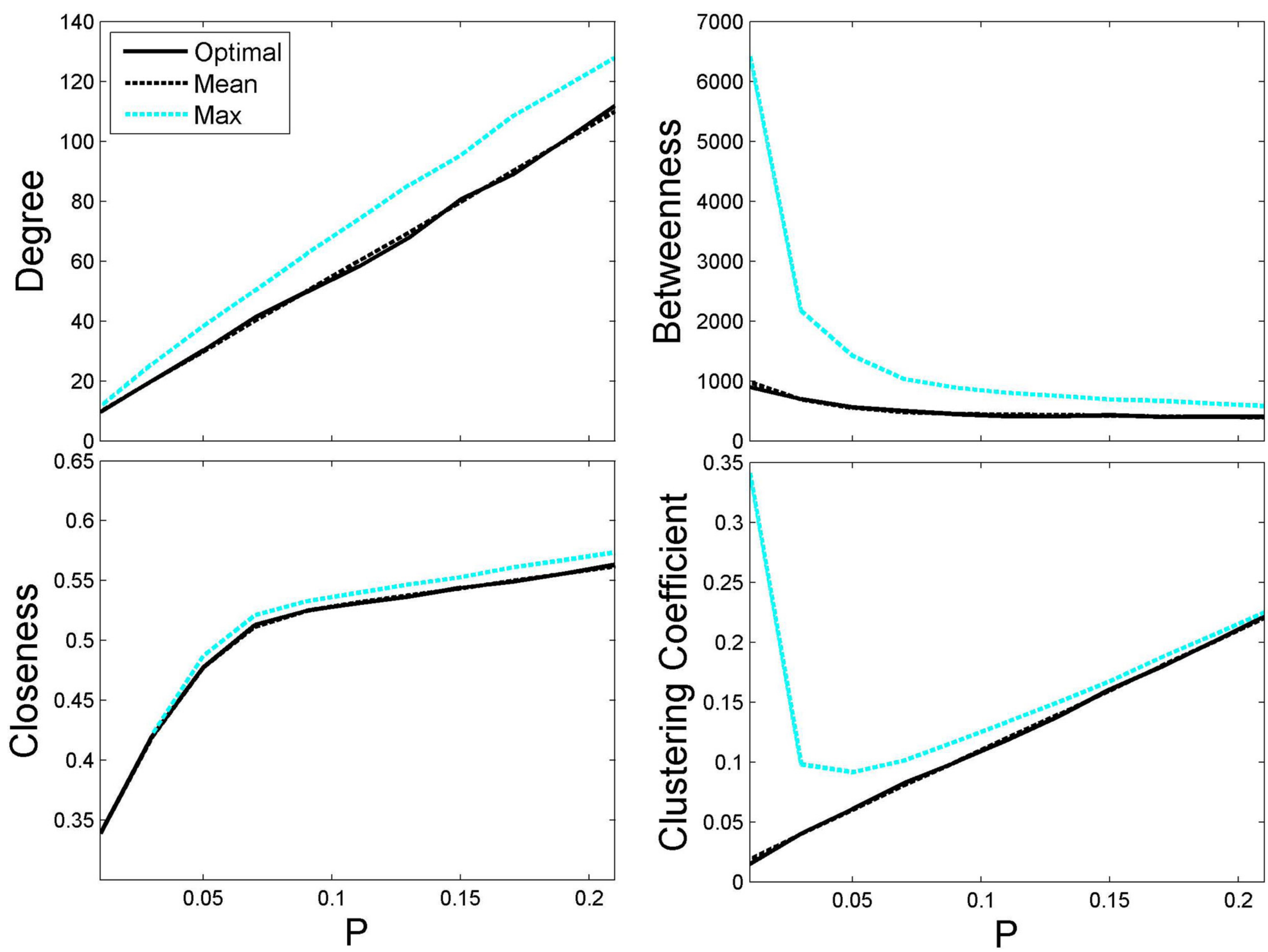

FIG. 6. (Color online) Various centrality properties of driver nodes as a function of connection probability P in Erdos-Renyi networks with $N = 500$. Designations are as in Fig. 4. Note that black and black-dashed lines are largely overlapped in this figure.

with m links that are connected to node i with probability

$$p_i = (k_i + B) \Big/ \sum_j (k_j + B), \tag{11}$$

where k_i is the degree of the node and B a tunable real parameter controlling the heterogeneity of the network; the higher the B, the less heterogeneous the network [32]. It can easily be shown that networks constructed with this algorithm have power-law degree distribution with exponent $\gamma = 3 + B/m$ [32].

Not all real networks have power-law degree distribution; some real networks show homogeneous distribution of degrees. We use two models in order to construct such networks. We use the Erdos-Renyi model in order to construct pure random networks; each pair of nodes in Erdos-Renyi networks is connected with probability P. Watts and Strogatz showed that many real networks are not random and have high transitivity [34]. They proposed a simple model in order to construct networks with such properties (short average path length and high clustering coefficient), which is used in this work. The model is as follows [34]. Starting with a regular ring graph with N nodes each connected to their m-nearest neighbors, the links are rewired with probability P. For some values of P, the resulting networks have both short average path length and high transitivity.

B. Numerical results

The effectiveness of pinning control strongly depends on the graph topology [20]; our results show that the performance of optimal pinning control also depends on the topology. Figs. 1–3 compare the performance of the proposed optimization process with heuristic methods (i.e., selecting the driver nodes based on their degree or betweenness centrality). In these experiments 1% of the nodes are considered as driver nodes ($N_d = 5$). It has been previously shown that in order to have an efficient pinning control, it is sufficient to control a small portion of the nodes [22]. Often, the driver nodes are selected among hub nodes with high degrees or betweenness centralities [15,16,22]; however, we show that one can always find a subset of optimal drivers that are not necessarily among hub nodes. The results show that the proposed optimization process results in much better controllability (i.e., less R) as compared to heuristic methods (i.e., choosing the high degree or betweenness nodes as drivers). For example, in scale-free networks, the optimal strategy results in about 17% less R than the other two strategies.

As the average degree of the networks increases, their controllability worsens, i.e., R increases. The average degree of scale-free networks increases as m increases, and our results show that all methods show declined controllability by

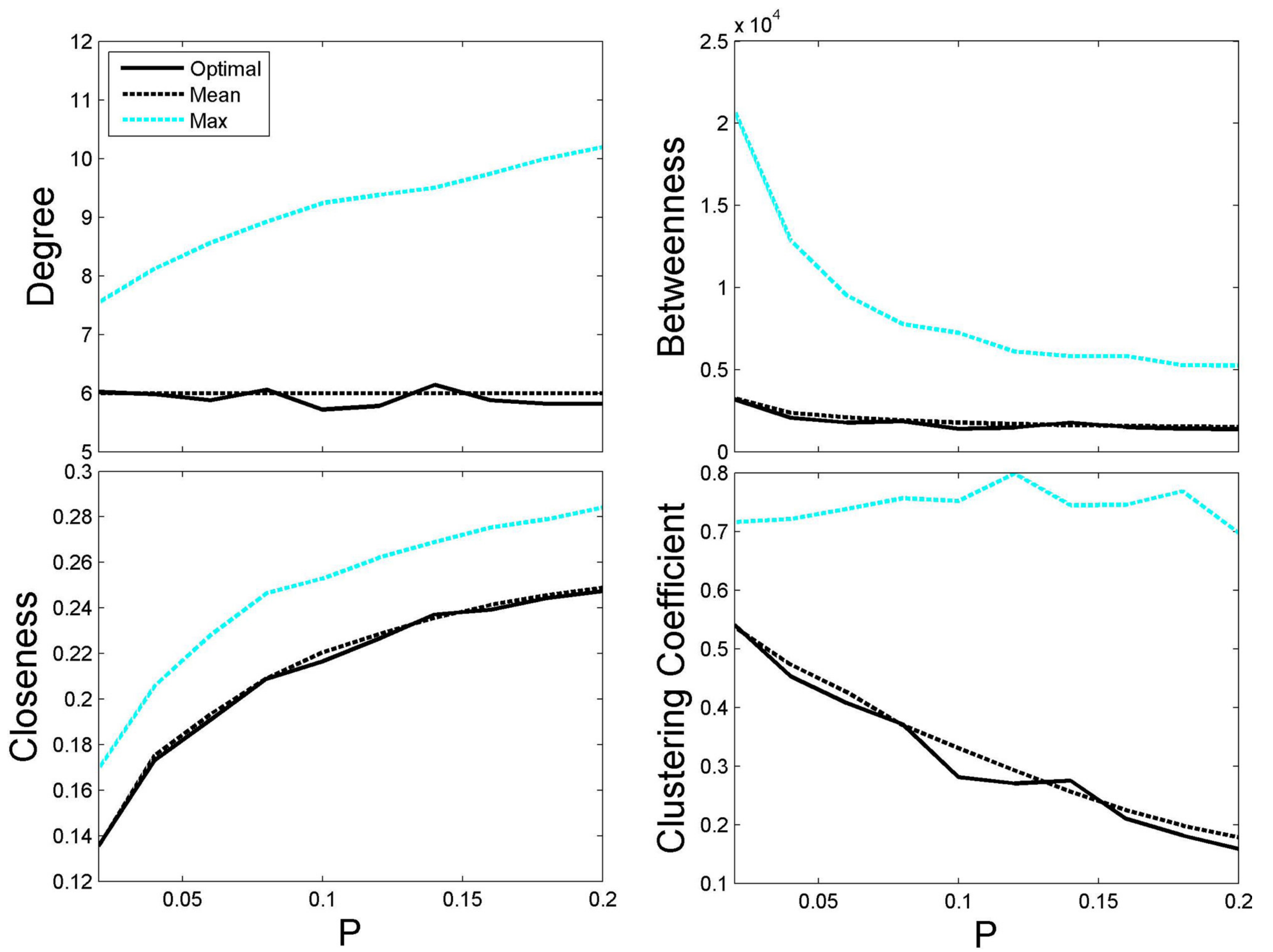

FIG. 7. (Color online) Various centrality properties of driver nodes as a function of rewiring probability P in Watts-Strogatz networks with $N = 500$ and $m = 3$. Designations are as Fig. 4.

increase of m (Fig. 1; left panel). In Erdos-Renyi networks, the connection probability P controls the average degree, in which the same behavior (i.e., declined controllability by increasing the average degree) is observed (Fig. 2). As the number of links increases, degree heterogeneity increases as well, resulting in worsening the ability to control the network. In order to further investigate the role of heterogeneity on the controllability, we change parameter B, which controls the heterogeneity in scale-free networks, i.e., the higher the B, the less the heterogeneity in degree distribution. It is seen that as B increases, R decreases, and thus the controllability improves (Fig. 1; right panel). However, heterogeneity is not the only factor in determining controllability of networks. It is well known that degree heterogeneity of Watts-Strogatz networks increases as the rewiring probability P increases. Note that for $P = 0$, we have an m-regular ring graph, while $P = 1$ results in pure random structure. Our results show that as P increases, the controllability of Watts-Strogatz networks improves, although their heterogeneity increases. Indeed, as P increases, more shortcuts are created between the nodes, and thus the network becomes better communicable. This makes it easier for driver nodes to communicate with other nodes, and thus to control the dynamics into the reference state.

We next investigate properties of optimal driver nodes. To this end, we study their properties in terms of different centrality measures including degree, betweenness, closeness, and clustering coefficient. We compare mean centrality of driver nodes with mean centrality of all nodes and that of five nodes with the highest centrality value (since $N_d = 5$). Figures 4–7 show the profile of these centrality measures in different networks. There is significant variability between the behaviors of network structures. In scale-free networks, for instance, the optimal driver nodes have centrality values close to those of hub nodes with high centralities (Figs. 4 and 5); centrality of optimal driver nodes is much closer to those of hub nodes than the mean population. As these networks become less heterogeneous (by increasing B), centrality scores decrease, but similarly proportioned high-centrality nodes remain favored as the driver nodes (Fig. 5). Indeed, the centrality values of optimally placed drivers get closer to the mean population. This means that as these networks become less heterogeneous, centrality scores decrease, but similarly proportioned high-centrality nodes remain favored as the driver nodes.

Figures 6 and 7 show the profiles of centrality measures in Erdos-Renyi (as a function of connection probability) and Watts-Strogatz networks (as a function of rewiring probability), respectively. These two models result in networks with almost homogeneous centrality distribution as compared to scale-free networks. Our results show that the centrality values of optimal drivers are almost similar to the average population in these homogeneous networks. Therefore, in homogeneous structures, optimal drivers do not have a specific property (in terms of centrality measures) as compared to other nodes. Indeed, in such networks, the average centrality of optimal drivers obeys the population average. However, this is not the case in heterogonous networks for which the optimal drivers have close centrality to the hub nodes.

In order to assess whether the results are sensitive to the number of driver nodes, we perform the experiments with $N_d = 10$. Figure 8 illustrates the performance of the proposed optimization method; the general profile is similar to the previous case with $N_d = 5$. For $N_d = 10$, we indeed found similar patterns as $N_d = 5$ (see figures in Supplemental Material [35]). Here we fix the location of the optimal driver nodes. More complex scenarios can also be applied where the location of the optimal drivers is not fixed and at each step, a certain number of the nodes are taken into account as drivers [23]. However, in many applications (such as power grids), a designer often would like to find a certain number of the nodes as optimal drivers and apply the control on them. Controlling a node requires substantial cost, and it might not be possible to include the possibility of control for many nodes.

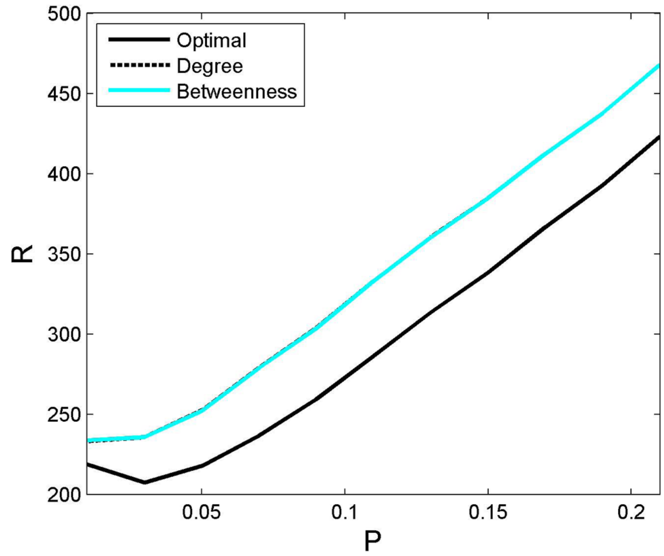

FIG. 8. (Color online) R as a function of connection probability P in Erdos-Renyi networks with $N = 500$. There are $N_d = 10$ driver nodes in the networks. Designations are as Fig. 1. Note that cyan (gray) and black-dashed lines are largely overlapped in this figure.

V. CONCLUSIONS

Pinning network dynamics to a reference state is often known as the network control problem. In pinning control, an input signal is fed into a number of nodes, known as driver nodes, and other nodes are forced to follow the reference state. The ultimate goal is to fully synchronize all the nodes to the reference state. In this paper, we have considered the concept of local synchronization to obtain a measure quantifying controllability of networks. The measure is based on the eigenvalues of a matrix that is obtained considering the Laplacian matrix of the connection graph and information on the driver nodes. In this work, we have used an optimization method based on differential evolution to find the best placement for a predetermined number of driver nodes. Applying the proposed placement strategy on a number of model networks including scale-free, pure random, and small-world, we have shown that it significantly outperforms heuristic methods including choosing the driver nodes as those with high degree or betweenness centrality.

We have studied the effect of heterogeneity in pinning controllability and showed its nonuniform behavior. In scale-free networks, decreasing the heterogeneity of the networks resulted in improving the controllability, while it had the inverse effect in Watts-Strogatz networks. Also, the controllability worsened by increasing the average degree of the networks. In an effort to understand the relation between network structure and its controllability, we studied properties of optimally placed driver nodes. To this end, we have considered a number of centrality values including degree, betweenness, closeness, and clustering coefficient. Centrality values of the driver nodes were strongly dependent on the network structure. In scale-free networks with heterogeneous degree distribution, the centrality values of optimal driver nodes were higher than the average value for the population and close to the hub nodes with high centrality, whereas, the centralities of driver nodes were almost the same as the population mean in homogeneous networks such as pure random and small-world ones.

The results obtained in this work are important in analyzing systems for which pinning synchronization is important. Examples of such real-world applications include synchronization in modern power grids [36] and parallel tasks in computer networks [37].

ACKNOWLEDGMENTS

This work was supported by Australian Research Council Project No. DE140100620.

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