Nonthermal and suprathermal distributions as a consequence of superstatistics

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We propose to put the well-known nonthermal and suprathermal empirical distributions, used in plasma physics, onto a more rigorous foundation. Their use is frequently criticized because of a lack of formal derivation and physical explanation. A connection between these non-Maxwellian distributions and the Beck-Cohen superstatistics is suggested. They are perceived as a consequence of typical temperature fluctuations. We show that the suprathermal distribution is generated by the Γ distribution of the inverse temperature, in the same way as the Tsallis q statistics. The nonthermal distribution also follows from the χ^2 distribution, with a small variance. Our contribution provides a possible physical meaning for these *ad hoc* distributions.

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I. INTRODUCTION

The most important concept—with regard to its omnipresence—in the study of complex systems, such as plasmas, is undoubtedly the concept of *distribution functions*. Indeed, the latter are involved in the derivation of thermodynamical measurable quantities, such as density or mean velocity and can be experimentally obtained by the direct observation of the system. Thus, the major applications of statistical physics relate to the use of distribution functions to describe various phenomena, e.g., electrons in a metal [1,2], Brownian motion of a particle in a fluid [3], or wave propagation in plasmas [4]. On the other hand, most of the developments in the foundations of statistical mechanics are motivated by the observation of distribution functions that cannot be explained by the usual first principles of statistical physics. The so-called nonextensive statistical mechanics proposed by Tsallis [5], a quarter century ago, has been introduced to deal with this problem. In fact, it leads to power-law distributions, observed in a variety of systems, that cannot be derived from the usual statistical mechanics. And so do the κ statistics proposed by Kaniadakis [6] or the Beck-Cohen superstatistics [7], to cite only few. The most common distribution function is the one derived by Boltzmann [8] in the kinetic theory of gases,

$$p_i = \frac{\exp(-\beta\varepsilon_i)}{Z}, \tag{1}$$

where ε_i is the energy of the ith state, β is the inverse temperature in energy units ($\beta = 1/k_BT$), and Z is the partition function that normalizes the distribution Eq. (1). Let us shed light on the procedure that leads to its establishment. Boltzmann considered an ideal gas of N particles in the phase space of one molecule, i.e., each one having a certain position $\overrightarrow{r}$ and a certain velocity $\overrightarrow{v}$, and he partitioned the phase-space into W cells ($W \ll N$) with a defined energy and a certain volume ω_i . He assumed equal the probability for any particle of the gas to be in any cell. Then, he maximized the probability under the constraint of a given number of particles,

$$\sum_{i=1}^{W} n_i = N, \tag{2}$$

and a given total energy,

$$\sum_{i=1}^{W} n_i\varepsilon_i = E. \tag{3}$$

This procedure leads to the Boltzmann distribution Eq. (1), which states that the probability of a given energy decreases exponentially with this energy. The distribution Eq. (1) is very common in nature. It describes, for example, velocity distribution of galaxy clusters in astrophysics [9]. However, numerous observations pointed out distributions that differ, more or less slightly, from it. Different empirical distribution functions have then been introduced to model such a behavior in a variety of systems out of thermal equilibrium. These distributions (frequently criticized because of a lack of formal derivation) could be a consequence of the inadequacy of one of the steps performed by Boltzmann, and can be a result of a yet unknown statistics. Among these distributions, the most commonly encountered in plasma physics are the nonthermal [10] and the suprathermal [11] distributions. The latter have been observed over and over and numerous *in situ* observations clearly indicate that they are ubiquitous in astrophysical plasma environments [12–15]. Empirical functions have then been introduced either in an *ad hoc* manner [10] or by curve fitting methods [11] to model such spatial environments. These functions are, nowadays, widely used in the investigation of various phenomena in plasma physics [16–19]. However, their use is frequently criticized because of a lack of formal derivation and physical explanation [20]. This paper is therefore aimed at putting the nonthermal and suprathermal distributions onto a more rigorous foundation, based upon the Beck-Cohen superstatistics [7].

II. BECK-COHEN SUPERSTATISTICS

The Beck-Cohen superstatistics [7] are a generalization of the ordinary Boltzmann statistics, for which it reduces in a certain limiting case. They constitute a powerful tool

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for studying systems with complex dynamics and out of equilibrium [21–24]. They rely on a more general formalism than the Tsallis q statistics, since they generate not only Tsallis-type distributions but can also generate other statistics. The main idea of the superstatistics is to consider a system with spatiotemporal fluctuations of an intensive quantity. In what follows, we will choose the temperature as our fluctuating quantity. This could be the inverse of another intensive quantity, such as the pressure, the chemical potential, or the energy dissipation rate in a turbulent fluid. Note that such an approach was first introduced in Refs. [25,26] to give an explanation of the q parameter appearing in the Tsallis q statistics, based on fluctuations of the physical quantities. We consider a nonequilibrium steady state of a macroscopic system, made up of many smaller cells that are temporarily in local equilibrium. Within each cell, the temperature is approximately constant. Each cell is large enough to obey statistical mechanics but has a different β assigned to it, according to a probability density $f(\beta)$. We assume that the local temperatures in the various cells change on a long time scale, $t_{\rm scale}$, much larger than the relaxation time a single cell needs to reach its local equilibrium. Then, the distribution $f(\beta)$ will shape the generalized Boltzmann distribution, which is given by Ref. [7],

$$B(E) = \int_0^\infty f(\beta)\exp(-\beta E)d\beta, \quad (4)$$

where B is an effective generalized Boltzmann factor for the nonequilibrium system. In a certain sense, it represents the statistics $[f(\beta)]$ of the statistics $[\exp(-\beta E)]$ of the cells constituting the system [7]. In absence of fluctuations, i.e., $f(\beta) = \delta(\beta - \beta_0)$, the ordinary Boltzmann factor is recovered. It has been shown that the generalized distributions Eq. (4) can describe Tsallis distributions or distributions that are nearly Tsallis with tiny corrections. We will show here that a judicious choice of $f(\beta)$ can lead to the so-called nonthermal and suprathermal distributions.

In principle, $f(\beta)$ could be any function. Nevertheless, it has to satisfy some conditions: (i) it must be a normalized probability density; (ii) it has to be chosen in such a way that the integral $\int_0^{+\infty} B(E)dE$ exists [in a more general way, the integral $\int_0^{+\infty} \rho(E)B(E)dE$, where $\rho(E)$ is the density of states, must exist], and (iii) it has to present a limit for which the usual Boltzmann factor is recovered. The Γ distribution is one of the most ubiquitous probability densities in nature. It reads as [27]

$$f(\beta) = \frac{1}{b\Gamma(c)}\left(\frac{\beta}{b}\right)^{c-1}\exp(-\beta/b), \quad (5)$$

where b and c are positive real parameters. This two-parameter continuous probability distribution contains as special cases the common exponential distribution and the χ^2 distribution. Figure 1 shows its behavior with different values of the parameters b and c. Making use of Eq. (5), the average value of the inverse temperature β is given by

$$\beta_0 = \int_0^\infty \beta f(\beta)d\beta = bc, \quad (6)$$

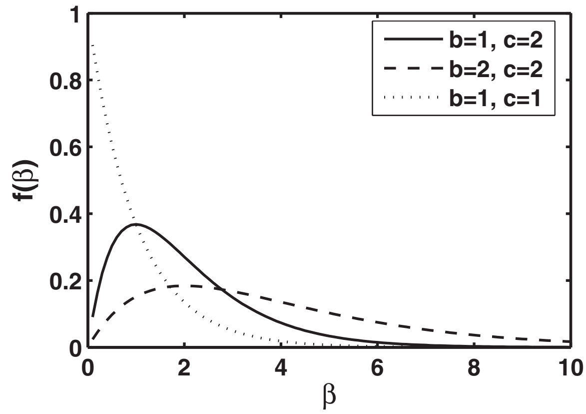

FIG. 1. Variation of the Γ distribution with the inverse temperature β for different values of the parameters b and c ($b = 1$, $c = 2$, solid line; $b = 2$, $c = 2$, dashed line; $b = 1$, $c = 1$, dotted line).

and the variance reads as

$$\sigma^2 = \int_0^\infty (\beta - \beta_0)f(\beta)d\beta = b^2c. \quad (7)$$

The generalized Boltzmann factor Eq. (4) becomes

$$B(E) = \int_0^\infty f(\beta)\exp(-\beta E)d\beta = (1 + bE)^{-c}. \quad (8)$$

By identifying $c = \kappa + 1$, Eq. (8) becomes

$$B(E) = \left(1 + \frac{\beta_0 E}{\kappa + 1}\right)^{-\kappa-1}. \quad (9)$$

The latter is the well-known suprathermal distribution. It has been introduced to fit the OGO1 and OGO2 solar wind electron data [11]. The parameter κ shapes the suprathermal tail of the distribution and measures its deviation from the Maxwell-Boltzmann equilibrium (which is recovered in the limit $\kappa \to \infty$). Figure 2 shows the behavior of the distribution Eq. (9) with different values of c (or equivalently the values of the so-called spectral index κ). As one may expect, as c or κ increases (a tendency toward Maxwellian equilibrium)

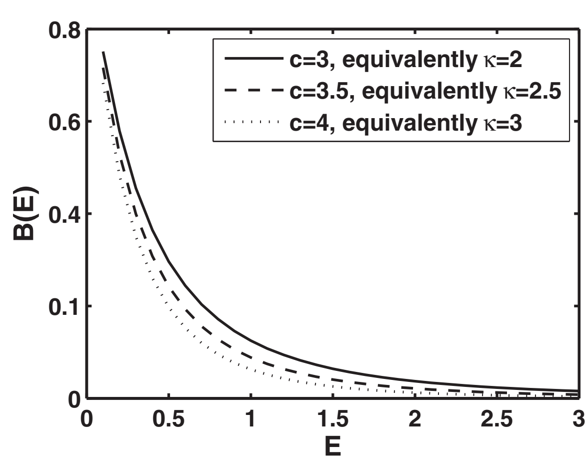

FIG. 2. Plot of the suprathermal distribution for different values of $c = 3$ ($\kappa = 2$, solid line), $c = 3.5$ ($\kappa = 2.5$, dashed line), and $c = 4$ ($\kappa = 3$, dotted line), with $b = 1$.

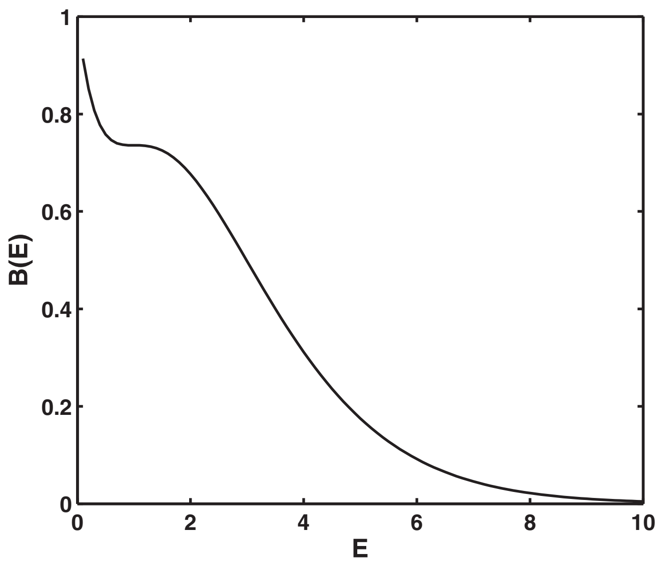

FIG. 3. Plot of the nonthermal distribution for $c = 1/2$ and $b = 1$.

high-energy states become less probable. Numerous observations clearly indicate that such distributions are ubiquitous in a variety of plasma environments. They have been observed, for instance, in the proton data near the Earth's bow shock [12] and IMP6 solar wind proton observations [13]. Nowadays, a large part of the plasma literature is devoted to their study and to the investigation of their influence on several plasma processes. The parameter κ, measuring the deviation from the usual Boltzmann equilibrium, is, in our interpretation, related to the fluctuations of the inverse temperature β. It follows from a Γ-distributed inverse temperatures β. It has been shown that the Tsallis q distributions can also be generated by a Γ distribution, with the identification $c = 1/q - 1$ [7]. This is in consistency with recent investigations stating that the suprathermal distribution is identical to the Tsallis q distribution, with $\kappa + 1 = 1/q - 1$ [20,28,29].

We can show that in a certain limit of fluctuations, the Γ distribution may also lead to the nonthermal or Cairns distribution. In fact, the distribution Eq. (8) can be written in the form

$$B(E) = (1 + bE)^{-c} = \exp\{-c \ln(1 + bE)\}. \tag{10}$$

In the case of small fluctuations $bE \ll 1$, the exponential can be expanded to keep only the first term on bE,

$$B(E) = \exp(-\beta_0 E)\left[1 + \frac{1}{2c}\beta_0^2 E^2\right]. \tag{11}$$

Introducing $\alpha = 1/(2c)$, Eq. (11) leads to the well- known nonthermal distribution

$$B(E) = \exp(-\beta_0 E)\left[1 + \alpha\beta_0^2 E^2\right], \tag{12}$$

where α is the nonthermal parameter. In Fig. 3, we show the plot of the distribution Eq. (12) for a maximum nonthermality ($\alpha = 1$, or equivalently $c = 1/2$). This distribution exhibits the presence of a flat shoulder, which is a characteristic of the nonthermal energetic particles. This distribution Eq. (12) has been introduced by Cairns *et al.* [10] to show that the presence of nonthermal electrons may change the nature of ion-sound solitary structures and allow the existence of rarefactive ion-acoustic solitary structures very much like those observed by the Freja and Viking satellites. In our interpretation, nonthermal effects are linked to Γ-distributed inverse temperatures β.

III. SUMMARY

In this paper, we aimed to deal with the physical explanation of the two famous empirical distributions, namely, the suprathermal and the nonthermal ones. A connection between these non-Maxwellian distributions and superstatistics is suggested. Using the Beck-Cohen superstatistics, we interpret these two distributions as a consequence of temperature fluctuations. We have shown that when the inverse temperature β is Γ-distributed, the corresponding distribution function (given by the Laplace transform) leads to the suprathermal distribution, with the parameter κ that underpins suprathermality given by $c = \kappa + 1$. Note that the quantity $2c$ is usually interpreted as the effective number of degrees of freedom contributing to this fluctuation. The parameter κ increases with the degree of freedom of the temperature fluctuation and (in a certain sense) measures it. This same Γ distribution leads to the nonthermal distribution in the case $bE \ll 1$ [small variance case, which is proportional to b, see Eq (7)]. When the variance σ is getting smaller and smaller, the distribution tends to have a Maxwellian behavior. In the limit $\sigma \to 0$, in which $f(\beta)$ leads to the Dirac function, the Maxwellian distribution is exactly recovered. It is interesting to mention that this interpretation provides somehow an unification of these different distributions encountered in plasma physics. In fact, they appear to belong to the same universal class of temperature fluctuations, with different order of the magnitude of the variance. It is well-known that temperature fluctuations occur often in plasma environments, and they contribute considerably to the particle flux [30] and drive a significant amount of the anomalously high electron heat transport [31]. Many observations indicate the existence of temperature fluctuations in astrophysical plasmas environments [32,33] and in laboratory plasma devices such as spray torches [34] or fusion plasmas [30,35]. Then, it appears very plausible to relate the anomalous distributions observed in plasmas to typical temperature fluctuations. We stress that a comparison of the experimental data of the temperature fluctuations is a problem of great importance, but beyond the scope of the present work.

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