Anomalous scaling of passive scalar fields advected by the Navier-Stokes velocity ensemble: Effects of strong compressibility and large-scale anisotropy

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The field theoretic renormalization group and the operator product expansion are applied to two models of passive scalar quantities (the density and the tracer fields) advected by a random turbulent velocity field. The latter is governed by the Navier-Stokes equation for compressible fluid, subject to external random force with the covariance $\propto\delta(t-t')k^{4-d-y}$, where d is the dimension of space and y is an arbitrary exponent. The original stochastic problems are reformulated as multiplicatively renormalizable field theoretic models; the corresponding renormalization group equations possess infrared attractive fixed points. It is shown that various correlation functions of the scalar field, its powers and gradients, demonstrate anomalous scaling behavior in the inertial-convective range already for small values of y. The corresponding anomalous exponents, identified with scaling (critical) dimensions of certain composite fields ("operators" in the quantum-field terminology), can be systematically calculated as series in y. The practical calculation is performed in the leading one-loop approximation, including exponents in anisotropic contributions. It should be emphasized that, in contrast to Gaussian ensembles with finite correlation time, the model and the perturbation theory presented here are manifestly Galilean covariant. The validity of the one-loop approximation and comparison with Gaussian models are briefly discussed.

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I. INTRODUCTION

In the past few decades, intermittent interest has been attracted to the problem of intermittency and anomalous scaling in fluid turbulence; see, e.g., Refs. [1–11] and the literature cited therein. The phenomenon manifests itself in singular (arguably powerlike) behavior of various statistical quantities as functions of the integral turbulence scales, with infinite sets of independent anomalous exponents [1]. In spite of considerable success, the problem remains essentially open: No regular calculational scheme, based on an underlying dynamical model and reliable perturbation expansion (like the famous ε expansion for critical exponents), was ever constructed for the anomalous exponents of the turbulent velocity field.

Both the natural experiments and the numerical simulations suggest that deviations from the classical Kolmogorov theory are even more strongly pronounced for passively advected scalar fields (like the temperature or the density of a pollutant) than for the velocity field itself [2–7]. For the most recent research of turbulent transport, see, e.g., Refs. [8] and references therein.

At the same time, various simplified models, describing passive advection by "synthetic" velocity fields with given statistics, appear easier tractable theoretically and allow analytical results to be derived [9]. Therefore, the problem of passive advection, being of practical importance in itself, may also be viewed as a starting point in studying intermittency and anomalous scaling in fluid turbulence on the whole.

The most remarkable progress was achieved for the Kraichnan's rapid-change model [10], where the advecting velocity field is taken Gaussian, not correlated in time, and having a powerlike correlation function of the form $\sim\delta(t-t')/k^{d+\xi}$, where d is the dimension of space, k is the wave number, and ξ is an arbitrary exponent. There, for the first time, the existence of anomalous scaling was firmly established on the basis of a microscopic model [10]; the corresponding anomalous exponents were calculated in controlled approximations [11] and, eventually, within a systematic perturbation expansion in a formal small parameter ξ [12]. A detailed review of the theoretical research on the passive scalar problem and the bibliography can be found in Ref. [9].

In the original Kraichnan's model, the velocity ensemble was taken to be Gaussian, decorrelated in time, and isotropic, and the fluid was implied to be incompressible. More realistic models should take into account finite correlation time and non-Gaussianity of the velocity ensemble, anisotropy of the experimental setup, compressibility of the fluid, etc.; see the discussion in [2,3]. Here two key issues arise: formulation of more realistic models and the possibility to treat them (more or less) analytically.

A most efficient way to study anomalous scaling is provided by the field theoretic renormalization group (RG) combined with the operator product expansion (OPE); see [13,14] for detailed exposition of these techniques and the references. In the RG + OPE scenario for the anomalous scaling in turbulence, proposed in [15], the singular dependence on the integral scales emerges as a consequence of the existence in the corresponding models of composite fields ("composite operators" in the quantum-field terminology) with *negative* scaling dimensions, termed as "dangerous operators"; for more detailed explanations and the references, see [14–17]. For Kraichnan's model, the anomalous exponents can be identified with the scaling dimensions ("critical dimensions" in the terminology of the theory of critical state) of certain individual Galilean-invariant composite operators [12]. This

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allows one to give a self-consistent derivation of the anomalous scaling, to construct a systematic perturbation expansion for the anomalous exponents in ξ, and to calculate the exponents in the second [12] and third [18] orders. The RG approach can be generalized to the case of finite correlation time [19] and to the non-Gaussian advecting velocity field, governed by the stochastic Navier-Stokes equation [20]. A general overview of the RG approach to Kraichnan's model and its descendants and more references can be found in [21].

Numerous studies were devoted to the effects of compressibility on the intermittency and anomalous scaling [22–32]. Analysis of simplified models suggests that compressibility strongly affects the passively advected fields. In particular, in contrast to the incompressible case, the diffusion can be depleted by the advection of a purely potential flow [24] and the phase transition from a turbulent to a certain purely chaotic state takes place when the degree of compressibility increases [27]. It was also shown that the anomalous exponents become nonuniversal due to dependence on the compressibility parameter, such that the anomalous scaling is enhanced, while the hierarchy of anisotropic contributions is suppressed [28–32]. For passive vector (e.g., magnetic) fields, the issues of anomalous scaling, hierarchy of anisotropic contributions, and the dependence on compressibility were discussed, e.g., in [33–40].

An important advantage of Kraichnan's model is the possibility to easily model compressibility [22–28]. Generalization to the case of a Gaussian ensemble with finite correlation time is also possible [31,32,38]. However, synthetic models with nonvanishing correlation time suffer from the lack of Galilean symmetry, which may lead to "interesting pathologies" (quoting Ref. [3]). In the RG approach, one of such a pathology manifests itself as an ultraviolet (UV) divergence in the vertex [31], which in more realistic models is forbidden by Galilean invariance and for the incompressible Gaussian model is absent because of rather technical reasons [19]. Thus, it is desirable to describe the advecting velocity field by the corresponding Navier-Stokes equations [41] with a random stirring force. However, this appeared to be a difficult task.

In Refs. [42,43], the leading-order correction in the Mach number Ma to the incompressible scaling regime was studied; generalization to all orders of the expansion in Ma was derived in [44]. The corrections are small for very small Ma and not very small momenta k, but become arbitrarily large (IR relevant in the sense of Wilson) and destroy the incompressible scaling regime if Ma is fixed and the momenta become small enough. Thus, the original incompressible regime becomes unstable, and a crossover to another unknown regime occurs. The case of strong compressibility was studied in Refs. [45–47]. The results are rather controversial, but all of those studies support the existence of a stationary resulting "compressible" regime, different from the original incompressible one.

In the present paper, we adopt the approach of Ref. [47], where the standard field theoretic RG was applied to the problem of stirred hydrodynamics of a compressible fluid, and the resulting stationary scaling regime was associated with the IR attractive fixed point of the corresponding multiplicatively renormalizable field theoretic model. That approach was later applied to the problem of mass distribution in the self-gravitating matter within the framework of a continuous stochastic formulation of the Vlasov-Poisson model [48]. The problem of anomalous scaling of the velocity field in that model remains open, as for its incompressible predecessors, but the passive scalar advection by such an ensemble can be treated analytically. This is the aim of the present work.

The plan of the paper is as follows.

In Sec. II we revisit the RG approach to the stochastic Navier-Stokes equation for a compressible fluid, following mostly Ref. [47], and introduce the basic notions (field theoretic formulation, canonical dimensions, renormalizability, and RG equations) needed for the further analysis of the passive advection. The RG equations possess an IR attractive fixed point, which implies the existence of a scaling regime in the inertial and energy-containing ranges. The one-loop explicit expressions for the renormalization constants and the RG functions (anomalous dimensions and β functions), calculated in [47], are presented. The corresponding scaling dimensions of the frequency and the velocity are known exactly and coincide with their analogs for the incompressible case. Another nontrivial fixed point is unstable (it is a saddle point) and corresponds to the incompressible fluid.

In Sec. III we introduce the diffusion-advection stochastic equations for the two types of passive scalar field: the tracer (temperature, entropy, or concentration of a pollutant) and the density of a conserved quantity (e.g., density of a pollutant). We present the field theoretic formulation of these models and show that they are multiplicatively renormalizable. Then the RG equations can be derived in a standard fashion. The renormalization constants and the RG functions are calculated in the leading (one-loop) approximation, which is consistent with the accuracy of the results derived in [47]. The full-scale models, involving the velocity field and the scalar field, possess an IR attractive fixed point. Thus, the existence of a scaling regime in the IR range is established. Exact expressions for the scaling dimensions of the scalar fields are obtained.

In Sec. IV we calculate, in the leading order of the expansion in y (one-loop approximation), critical dimensions of the composite operators built of the scalar field and its spatial derivatives, including some tensor operators. In the next section, those dimensions are identified with various anomalous exponents.

In Sec. V we apply the OPE to the analysis of the inertial-range behavior of various correlation functions: the correlation functions of the scalar fields and their powers for the density case and of the structure functions for the tracer case. We show that, for the density case, leading terms of the inertial-range behavior are determined by the contributions of the operators built solely of the scalar fields. Their critical dimensions are negative, which leads to strong dependence on the integral scale and to the anomalous scaling, with the anomalous exponents identified with those dimensions.

For the tracer case, more interesting quantities are the structure functions that involve differences of the values of the scalar field at different points. Their anomalous behavior is determined by the scalar operators built of the gradients of the scalar field, whose negative dimensions are identified with the corresponding anomalous exponents.

In the presence of anisotropy introduced into the system at large scales, contributions of the tensor operators in the

OPEs come into play: lth rank tensor operators determine the contribution in the correlation functions with nontrivial angular dependence described by the lth-order spherical harmonics. Like for the Kraichnan model, those anisotropic contributions organize a kind of hierarchy related to the degree of anisotropy: They become less important as l grows, so that the leading term of the inertial-range asymptotic behavior is given by the isotropic contribution ($l = 0$), in agreement with Kolmogorov's hypothesis of the local isotropy restoration. This issue is discussed for the pair correlation function in the both models and for the structure functions of arbitrary order for the tracer.

Section VI is reserved for the discussion, comparison with the Gaussian models, and the conclusion.

II. RG ANALYSIS OF THE STOCHASTIC NS EQUATION WITH STRONG COMPRESSIBILITY

A. Description of the model

The Navier-Stokes equation for a viscid compressible fluid has the form [41]

$$\rho\nabla_t v_i = \nu_0[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + \mu_0\partial_i\partial_k v_k - \partial_i p + \eta_i, \quad (2.1)$$

where

$$\nabla_t = \partial_t + v_k\partial_k \quad (2.2)$$

is the Lagrangian (Galilean covariant) derivative, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, and $\partial^2 = \partial_i\partial_i$ is the Laplace operator.

Equation (2.1) is obtained by combining the momentum balance equation

$$\partial_t(\rho v_i) + \partial_k\Pi_{ik} = \eta_i, \quad (2.3)$$

where

$$\Pi_{ik} = \rho v_i v_k + \delta_{ik}p - \nu_0(\partial_i v_k + \partial_k v_i) - \delta_{ik}(\mu_0 - 2\nu_0)\,\partial_l v_l \quad (2.4)$$

is the stress tensor, with the continuity equation

$$\partial_t\rho + \partial_i(\rho v_i) = 0. \quad (2.5)$$

In those equations, v_i is the velocity, ρ is the mass density, p is the pressure, and η_i is the density of the external force (per unit volume). All these quantities depend on $x = \{t,\mathbf{x}\}$ with $\mathbf{x} = \{x_i\}$, $i = 1,\ldots,d$, where d is an arbitrary (for generality) dimensionality of space. The constants ν_0 and μ_0 are two independent molecular viscosity coefficients; in the viscous terms in (2.1) we explicitly separated the transverse and the longitudinal parts. Summations over repeated vector indices are always implied.

The problem (2.1) and (2.5) should be augmented by the equation of state, $p = p(\rho)$. It will be taken in the simplest form of the linear relation,

$$(p - \bar{p}) = c_0^2(\rho - \bar{\rho}), \quad (2.6)$$

between the deviations of the pressure and the density from their mean values. The constant c_0 has the meaning of the (adiabatic) speed of sound.

Following [47], we divide Eq. (2.1) with ρ and in the viscous terms replace ρ with its mean value. This approximation (which is needed to obtain a renormalizable local field theoretic model) is implicitly justified by the analysis of Ref. [44]; we also note that the viscosity plays a little role in the inertial range. We retain the same notation ν_0 and μ_0 for the resulting constant kinematic viscosity coefficients. Then the equations (2.1) and (2.5) can be rewritten in the form

$$\nabla_t v_i = \nu_0[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + \mu_0\partial_i\partial_k v_k - \partial_i\phi + f_i, \quad (2.7)$$

$$\nabla_t\phi = -c_0^2\partial_i v_i, \quad (2.8)$$

where we have introduced the new scalar field

$$\phi = c_0^2\ln(\rho/\bar{\rho}) \quad (2.9)$$

and $f_i = f_i(x)$ is the density of the external force (per unit mass).

In the stochastic formulation of the problem, the external force becomes a random field that models the energy input into the system from the large-scale stirring. The details of its statistics are believed to be unessential, so it is taken to be Gaussian with zero mean, white in time (this is required by the Galilean symmetry), and involving some typical IR scale L (the integral scale). On the other hand, for the use of the standard RG technique it is important that its correlation function have a power-law tail at large wave numbers. More detailed discussion can be found in [16,17,49]. In the present case one chooses the correlation function in the form [47]

$$\langle f_i(x)f_j(x')\rangle = \delta(t-t')\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,D^f_{ij}(\mathbf{k})\exp\{i\mathbf{k}\cdot\mathbf{x}\}, \quad (2.10)$$

where

$$D^f_{ij}(\mathbf{k}) = D_0\,k^{4-d-y}\,\{P^{\perp}_{ij}(\mathbf{k}) + \alpha P^{\parallel}_{ij}(\mathbf{k})\}. \quad (2.11)$$

Here $P^{\perp}_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$ and $P^{\parallel}_{ij}(\mathbf{k}) = k_ik_j/k^2$ are the transverse and longitudinal projectors, respectively, $k = |\mathbf{k}|$ is the wave number, and D_0 and α are positive amplitudes. It is convenient to write $D_0 = g_0\nu_0^3$: The parameter g_0 plays the role of the coupling constant (formal expansion parameter in the ordinary perturbation theory). The relation $g_0 \sim \Lambda^y$ sets in the typical UV momentum scale (reciprocal of the dissipation length scale). The parameter $m = L^{-1}$ provides IR regularization; its precise form is unessential and the sharp cutoff is the simplest choice for calculational reasons. The exponent $0 < y \leqslant 4$ plays the role analogous to that played by $\varepsilon = 4 - d$ in the RG theory of critical behavior [13,14]: It provides UV regularization (so that the UV divergences have the form of the poles in y) and the coordinates of fixed points and various scaling dimensions are calculated as series in y. The most realistic (physical) value is given by the limit $y \to 4$, when the functions in (2.11) can be viewed (with the proper choice of the amplitude) as powerlike models of the function $\delta(\mathbf{k})$: It corresponds to the idealized picture of the energy input from infinitely large scales.

B. Field theoretic formulation and Feynman rules

According to the general theorem [13,14], the stochastic problem (2.7), (2.8), (2.10), and (2.11), is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{v'_i,\phi',v_i,\phi\}$

and the action functional,

$$\mathcal{S}(\Phi) = \tfrac{1}{2} v_i' D_{ik}^f v_k' + v_i'\{-\nabla_t v_i + \nu_0[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + u_0\nu_0\partial_i\partial_k v_k - \partial_i\phi\} + \phi'\left[-\nabla_t\phi + v_0\nu_0\partial^2\phi - c_0^2(\partial_i v_i)\right], \qquad (2.12)$$

where D^f is the correlation function (2.10) and (2.11), and all the needed summations over the vector indices and integrations over $x = \{t,\mathbf{x}\}$ are implied, for example,

$$v_i'\nabla_t v_i = \int dt \int d\mathbf{x}\, v_i'(x)[\partial_t + v_k(x)\partial_k]v_i(x). \qquad (2.13)$$

In (2.12) we passed to the new dimensionless parameter $u_0 = \mu_0/\nu_0 > 0$ and introduced a new term $v_0\nu_0\phi'\partial^2\phi$ with another positive dimensionless parameter v_0. This term is not forbidden by the symmetry or dimensionality considerations, so it will necessarily appear in the renormalization procedure. From the physics viewpoints, it corresponds to some redefinition of the relation between the velocity and the momentum [41]. From a more technical point of view, it is needed to ensure multiplicative renormalizability of the model (2.12), which allows one to easily derive the RG equations. One can insist on studying the original model (2.7) and (2.8) without such a term. Then the RG equations must be solved with the initial condition $v_0 = 0$. In renormalized variables, this corresponds to a general situation with a nonzero value of the corresponding renormalized parameter v. Since the IR attractive fixed point is unique (see below), the specific initial condition is unessential.

The field theoretic formulation means that various correlation functions and response (Green's) functions of the original stochastic problem are represented by functional averages over the full set of fields with weight $\exp\mathcal{S}(\Phi)$, and in this sense they can be viewed as the Green's functions of the field theoretic model with action (2.12). The model corresponds to standard Feynman diagrammatic techniques with two vertices $-v'(v\partial)v$ and $-\phi'(v\partial)\phi$ and the free (bare) propagators, determined by the quadratic part of the action functional; in the frequency-momentum (ω-$\mathbf{k}$) representation, they have the forms

$$\begin{aligned}
\langle vv'\rangle_0 &= \langle v'v\rangle_0^* = P^\perp\epsilon_1^{-1} + P^\parallel\epsilon_3 R^{-1},\\
\langle vv\rangle_0 &= P^\perp\frac{d^f}{|\epsilon_1|^2} + P^\parallel\alpha d^f\left|\frac{\epsilon_3}{R}\right|^2,\\
\langle\phi v'\rangle_0 &= \langle v'\phi\rangle_0^* = -\frac{ic_0^2\mathbf{k}}{R}, \quad \langle v\phi'\rangle_0 = \langle\phi' v\rangle_0^* = \frac{i\mathbf{k}}{R},\\
\langle\phi\phi'\rangle_0 &= \langle\phi'\phi\rangle_0^* = \frac{\epsilon_2}{R}, \quad \langle\phi\phi\rangle_0 = \frac{\alpha c_0^4 k^2 d^f}{|R|^2},\\
\langle v\phi\rangle_0 &= \langle\phi v\rangle_0^* = \frac{i\alpha c_0^2 d^f\epsilon_3\mathbf{k}}{|R|^2},\\
\langle\phi'\phi'\rangle_0 &= \langle v'\phi'\rangle_0 = \langle v'v'\rangle_0 = 0,
\end{aligned} \qquad (2.14)$$

where we have denoted

$$\begin{aligned}
\epsilon_1 &= -i\omega + \nu_0 k^2, \quad \epsilon_2 = -i\omega + u_0\nu_0 k^2,\\
\epsilon_3 &= -i\omega + v_0\nu_0 k^2, \quad R = \epsilon_2\epsilon_3 + c_0^2 k^2,\\
d^f &= g_0\nu_0^3 k^{4-d-y},
\end{aligned} \qquad (2.15)$$

and omitted the vector indices of the fields and the projectors.

In the limit $c_0 \to \infty$, the propagators $\langle vv'\rangle_0$ and $\langle vv\rangle_0$ become purely transverse, while the mixed propagator $\langle v\phi\rangle_0$ vanishes. Then the scalar field ϕ decouples from v, v' [it does not enter the vertex in (2.7)], and we arrive at the well-known Feynman rules for the incompressible fluid [14,16,17].

C. UV divergences, renormalization, and multiplicative renormalizability

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions; see, e.g., [13,14]. Dynamical models like (2.12) have two independent scales: the time scale T and the length scale L. Thus, the canonical dimension of any quantity F (a field or a parameter) is described by two numbers, the frequency dimension d_F^ω and the momentum dimension d_F^k, defined such that $[F] \sim [T]^{-d_F^\omega}[L]^{-d_F^k}$. The obvious consequences of the definition are the relations

$$\begin{aligned}
d_k^k &= -d_{\mathbf{x}}^k = 1, \quad d_k^\omega = d_{\mathbf{x}}^\omega = 0,\\
d_\omega^k &= d_t^k = 0, \quad d_\omega^\omega = -d_t^\omega = 1.
\end{aligned} \qquad (2.16)$$

The other dimensions are found from the requirement that each term of the action functional be dimensionless (with respect to the momentum and the frequency dimensions separately). Then one introduces the total canonical dimension

$$d_F = d_F^k + 2d_F^\omega, \qquad (2.17)$$

which plays in the theory of renormalization of dynamical models the same part as the conventional canonical dimension does in static problems. The canonical dimensions for the model (2.12) are given in Table I, including renormalized parameters (without the subscript "o"), which appear a bit later.

The choice (2.17) for the total canonical dimension deserves a more careful explanation. It means that all the viscosity or diffusion coefficients in the model are pronounced dimensionless (with respect to the new total dimension), and the time and the space variables are measured in the same units; cf., [13,14]. The experienced reader recalls the $c = 1$ system of units in relativistic physics, where all the distances are measured in the time units (light years). Here we relate the dimensions by Eq. (2.17) because the dispersion law for diffusion modes is $\omega \sim k^2$. However, our model involves another dispersion law, $\omega \sim k$, related to the sound modes. If we decided to set

TABLE I. Canonical dimensions of the fields and parameters in the models (2.12), (3.4), (3.5), and (3.8).

F	v'	v	ϕ'	ϕ	θ'	θ	m, μ, Λ	ν_0, ν	c_0, c	g_0	$u_0, v_0\ w_0, u, v, w, g, \alpha$
d_F^k	$d+1$	-1	$d+2$	-2	d	0	1	-2	-1	y	0
d_F^ω	-1	1	-2	2	$1/2$	$-1/2$	0	1	1	0	0
d_F	$d-1$	1	$d-2$	2	$d+1$	-1	1	0	1	y	0

the speed of sound c_0 dimensionless, we would have to set $d_F = d_F^k + d_F^\omega$.

A similar alternative exists in the so-called model H of equilibrium dynamical critical behavior, where the motion of the fluid is taken into account and several dispersion laws are simultaneously present; see, e.g., p. 552 in the monograph [14]. The choice (2.17) means that we are interested in the asymptotic behavior of the Green's functions where $\omega \sim k^2 \to 0$; the RG treatment will modify it to the Kolmogorov law $\omega \sim k^{2/3} \to 0$ (see below). The same choice is made in the models of incompressible fluid (where it is the only possible one because the speed of sound is infinite). The alternative choice $d_F = d_F^k + d_F^\omega$ would mean that we were interested in the asymptotic behavior of the (same) Green's functions for $\omega \sim k \to 0$ (sound modes in turbulent medium); this problem is, of course, extremely interesting, but so far it is not accessible by the RG treatment and is not discussed in the present paper.

From Table I it follows that the model becomes logarithmic (the coupling constant $g_0 \sim \Lambda^y$ becomes dimensionless) at $y = 0$, so that the UV divergences have the form of poles in y in the Green's functions. The total canonical dimension of any 1-irreducible Green's function Γ (the formal index of UV divergence) is

$$\delta_\Gamma = d + 2 - \sum_\Phi N_\Phi d_\Phi, \tag{2.18}$$

where N_Φ denotes the numbers of the fields entering into the function Γ, d_Φ denotes their total canonical dimensions, and the summation over all types of the fields Φ is implied. Superficial UV divergences, whose removal requires counterterms, can be present only in the functions Γ with a non-negative integer δ_Γ. The counterterm is a polynomial in frequencies and momenta of degree δ_Γ, with the convention that $\omega \sim k^2$.

For the model (2.12), dimensional analysis should be augmented by the following additional considerations [47].

(i) All the 1-irreducible Green's functions without the response fields ($N_{v'} = N_{\phi'} = 0$) involve closed circuits of retarded propagators, vanish identically, and therefore require no counterterms [14].

(ii) If for some reason a number of external momenta occurs as an overall factor in all the diagrams of a given Green's function, the real index of divergence δ'_Γ is smaller than δ_Γ by the corresponding number of unities [14,17]. In the model (2.12) the field ϕ enters the vertex $\phi'(v\partial)\phi$ only in the form of spatial derivative, which reduces the real index of divergence:

$$\delta'_\Gamma = \delta_\Gamma - N_\phi. \tag{2.19}$$

The field ϕ enters the counterterms only in the form of the derivative $\partial\phi$. In particular, for the 1-irreducible function $\langle\phi'\phi\rangle_{1-\text{ir}}$ one obtains $\delta_\Gamma = 2$, $\delta'_\Gamma = 0$. Thus, the counterterm $\phi'\partial_t\phi$, allowed by dimensional analysis, is, in fact, forbidden, and the only possible structure is $\phi'\partial^2\phi$.

(iii) Galilean invariance of the model (2.12) requires that the contributions of the counterterms be also invariant. In particular, this means that the covariant derivative (2.2) enters the counterterms as a whole. As a consequence, the counterterm required for the 1-irreducible function $\langle\phi' v\phi\rangle_{1-\text{ir}}$ with $\delta_\Gamma = 1$, $\delta'_\Gamma = 0$, necessarily has the form $\phi'(v\partial)\phi$ and appears in the combination $\phi'\nabla_t\phi$ with the counterterm $\phi'\partial_t\phi$ discussed above. Hence, it is also forbidden.

Similarly, the divergences in the functions $\langle v'v\rangle_{1-\text{ir}}$ with $\delta_\Gamma = 2$ and $\langle v'vv\rangle_{1-\text{ir}}$ with $\delta_\Gamma = 1$ can be eliminated by the two counterterms: $v'\partial^2 v$ and the combination $v'\nabla_t v$. In fact, the latter is also forbidden by the *generalized* Galilean invariance with the time-dependent transformation velocity parameter $\mathbf{w}(t)$ [50,51]:

$$\begin{aligned} \mathbf{v}_w(x) &= \mathbf{v}(x_w) - \mathbf{w}(t), \quad \Phi_w(x) = \Phi(x_w), \\ x &= \{t,\mathbf{x}\}, \quad x_w = \{t,\mathbf{x} + \mathbf{u}(t)\}, \\ \mathbf{u}(t) &= \int^t \mathbf{w}(t')dt'. \end{aligned} \tag{2.20}$$

Here Φ denotes the three fields v',ϕ',ϕ. The action functional is *not* invariant with respect to such a transformation: $\mathcal{S}(\Phi_w) = \mathcal{S}(\Phi) + v'\partial_t w$. One can show, however, that the generating functional of the 1-irreducible Green's functions transforms in the identical way, $\Gamma(\Phi_w) = \Gamma(\Phi) + v'\partial_t w$. Since, in general, $\Gamma(\Phi) = \mathcal{S}(\Phi)$ plus the diagrams with loops (which contain all the UV divergences), the counterterms appear invariant under (2.20). This excludes the counterterm $v'\nabla_t v$, invariant with respect to conventional Galilean transformation with a constant $\mathbf{w}$, but not invariant with respect to (2.20). More detailed discussion of the uses of the generalized Galilean transformation, especially for composite fields, can be found in [14,17,51].

(iv) Expressions (2.14) show that the propagators $\langle v'\phi\rangle_0$ and $\langle v\phi\rangle_0$ contain the factor c_0^2, while $\langle v'\phi\rangle_0$ contains c_0^4. These factors appear as external numerical factors in any diagram involving these propagators, and its real index of divergence reduces by the corresponding number of unities. In particular, any diagram of the 1-irreducible function with $N_{\phi'} > N_\phi$ contains the factor $c_0^{2(N_{\phi'}-N_\phi)}$. It then follows that the counterterm to the 1-irreducible function $\langle\phi' v\rangle_{1-\text{ir}}$ with $\delta_\Gamma = 3$ necessarily reduces to $c_0^2\phi'(\partial v)$, while the structures $\phi'\partial^2(\partial v)$, etc., are forbidden. Another consequence is finiteness of the function $\langle\phi' vv\rangle_{1-\text{ir}}$ with $\delta_\Gamma = 2$. Each diagram of this function contains the factor c_0^2, which forbids the counterterms like $\phi'(\partial v)(\partial v)$, etc., while the remaining structure $c_0^2\phi' v^2$ is forbidden by the Galilean symmetry.

Using all these considerations one can check that all the UV divergences in the model (2.12) are removed by the counterterms of the form

$$v'_i\partial^2 v_i,\; v'_i\partial_i\partial_k v_k,\; v'_i\partial_i\phi,\; c_0^2\phi'\partial_i v_i,\; \phi'\partial^2\phi. \tag{2.21}$$

All these structures are present in the extended action functional (2.12) with $v_0 > 0$, so the model is multiplicatively renormalizable.

Like for the incompressible case [52], for $d = 2$ a new UV divergence arises in the function $\langle v'v'\rangle_{1-\text{ir}}$, and a new counterterm $v'\partial^2 v'$ should be included. This case requires special treatment, and in the following we assume $d > 2$. Then the renormalized action functional has the form

$$\begin{aligned} \mathcal{S}^R(\Phi) = {}& \tfrac{1}{2}v'_i D^f_{ik} v'_k + v'_i\{-\nabla_t v_i + Z_1\nu[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k \\ & + Z_2 u\nu\partial_i\partial_k v_k - Z_4\partial_i\phi\} \\ & + \phi'[-\nabla_t\phi + Z_3 v\nu\partial^2\phi - Z_5 c^2(\partial_i v_i)]. \end{aligned} \tag{2.22}$$

Here g, ν, u, v, and c are renormalized counterparts of the original (bare) parameters (with the subscript "o"), the function D^f is expressed in renormalized parameters using the relation $g_0\nu_0^3 = g\mu^y\nu^3$, the reference scale (or the "normalization mass") μ is an additional free parameter of the renormalized theory; the renormalization constants Z_i depend only on the completely dimensionless parameters g, u, v, α, d, and y. The renormalized action (2.22) is obtained from the original one (2.12) by the renormalization of the fields $\phi \to Z_\phi\phi$, $\phi' \to Z_{\phi'}\phi'$ and the parameters

$$g_0 = g\mu^y Z_g, \quad \nu_0 = \nu Z_\nu, \quad u_0 = uZ_u,$$
$$v_0 = vZ_v, \quad c_0 = cZ_c. \tag{2.23}$$

The renormalization constants in (2.22) and (2.23) are related as

$$Z_\nu = Z_1, \quad Z_u = Z_2 Z_1^{-1},$$
$$Z_v = Z_3 Z_1^{-1}, \quad Z_\phi = Z_{\phi'}^{-1} = Z_4,$$
$$Z_c = (Z_4 Z_5)^{1/2}, \quad Z_g = Z_\nu^{-3}. \tag{2.24}$$

The last relation follows from the absence of renormalization of the nonlocal term of the random force in (2.22); for the same reason the parameters m,α from the correlation function (2.10) are not renormalized: $Z_m = Z_\alpha = 1$. No renormalization of the fields v,v' is needed: $Z_v = Z_{v'} = 1$ due to the absence of renormalization of the term $v'\nabla_t v$.

The renormalization constants are found from the requirement that the Green's functions of the renormalized model (2.22), when expressed in renormalized variables, be UV finite (in our case, be finite at $y \to 0$). In the minimal subtraction (MS) scheme, which is always used in what follows, they have the form "$Z = 1+$ only poles in y." The calculation in the first order in g (one-loop approximation) gives [47]

$$Z_1 = 1 + \frac{\hat g}{y}A, \quad Z_2 = 1 + \frac{\hat g}{uy}B,$$
$$Z_3 = 1 + \frac{\hat g}{y}\frac{(d-1)}{2dv(v+1)} - \frac{\alpha\hat g}{y}\frac{(u-v)}{2duv(u+v)^2}, \tag{2.25}$$
$$Z_4 = 1 + \frac{\hat g}{y}\frac{(d-1)}{2d(u+1)(v+1)}, \quad Z_5 = 1,$$

with corrections of order $\hat g^2$ and higher. Here we passed to the new coupling constant

$$\hat g = gS_d/(2\pi)^d, \tag{2.26}$$

where

$$S_d = 2\pi^{d/2}/\Gamma(d/2) \tag{2.27}$$

is the surface area of the unit sphere in d-dimensional space and $\Gamma(\cdots)$ is Euler's Γ function and denoted

$$A = \frac{d(d-1)u^2 - 2(d^2+d-4)u - d(d+3)}{4d(d+2)(1+u)^2} + \frac{\alpha(1-u)}{2du(1+u)^2},$$
$$B = (1-d)\frac{(d-1)u^2 + (d+4)u + 1}{2d(d+2)(1+u)^2}. \tag{2.28}$$

One important technical remark follows. The renormalization constants in the MS scheme do not depend on the dimensional parameter c_0. On the other hand, all the propagators (2.14), and hence all the Feynman diagrams, have a well-defined limit for $c_0 \to 0$. Thus, in the calculation of the constants Z_1–Z_4 one can simply set $c_0 = 0$ in (2.14) and (2.15). Then the propagators $\langle\phi v'\rangle_0$, $\langle v\phi\rangle_0$, $\langle\phi\phi\rangle_0$ vanish, while the form of the others drastically simplifies. In the calculation of the constant Z_5 in front of the term $c_0^2\phi'(\partial v)$ it is sufficient to take into account the diagrams with one and only one propagator $\langle\phi v'\rangle_0$ or $\langle v\phi\rangle_0$. Then the needed c_0^2 appears as an external factor, and in the remaining expression one can set $c_0 = 0$.

To avoid possible misunderstanding, we stress that we are interested in the model with finite and arbitrary c_0 and that more involved calculation with the full-scale propagators (2.14) would give the same results (2.25) and (2.28) for the renormalization constants. In this respect, the parameter c_0 is similar to $\tau \propto T - T_c$, deviation of the temperature from its critical value, in models of critical behavior: In the MS scheme, the renormalization constants do not depend on it and can be calculated directly at the critical point $\tau = 0$.

The simple expression $Z_5 = 1$ results from the cancellation of nontrivial contributions from three Feynman diagrams; we see no reason to expect that it is valid to all orders in g.

D. RG equations and RG functions

Let us recall a simple derivation of the RG equations; detailed discussion can be found in [13,14]. The RG equations are written for the renormalized correlation functions $G^R = \langle\Phi\cdots\Phi\rangle_R$, which differ from the original (unrenormalized) ones $G = \langle\Phi\cdots\Phi\rangle$ only by normalization and choice of the parameters and thus can be equally used for the analysis of the critical behavior. The relation $\mathcal{S}_R(\Phi,e,\mu) = \mathcal{S}(Z_\Phi\Phi,e_0)$ between the functionals (2.12) and (2.22) results in the relations

$$G(e_0,\dots) = Z_\phi^{N_\phi} Z_{\phi'}^{N_{\phi'}} G^R(e,\mu,\dots) \tag{2.29}$$

between the correlation functions. Here, as usual, N_ϕ and $N_{\phi'}$ are the numbers of corresponding fields entering into G (we recall that in our model $Z_v = Z_{v'} = 1$); $e_0 = \{\nu_0,g_0,u_0,v_0\}$ is the full set of bare parameters and $e = \{\nu,g,u,v\}$ are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta, etc.).

We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of Eq. (2.29) with it. This gives the basic RG differential equation,

$$\{\mathcal{D}_{RG} + N_\phi\gamma_\phi + N_{\phi'}\gamma_{\phi'}\}\, G^R(e,\mu,\dots) = 0, \tag{2.30}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g\partial_g + \beta_u\partial_u + \beta_v\partial_v - \gamma_\nu\mathcal{D}_\nu - \gamma_c\mathcal{D}_c. \tag{2.31}$$

Here we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x. The anomalous dimension γ_F of a certain quantity F (a field or a parameter) is defined as

$$\gamma_F = Z_F^{-1}\widetilde{\mathcal{D}}_\mu Z_F = \widetilde{\mathcal{D}}_\mu \ln Z_F, \tag{2.32}$$

and the β functions for the three dimensionless coupling constants g, u, and v are

$$\beta_g = \widetilde{\mathcal{D}}_\mu g = g\,[-y - \gamma_g],\quad \beta_u = \widetilde{\mathcal{D}}_\mu u = -u\gamma_u,\quad \beta_v = \widetilde{\mathcal{D}}_\mu v = -v\gamma_v, \tag{2.33}$$

where the second equalities result from the definitions and the relations (2.29).

From the relations (2.24) we obtain

$$\beta_g = g\,[-y + 3\gamma_1],\quad \beta_u = u\,[\gamma_1 - \gamma_2],\quad \beta_v = v\,[\gamma_2 - \gamma_3], \tag{2.34}$$

and for the anomalous dimensions we have

$$\gamma_\phi = -\gamma_{\phi'} = \gamma_4,\quad \gamma_c = (\gamma_4 + \gamma_5)/2,\quad \gamma_\nu = \gamma_1,\quad \gamma_v = \gamma_{v'} = \gamma_\alpha = \gamma_m = 0. \tag{2.35}$$

The relations in the second line follow from the absence of renormalization of the corresponding fields and parameters; see the remarks below Eq. (2.24).

In the MS scheme all the renormalization constants have the form

$$Z_F = 1 + \sum_{n=1}^{\infty} z^{(n)} y^{-n}, \tag{2.36}$$

where the coefficients $z^{(n)}$ do not depend on y. Then from the definition and the expressions (2.33) it follows that the corresponding anomalous dimension is determined solely by the first-order coefficient,

$$\gamma_F = -\mathcal{D}_g z^{(1)}, \tag{2.37}$$

see, e.g., the discussion [13,14]. Then in the one-loop approximation from the explicit expressions (2.25) one finds

$$\gamma_1 = -A\hat{g},\quad \gamma_2 = -B\hat{g}/u,\quad \gamma_3 = \hat{g}\,\frac{(d-1)}{2dv(v+1)} + \alpha\hat{g}\,\frac{(u-v)}{2duv(u+v)^2},\quad \gamma_4 = \hat{g}\,\frac{(1-d)}{2d(u+1)(v+1)},\quad \gamma_5 = 0, \tag{2.38}$$

with A and B from (2.28) and the corrections of order $\hat{g}^2$ and higher.

E. The IR attractive fixed point

It is well known that possible IR asymptotic regimes of a renormalizable field theoretic model are associated with IR attractive fixed points of the corresponding RG equations. The coordinates g_* of the fixed points are found from the equations

$$\beta_i(g_*) = 0, \tag{2.39}$$

where $g = \{g_i\}$ is the full set of coupling constants and β_i are the corresponding β functions. The type of a fixed point is determined by the matrix

$$\Omega_{ij} = \partial\beta_i/\partial g_j|_{g=g_*}. \tag{2.40}$$

For the IR stable fixed points the matrix Ω is positive; i.e., the real parts of all its eigenvalues are positive.

In our model, $g = \{\hat{g},u,w\}$, and the β functions are given be the relations (2.33) and the explicit one-loop expressions (2.38). We do not include the dimensionless parameter α into the list of coupling constants, because it is not renormalized ($\alpha_0 = \alpha$ and $Z_\alpha = 1$) and the corresponding function $\beta_\alpha = -\alpha\gamma_\alpha$ vanishes identically. Thus, the equation $\beta_\alpha = 0$ imposes no restriction on the value of α, and it remains a free parameter.

Analysis of the expressions (2.33), (2.38), and (2.28) shows that in the physical range of parameters ($\hat{g},u,v,\alpha > 0$) there is only one IR attractive fixed point with the coordinates

$$\hat{g}_* = \frac{4dy}{3(d-1)},\quad u_* = v_* = 1, \tag{2.41}$$

with possible higher-order corrections in y.

Let us briefly explain the derivation of (2.41). Any fixed point with $\hat{g}_* = 0$ cannot be IR attractive, because one of the eigenvalues of the matrix Ω coincides with the diagonal element $\partial_g\beta_g = -y < 0$. For $\hat{g}_* \neq 0$ from the equation $\beta_g = 0$ we immediately find the relation $\gamma_1^* = \gamma_\nu^* = y/3$, valid to all orders in y [here and below $\gamma_F^* = \gamma_F(g_*)$ for any F is the value of the anomalous dimension at the fixed point in question]. Substituting this relation into the equation $\beta_u = 0$ gives the equation for u_* with the only positive solution $u_* = 1$. Substituting it into the equation $\beta_g = 0$ gives the value of $\hat{g}_*$ (it is important here that the functions β_g and β_u in the one-loop approximation do not depend on v). Finally, substituting the known values of $\hat{g}_*$ and u_* into the relation $\beta_v = 0$ gives the equation for v_* with the only positive solution $v_* = 1$. Now it is easy to see that the matrix (2.40) at the fixed point (2.41) is triangular, so that its eigenvalues coincide with the diagonal elements and are easily calculated from the explicit expressions (2.38). They are positive for all $y > 0$, $\alpha > 0$, and $d > 2$.

It is also worth noting that the so-called "RG flows" (solutions to the RG equations for the RG-invariant or "running" coupling constants) cannot leave the physical range $\hat{g},u,v > 0$ (for the physical initial data). This follows from the fact that all the β functions vanish for $g = 0$ and that the functions β_u and β_v are negative for $u = 0$ and $v = 0$, respectively:

$$\beta_u|_{u=0} = -\hat{g}\,\frac{(d-1)}{2d(d+2)},\quad \beta_v|_{v=0} = -\hat{g}\,\left\{\frac{(d-1)}{2d} + \frac{1}{du^2}\right\}.$$

It then follows that the IR asymptotic behavior of the Green's functions in our model can be governed only by the fixed point (2.41): Even if some other fixed points exist in the unphysical range, they cannot be reached by the RG flow.

Changing to the new variable $f = 1/u$ one can find another fixed point with $f_* = 0$ and $\hat{g}_* = 4(d+2)y/3(d-1)$. From the explicit form of the propagators (2.14) it follows that the limit $u \to \infty$ corresponds to the purely transverse velocity field, while the scalar field decouples. The point is unstable (it is a saddle point) in agreement with the analysis of Refs. [42–44], which shows that the leading-order correction in the Mach number to the incompressible scaling regime destroys its stability (in the RG terminology, it is relevant in the sense of Wilson).

F. IR behavior and the critical dimensions

It follows from the solution of the RG equation (2.30) that when an IR fixed point is present, the leading term of the IR asymptotic behavior of the Green's function G^R satisfies Eq. (2.30) with the replacement $g \to g_*$ for the full set of the couplings; see, e.g., [14]. In our case this gives

$$\left\{\mathcal{D}_\mu - \gamma_\nu^* \mathcal{D}_\nu - \gamma_c^* \mathcal{D}_c + \sum_\Phi N_\Phi \gamma_\Phi^*\right\} G^R = 0. \qquad (2.42)$$

We recall that $\mathcal{D}_x \equiv x\partial_x$ for any variable x, γ_F^* is the fixed-point value of the anomalous dimension γ_F, and the summation over all types of the fields Φ is implied. In the one-loop approximation, from (2.38) and (2.41) we obtain

$$\gamma_\nu^* = y/3 \text{ (exact)}, \quad \gamma_\phi^* = -\gamma_{\phi'}^* = -y/6 + O(y^2),$$
$$\gamma_c^* = -y/12 + O(y^2). \qquad (2.43)$$

Canonical scale invariance is expressed by the two equations

$$\left\{\sum_F d_F^k \mathcal{D}_F - d_G^k\right\} G^R = 0,$$
$$\left\{\sum_F d_F^\omega \mathcal{D}_F - d_G^\omega\right\} G^R = 0, \qquad (2.44)$$

with the summations over all the arguments of the function G^R. From Table I we obtain

$$\left\{-\mathcal{D}_{\mathbf{x}} + \mathcal{D}_\mu + \mathcal{D}_m - 2\mathcal{D}_\nu - \mathcal{D}_c - \sum_\Phi N_\Phi d_\Phi^k\right\} G^R = 0,$$
$$\left\{-\mathcal{D}_t + \mathcal{D}_\nu + \mathcal{D}_c - \sum_\Phi N_\Phi d_\Phi^\omega\right\} G^R = 0, \qquad (2.45)$$

where the dimensions $d_\Phi^{k,\omega}$ of the fields are also given in the table. Each of the equations (2.42) and (2.45) describes the scaling with dilatation of the variables whose derivatives enter the differential operator. One is interested in the scaling with fixed "IR irrelevant" parameters μ and ν; see [14,16,17]. In order to derive the corresponding scaling equation, one has to combine (2.42) and (2.45) such that the derivatives with respect to these parameters be eliminated; this gives

$$\left\{-\mathcal{D}_{\mathbf{x}} + \Delta_t \mathcal{D}_t + \Delta_c \mathcal{D}_c + \Delta_m \mathcal{D}_m - \sum_\Phi N_\Phi \Delta_\Phi\right\} G^R = 0, \qquad (2.46)$$

with

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = -\Delta_t = 2 - \gamma_\nu^*. \qquad (2.47)$$

Here Δ_F is the critical dimension of the quantity F (following [14,16,17] we use this term to distinguish it from canonical dimensions), while Δ_t and Δ_ω are the critical dimensions of the time and the frequency.

From Table I and expressions (2.43) we obtain

$$\Delta_v = 1 - y/3, \quad \Delta_{v'} = d - \Delta_v, \quad \Delta_\omega = 2 - y/3, \quad \Delta_m = 1 \qquad (2.48)$$

(these results are exact due to $\gamma_\nu^* = y/3$ and $\gamma_{v,v',m}^* = 0$) and

$$\Delta_\phi = d - \Delta_{\phi'} = 2 - 5y/6 + O(y^2),$$
$$\Delta_c = 1 - 5y/12 + O(y^2). \qquad (2.49)$$

We note that the analogs of the expressions (2.48) and (2.49) in Ref. [47] contain a few misprints.

Surprisingly enough, all the results (2.41), (2.43), (2.48), and (2.49) are independent on α (and some of them do not depend on d). They are valid for all $\alpha > 0$, but the case $\alpha \to \infty$ (purely potential random force) requires special attention. To study this limit, one should pass to the new couplings $g' = g\alpha$, $b = 1/\alpha$ and then set $b = 0$ at finite g'. This gives

$$\beta_{g'} = -yg', \quad \beta_u = g'\frac{(u-1)}{2du(1+u)^2}, \quad \beta_v = g'\frac{(v-u)}{du(u+v)^2}. \qquad (2.50)$$

The system (2.50) has no IR attractive fixed point, because from $\beta_{g'} = 0$ it necessarily follows that $g' = 0$, and such a point cannot be IR attractive due to $\partial_{g'}\beta_{g'} = -y < 0$. In principle, the needed fixed point with $g'_* \sim y^{1/2}$ can appear on the two-loop level, if the term of order $(g')^3$ appears in $\beta_{g'}$. Then the results (2.48) remain valid, while (2.49) should be revised.

III. PASSIVE SCALAR FIELDS: RENORMALIZATION, RG FUNCTIONS, AND FIXED POINT

A. The models and their field theoretic formulation

There are two main types of diffusion-advection problems for the compressible velocity field [41]. Passive advection of a density field $\theta(x) \equiv \theta(t,\mathbf{x})$ (say, the density of a pollutant) is described by the equation

$$\partial_t\theta + \partial_i(v_i\theta) = \kappa_0\partial^2\theta + f, \qquad (3.1)$$

while the advection of a "tracer" (temperature, specific entropy, or *concentration* of the impurity particles) is described by

$$\partial_t\theta + (v_i\partial_i)\theta = \kappa_0\partial^2\theta + f. \qquad (3.2)$$

Here $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, κ_0 is the molecular diffusivity coefficient, $\partial^2 = \partial_i\partial_i$ is the Laplace operator, $\mathbf{v}(x)$ is the velocity field, and $f \equiv f(x)$ is a Gaussian noise with zero mean and given covariance,

$$\langle f(x)f(x')\rangle = \delta(t-t')\,C(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'. \qquad (3.3)$$

Here $C(\mathbf{r}/L)$ is some function finite at $(\mathbf{r}/L) \to 0$ and rapidly decaying for $(\mathbf{r}/L) \to \infty$. In the following, we do not distinguish the integral scale L, related to the noise, and its analog $L = m^{-1}$ in the correlation function of the stirring force (2.11). Without loss of generality, one can set $C(0) = 1$ (the coefficient can be absorbed by rescaling of θ and f). The noise mimics the effects of initial and/or boundary conditions: It maintains the steady state of the system and serves as the source of the large-scale anisotropy. (The latter term means that the anisotropy is introduced at scales of order L, while the statistics of the velocity field remains isotropic. The case of anisotropic velocity statistics is discussed, within the RG + OPE approach, in Refs. [53].) In more realistic formulations, the noise can arise from an imposed linear gradient of the (temperature) field. It turns out, however, that the specific form of the random stirring is unimportant, and in the following we use the artificial noise with the correlation function (3.3).

In the absence of the noise, Eq. (3.1) has the form of a continuity equation (conservation law); θ being the density of a corresponding conserved quantity. For (3.2), the conserved quantity is the auxiliary (response) field θ', which appears in the field theoretic formulation of the problem; see below. If the function in (3.3) is chosen such that its Fourier transform $C(\mathbf{k})$ vanishes at $\mathbf{k}=0$, the fields θ or θ' remain to be conserved in the statistical sense in the presence of the external stirring.

The models (3.1) and (3.2) were thoroughly studied for the case of Kraichnan's rapid-change model [23–30]; the case of Gaussian velocity statistics with finite correlation time was studied in [31,32].

The stochastic problem (3.1) and (3.3) is equivalent to the field theoretic model of the full set of fields $\Phi \equiv \{\theta',\theta,v',v,\phi',\phi\}$ with the action functional

$$\mathcal{S}_\Phi(\Phi) = \mathcal{S}_\theta(\theta',\theta,v) + \mathcal{S}(v',v,\phi',\phi), \tag{3.4}$$

where

$$\mathcal{S}_\theta(\theta',\theta,v) = \tfrac{1}{2}\theta' D_f \theta' + \theta'\{-\partial_t\theta - \partial_i(v_i\theta) + \kappa_0\partial^2\theta\} \tag{3.5}$$

is the De Dominicis-Janssen action for the stochastic problem (3.1) and (3.3) at fixed $\mathbf{v}$, while the second term is given by (2.12) and represents the velocity statistics; D_f is the correlation function (3.3), and, as usual, all the required integrations and summations over the vector indices are implied.

In addition to (2.14), the diagrammatic technique in the full problem involves two propagators,

$$\begin{aligned} \langle\theta\theta'\rangle_0 &= \langle\theta'\theta\rangle_0^* = \frac{1}{-i\omega+\kappa_0 k^2}, \\ \langle\theta\theta\rangle_0 &= \frac{C(\mathbf{k})}{\omega^2+\kappa_0^2 k^4}, \end{aligned} \tag{3.6}$$

and the new vertex $-\theta'\partial_i(v_i\theta) = V_i\theta' v_i\theta$. In the momentum representation, the vertex factor V_i in the diagrams has the form

$$V_i(\mathbf{k}) = ik_i, \tag{3.7}$$

where $\mathbf{k}$ is the momentum argument of the field θ' (using integration by parts, the derivative at the vertex can be moved onto the field θ').

The problem (3.2) and (3.3) corresponds to the action (3.4), where the part $\mathcal{S}_\theta$ is given by

$$\mathcal{S}_\theta(\theta',\theta,v) = \tfrac{1}{2}\theta' D_f \theta' + \theta'\{-\partial_t\theta - (v_i\partial_i)\theta + \kappa_0\partial^2\theta\}. \tag{3.8}$$

The propagators are given by the same expressions (3.6), while the vertex factor (3.7) is replaced with

$$V_i(\mathbf{k}) = -ik_i, \tag{3.9}$$

where $\mathbf{k}$ is the momentum argument of the field θ.

B. UV renormalization and related subjects

Canonical dimensions of the new fields and parameters that appear in the models (3.4), (3.5), and (3.8) are given in Table I, where we introduced a new dimensionless parameter $w_0 = \kappa_0/\nu_0$ with ν_0 from (2.1).

Now in the expression (2.18) for the formal index of UV divergence the summation runs over the full set of fields $\Phi \equiv \{\theta',\theta,v',v,\phi',\phi\}$. Rules (i)–(iv) from Sec. II C should be extended and augmented as follows.

(i) All the 1-irreducible Green's functions without the response fields v',ϕ',θ' vanish identically and require no counterterms.

(ii) In the model (3.8), the field θ enters the vertex $-\theta'(v_i\partial_i)\theta$ only in the form of derivative. Then the expression (2.19) for the real index of divergence should be modified as

$$\delta'_\Gamma = \delta_\Gamma - N_\phi - N_\theta. \tag{3.10}$$

In the model (3.5), the derivative at the vertex $-\theta'\partial_i(v_i\theta)$ can be moved onto the field θ' using integration by parts, and the real index becomes

$$\delta'_\Gamma = \delta_\Gamma - N_\phi - N_{\theta'}. \tag{3.11}$$

Since the field θ in model (3.8) and θ' in model (3.8) can enter the counterterms only in the form of spatial derivatives, the counterterm $\theta'\partial_t\theta$ to the 1-irreducible Green's function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ with $\delta_\Gamma = 2$, $\delta'_\Gamma = 1$ is forbidden for the both models.

(iii) Another consequence of (ii) is that the counterterms to the 1-irreducible function $\langle\theta' v\theta\rangle_{1-\mathrm{ir}}$ with $\delta_\Gamma = 1$, $\delta'_\Gamma = 0$ necessarily reduce to the form $\theta'\partial_i(v_i\theta)$ for the model (3.5) and $\theta'(v_i\partial_i)\theta$ for the model (3.8). Galilean symmetry requires, however, that these monomials enter the counterterms in the form of invariant combinations $\theta'[\partial_t\theta + \partial_i(v_i\theta)]$ and $\theta'\nabla_t\theta$. Hence, they are also forbidden.

(iv) From the straightforward analysis of the Feynman diagrams it follows that, for any 1-irreducible function, $N_{\theta'} - N_\theta = 2N_0$, where N_0 is the total number of bare propagators $\langle\theta\theta\rangle_0$ entering the diagram. Clearly, no diagram with $N_0 < 0$ can be constructed, so that the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any nontrivial Green's function. This fact, a consequence of the linearity of the original stochastic equations (3.1) and (3.2) in the field θ, appears crucial for the renormalizability of the models (3.5) and (3.8). Indeed, the total canonical dimension $d_\theta = -1$ is negative (in contrast to most conventional field theoretic models), so that the index (3.11) increases with N_θ, while (3.10) does not depend on N_θ. Without the restriction $N_\theta \leqslant N_{\theta'}$, we would face the infinity of superficially divergent functions $\langle\theta'\theta\cdots\theta\rangle_{1-\mathrm{ir}}$ and hence the lack of renormalizability.

Finally, we are left with the only superficially divergent 1-irreducible Green's function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ with the only counterterm $\theta'\partial^2\theta$. It is naturally reproduced as multiplicative renormalization of the diffusion coefficient, $\kappa_0 = \kappa Z_\kappa$. No renormalization of the fields θ', θ is needed: $Z_{\theta'} = Z_\theta = 1$. The renormalized analog of the action functional (3.5) has the form

$$\mathcal{S}^R_\Phi(\Phi) = \mathcal{S}^R_\theta(\theta',\theta,v) + \mathcal{S}^R(v',v,\phi',\phi), \tag{3.12}$$

with $\mathcal{S}^R$ from (2.22) and

$$\mathcal{S}^R_\theta(\theta',\theta,v) = \tfrac{1}{2}\theta' D_f \theta' + \theta'\{-\partial_t\theta - \partial_i(v_i\theta) + \kappa Z_\kappa \partial^2\theta\}, \tag{3.13}$$

and similarly for (3.8):

$$\mathcal{S}^R_\theta(\theta',\theta,v) = \tfrac{1}{2}\theta' D_f \theta' + \theta'\{-\partial_t\theta - (v_i\partial_i)\theta + \kappa Z_\kappa \partial^2\theta\}. \tag{3.14}$$

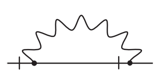
It remains to note that, if the term with D_f is omitted, the models (3.5) and (3.8) can be mapped onto each other by means of the interchange $\theta(t,\mathbf{x}) \leftrightarrow \theta'(t,\mathbf{x})$ and the reflection $t \to -t$. In particular, this means that the renormalization constants Z_κ in (3.13) and (3.14) coincide in all orders of the perturbation theory, because the correlator D_f does not appear in the relevant diagrams; see the next section.

C. Explicit leading-order results: Fixed points and scaling dimensions

Let us turn to the explicit calculation of the renormalization constant Z_κ in the leading one-loop order; for definiteness, consider the case of the density field (3.13). The constant is found from the requirement that the 1-irreducible Green's function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ be UV finite (that is, finite at $y \to 0$) when expressed in renormalized parameters. The corresponding Dyson equation in the frequency-momentum representation reads

$$\langle\theta'\theta\rangle_{1-\mathrm{ir}}(\omega,p) = +i\omega - \kappa_0 p^2 + \Sigma_{\theta'\theta}(\omega,p), \tag{3.15}$$

where the "self-energy operator" $\Sigma_{\theta'\theta}$ is given by the infinite sum of 1-irreducible graphs. In the one-loop approximation it has the form

$\Sigma_{\theta'\theta} =$  , (3.16)

where the wavy line denotes the bare propagator $\langle vv\rangle_0$ from (2.14) and the solid line with a slash denotes the bare propagator $\langle\theta\theta'\rangle_0$ from (3.6), the slashed end corresponding to the field θ'. The dots with three attached fields θ', θ, v denote the vertex (3.7).

In the leading-order approximation, the renormalization constant in the bare term of (3.15) is taken only in the first order in g, that is, $\kappa_0 = \kappa Z_\kappa \simeq \kappa(1 + z^{(1)}g/y)$, while in the diagram (3.16) all Z's are replaced with unities. Furthermore, we only need to know the divergent part of (3.16), which is proportional to p^2 (see the preceding section). Thus, we can set $\omega = 0$ in (3.15) and keep in the expansion in $\mathbf{p}$ of the resulting integrand only the p^2 term. Like for the original NS model, its divergent part is independent on $c_0 \sim c$ and can be calculated directly at $c = 0$; see the discussion in Sec. II C. Then the expression for (3.16) becomes

$$\Sigma_{\theta'\theta} = ip_s \int \frac{d\omega'}{2\pi} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d}\, i(p+k)_l \frac{D_{sl}(\omega',\mathbf{k})}{-i\omega' + wv|\mathbf{p}+\mathbf{k}|^2}, \tag{3.17}$$

where

$$D_{sl}(\omega',\mathbf{k}) = g\mu^y \nu^3 \left\{ \frac{P^\perp_{sl}(\mathbf{k})}{(\omega')^2 + \nu^2 k^4} + \frac{\alpha P^\parallel_{sl}(\mathbf{k})}{(\omega')^2 + u^2\nu^2 k^4} \right\} \tag{3.18}$$

is the velocity correlation function from (2.14) with the proper substitutions, including $c = 0$.

Integrations over the frequency are easily performed, for example,

$$\int \frac{d\omega'}{2\pi} \frac{1}{-i\omega' + w\nu|\mathbf{p}+\mathbf{k}|^2} \frac{1}{(\omega')^2 + u^2\nu^2k^4} = \frac{1}{2u\nu^2k^2(uk^2 + w|\mathbf{p}+\mathbf{k}|^2)}. \tag{3.19}$$

In the terms containing the factor $p_s p_l$ one can immediately set $\mathbf{p} = 0$ in (3.19), while in the exceptional term with $p_s k_l P^\parallel_{sl}(\mathbf{k}) = p_s k_s$ one should expand (3.19) up to the linear term in $\mathbf{p}$:

$$\frac{1}{uk^2 + w|\mathbf{p}+\mathbf{k}|^2} = \frac{1}{(u+w)k^2}\left\{1 - \frac{2w}{(u+w)}\frac{(\mathbf{p}\cdot\mathbf{k})}{k^2}\right\}.$$

With the aid of the formulas

$$\int d\mathbf{k}\, k_i f(k) = 0, \quad \int d\mathbf{k} \frac{k_i k_s}{k^2} f(k) = \frac{\delta_{is}}{d}\int d\mathbf{k}\, f(k), \quad \int d\mathbf{k} \frac{k_i k_s k_l k_p}{k^4} f(k) = \frac{\delta_{is}\delta_{lp} + \delta_{il}\delta_{sp} + \delta_{ip}\delta_{sl}}{d(d+2)} \int d\mathbf{k} f(k), \tag{3.20}$$

where $f(k)$ is any function depending only on $k = |\mathbf{k}|$, all the resulting integrals are reduced to the scalar integral

$$J(m) = \int_{k>m} d\mathbf{k} \frac{1}{k^{d+y}} = S_d \frac{m^{-y}}{y}, \tag{3.21}$$

with S_d from (2.27).

Collecting all the terms gives

$$\Sigma_{\theta'\theta} = -\frac{\hat{g}}{2dy}\left(\frac{\mu}{m}\right)^y \left\{ \frac{(d-1)}{(1+w)} + \frac{\alpha}{u(u+w)} - \frac{2\alpha w}{u(u+w)^2} \right\}, \tag{3.22}$$

with $\hat{g}$ defined in (2.26). Then the renormalization constant, needed to cancel the pole in y in (3.15), in the MS scheme should be chosen as

$$Z_\kappa = 1 - \frac{\hat{g}}{2dwy}\left\{ \frac{(d-1)}{(1+w)} + \frac{\alpha(u-w)}{u(u+w)^2} \right\}, \tag{3.23}$$

while the corresponding anomalous dimension is

$$\gamma_\kappa = \frac{\hat{g}}{2dw}\left\{ \frac{(d-1)}{(1+w)} + \frac{\alpha(u-w)}{u(u+w)^2} \right\}, \tag{3.24}$$

with the corrections of the order $\hat{g}^2$ and higher.

The function $\beta_w = \widetilde{\mathcal{D}}_\mu w$ for the new dimensionless parameter w has the form

$$\beta_w = -w\gamma_w = w[\gamma_\nu - \gamma_\kappa], \tag{3.25}$$

cf. Eq. (2.33). Substituting the one-loop expressions (2.41) and (3.24) and the exact relation (2.43) into the equation $\beta_w = 0$ gives, after some simple algebra, the equation

$$(w-1)[(d-1)(w+1)(w+2) + 2\alpha] = 0, \tag{3.26}$$

with the only positive solution $w_* = 1$.

The corresponding new eigenvalue of the matrix (2.40) coincides with the diagonal element

$$\partial\beta_w/\partial w|_{g=g_*} = y[3(d-1)+\alpha]/6(d-1) > 0,$$

because the functions (2.33) do not depend on w. We conclude that the fixed point with the coordinates (2.41) and $w_* = 1$ is IR attractive in the full space of couplings g, u, v, and w and governs the IR asymptotic behavior of the full-scale models (3.5) and (3.8).

The critical dimensions of the fields θ, θ' are obtained from the data in Table I and the expression (2.47) for Δ_ω:

$$\Delta_\theta = -1 + y/6, \quad \Delta_{\theta'} = d + 1 - y/6. \tag{3.27}$$

These expressions are exact due to the absence of renormalization of the fields θ and θ'.

IV. COMPOSITE FIELDS AND THEIR DIMENSIONS

The key role in the following is played by certain composite fields ("composite operators" in the quantum-field terminology). A local composite operator is a monomial or polynomial constructed from the primary fields $\Phi(x)$ and their finite-order derivatives at a single space-time point $x = \{t,\mathbf{x}\}$. In the Green's functions with such objects, new UV divergences arise due to coincidence of the field arguments. They are removed by additional renormalization procedure. As a rule, operators mix in renormalization: Renormalized operators are given by certain finite linear combinations of the original monomials. However, in the following only a simpler situation will be encountered, when the original operator $F(x)$ and the renormalized one $F^R(x)$ are related by multiplicative renormalization $F(x) = Z_F F^R(x)$ with the renormalization constant of the form (2.36). Then the critical dimension of the operator is given by the same expression (2.47) and, in general, differs from the simple sum of the dimensions of the fields and derivatives that enter the operator.

The total canonical dimension of any 1-irreducible Green's function Γ with one operator $F(x)$ and arbitrary number of primary fields (the formal index of UV divergence) is given by

$$\delta_\Gamma = d_F - \sum_\Phi N_\Phi d_\Phi, \tag{4.1}$$

where N_Φ are the numbers of the fields entering into Γ, d_Φ are their total canonical dimensions, d_F is the canonical dimension of the operator, and the summation over all types of the fields is implied. Superficial UV divergences can be present only in the functions Γ with a non-negative integer δ_Γ.

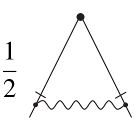
A. Renormalization of the composite fields θ^n: Explicit leading-order results

Let us begin with the simplest case of the operators $F(x) = \theta^n(x)$ in the density model. Then $d_F = -n$ in (4.1). Due to the linearity of the stochastic equation (3.1) in θ, the number of fields θ in any 1-irreducible function with the operator $F(x)$ cannot exceed their number in the operator itself. This is easily seen from the fact that the chains of the propagators $\langle\theta'\theta\rangle_0$, $\langle\theta\theta\rangle_0$ in any diagram cannot branch; cf. item (iv) in Sec. II C. Then the analysis of expression (4.1) shows that the superficial divergence can only be present in the 1-irreducible function with $N_\theta = n$ and $N_\Phi = 0$ for the fields Φ other than θ. For this function $\delta_\Gamma = 0$, the divergence is logarithmic, and the corresponding counterterm has the form $\theta^n(x)$. Hence, our operators are multiplicatively renormalizable: $F(x) = Z_n F^R(x)$ with certain renormalization constants of the form (2.36).

Now we turn to the calculation of the constants Z_n in the leading (one-loop) approximation. Let $\Gamma(x;\theta)$ be the generating functional of the 1-irreducible Green's functions with one composite operator $F(x)$ and any number of fields θ. Here $x = \{t,\mathbf{x}\}$ is the argument of the operator and θ is the functional argument, the "classical analog" of the random field θ. We are interested in the θ^n term of the expansion of $\Gamma(x;\theta)$ in $\theta(x)$, which we denote $\Gamma_n(x;\theta)$. It can be written as

$$\Gamma_n(x;\theta) = \int dx_1 \cdots \int dx_n\, \theta(x_1)\cdots\theta(x_n) \times \langle F(x)\theta(x_1)\cdots\theta(x_n)\rangle_{1-\text{ir}}. \tag{4.2}$$

In the one-loop approximation the function (4.2) is represented diagramatically as follows:

$$\Gamma_n(x;\theta) = F(x) + \frac{1}{2}$$  . (4.3)

The first term is the tree (loopless) approximation, and the thick dot with the two attached lines in the diagram denotes the operator vertex, that is, the variational derivative

$$V(x;x_1,x_2) = \delta^2 F(x)/\delta\theta(x_1)\delta\theta(x_2). \tag{4.4}$$

In the present case, the vertex

$$V(x;x_1,x_2) = n(n-1)\,\theta^{n-2}(x)\,\delta(x-x_1)\delta(x-x_2) \tag{4.5}$$

contains $(n-2)$ fields θ. [We recall that $\delta\theta(x)/\delta\theta(x') = \delta(x - x') \equiv \delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$.] Two more fields are attached to the plain vertices $\theta'\partial(v\theta)$ at the bottom of the diagram.

Since the divergence is logarithmic, one can set all the external frequencies and momenta equal to zero. Then all θ's acquire the common argument x and the diagram becomes proportional to the operator $\theta^n(x)$ with the coefficient, given by the "core" of the diagram,

$$\int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d}\, k_s k_l\, \frac{1}{\omega^2 + w^2\nu^2 k^4}\, D_{sl}(\omega,\mathbf{k}), \tag{4.6}$$

where the first factor in the integrand comes from the vertices (3.7), the second one comes from the propagators $\langle\theta'\theta\rangle_0$ in (3.6) with the replacement $\kappa_0 \to w\nu$, and the last factor is the velocity propagator from (3.18). Note that only the second term from D_{sl} gives nonvanishing contribution to (4.6). Integration over the frequency is easily performed using the formula

$$\int \frac{d\omega}{2\pi} \frac{1}{(\omega^2+a^2)(\omega^2+b^2)} = \frac{1}{2ab(a+b)}, \tag{4.7}$$

and after the contraction of the tensor indices the integral over the momentum reduces to (3.21). Collecting all the factors gives

$$\Gamma_n(x;\theta) = \theta^n(x)\left\{1 + \frac{n(n-1)}{2}\,\frac{\alpha\hat{g}}{2wu(u+w)}\left(\frac{\mu}{m}\right)^y \frac{1}{y}\right\}, \tag{4.8}$$

with $\hat{g}$ defined in (2.26) and up to a finite part and higher-order corrections.

The renormalization constant Z_n is found from the requirement that the renormalized analog $\Gamma_n^R = Z_n^{-1}\Gamma_n$ of the function (4.2) be UV finite in terms of renormalized parameters (mind the minus sign in the exponent). In our approximation, it is sufficient to replace $\theta^n \to Z_n^{-1}\theta^n$ only in the first term of the expression (4.8) and then to choose Z_n to cancel the pole in the second term. In the MS scheme this gives

$$Z_n = 1 + \frac{n(n-1)}{2}\,\frac{\alpha\hat{g}}{2wu(u+w)}\,\frac{1}{y}. \tag{4.9}$$

Then for the corresponding anomalous dimension Eq. (2.37) gives

$$\gamma_n = -\frac{n(n-1)}{2}\,\frac{\alpha\hat{g}}{2wu(u+w)}, \tag{4.10}$$

with the higher-order corrections in $\hat{g}$.

For the critical dimensions of the operators θ^n from the expression (2.47) one obtains

$$\Delta[\theta^n] = n\Delta_\theta + \gamma_n^*, \tag{4.11}$$

and substituting the fixed-point values (2.41) and $w_* = 1$ into (4.10) finally gives

$$\Delta[\theta^n] = -n + \frac{ny}{6} - \frac{n(n-1)\,\alpha\,dy}{6(d-1)}, \tag{4.12}$$

with the higher-order corrections in y. These dimensions are negative ("dangerous" in the terminology of [14–17]) and decrease as n grows. One can argue that dangerous operators can always appear in a field theoretic only as infinite families with the spectrum of dimensions not bounded from below.

Now let us turn to the same operators θ^n in the tracer model. From the expression (4.1) and the linearity of the stochastic equation (3.2) it follows that, like for the density case, the superficial UV divergences can only be present in the 1-irreducible function $\langle\theta^n(x)\theta(x_1)\cdots\theta(x_n)\rangle_{1-\mathrm{ir}}$. Clearly, at least one of the external tails of the field θ is attached to a vertex $\theta'(v\partial)\theta$: It is impossible to construct a nontrivial diagram of the desired type with all the external tails attached only to the vertex (4.5) of the operator $F(x)$. Therefore, at least one derivative ∂, acting on a tail θ, appears as an external factor in the diagram. Consequently, its real index of divergence δ'_Γ is necessarily negative, and the diagram is, in fact, UV convergent; cf. item (iii) in Sec. II C.

This means that the operators θ^n are, in fact, UV finite, $Z_n = 1$, and their scaling dimensions are given by the expression

$$\Delta[\theta^n] = n\Delta_\theta = -n + ny/6 \tag{4.13}$$

exactly, that is, with no higher-order corrections in y.

B. Renormalization of the composite fields $(\partial\theta)^n$ in the tracer model: Explicit leading-order results

In the tracer model, of special importance are tensor operators, constructed solely of the gradients of the passive scalar field. Such operators with the lowest canonical dimension contain the minimal number of derivatives (one derivative per each field) and have the form

$$F^{(n,l)}_{i_1\ldots i_l} = \partial_{i_1}\theta\cdots\partial_{i_l}\theta\,(\partial_i\theta\partial_i\theta)^s + \cdots. \tag{4.14}$$

Here l is the number of the free vector indices (the rank of the tensor) and $n = l + 2s$ is the total number of the fields θ entering into the operator. The ellipsis stands for the subtractions with Kronecker's δ symbols that make the operator irreducible (so that contraction with respect to any pair of the free tensor indices vanish); for example,

$$F^{(2,2)}_{ij} = \partial_i\theta\partial_j\theta - \frac{\delta_{ij}}{d}\,(\partial_k\theta\partial_k\theta). \tag{4.15}$$

For all these operators $d_F = 0$, and the real index of divergence is $\delta'_\Gamma = \delta_\Gamma - N_\theta$ with δ_Γ from (4.1). Indeed, now one derivative ∂ appears as an external factor in a diagram for any external tail θ, no matter if it is attached to the ordinary vertex $\theta'(v\partial)\theta$ or to the vertex (4.5) for the operator (4.14). Like for the operators θ^n, the number of the fields θ in any 1-irreducible function cannot exceed their number in the operator itself: $N_\theta \leqslant n$; cf. the discussion in Sec. IV A. It then follows that superficial UV divergences can only be present in the 1-irreducible functions $\langle F^{(n,l)}(x)\theta(x_1)\cdots\theta(x_k)\rangle_{1-ir}$ with $k \leqslant n$. For such functions $\delta'_\Gamma = 0$ and $\delta_\Gamma = k$, so that the corresponding counterterm can only involve the monomials $F^{(k,p)}$ from (4.14) with certain values of the rank p. We conclude that the family of the operators (4.14) is closed with respect to renormalization in the sense that $F^{(n,l)} = Z_{(n,l)(k,p)}F_R^{(k,p)}$ with a certain matrix of renormalization constants. Since $Z_{(n,l)(k,p)} = 0$ for $k > n$, this matrix is block triangular with the diagonal subblocks corresponding to $n = k$, and so is the corresponding matrix Δ_F in (2.47).

We are interested presumably in the scaling dimensions, associated with the operators (4.14). They are given by the eigenvalues of the matrix Δ_F, which are completely defined by its diagonal subblocks. A simple analysis shows that the corresponding diagrams do not involve the propagator $\langle\theta\theta\rangle_0$ from (3.6); this is again a consequence of the linearity of the original stochastic equation (3.2). Hence, the diagonal blocks can be calculated directly in the model without the random noise in (3.2), because the correlation function of the noise (3.3) enters the diagrams only via the propagator $\langle\theta\theta\rangle_0$. The function (3.3) is the only source of the anisotropy in the problem. Without the noise, the model becomes SO(d) covariant, and the irreducible tensor operators with different ranks cannot mix in renormalization. This means that the diagonal subblocks of the matrix Δ_F are, in fact, diagonal, and their diagonal elements coincide with the eigenvalues of the full matrix Δ_F.

We finally conclude that, as long as the scaling dimensions are concerned, the operators (4.14) can be treated as multiplicatively renormalizable, $F^{(n,l)} = Z_{(n,l)}F_R^{(n,l)}$, with certain renormalization constants $Z_{(n,l)}$, the diagonal elements of the full matrix $Z_{(n,l)(k,p)}$.

For practical calculations, it is convenient to contract the tensors (4.14) with an arbitrary constant vector $\boldsymbol{\lambda} = \{\lambda_i\}$. The resulting scalar operator has the form

$$F^{(n,l)} = (\lambda_i w_i)^l (w_i w_i)^s + \cdots, \quad w_i \equiv \partial_i\theta, \tag{4.16}$$

where the subtractions, denoted by the ellipsis, necessarily involve the factors of $\lambda^2 = \lambda_i\lambda_i$. The counterterm to $F^{(n,l)}$ is proportional to the same operator, and in order to find the constant $Z_{(n,l)}$, it is sufficient to retain only the principal monomial, explicitly shown in (4.16), and to discard in the result all the terms with factors of λ^2. Then, using the chain rule, the vertex (4.4) for the operator $F^{(n,l)}$ can be written in the form

$$V(x;x_1,x_2) = \frac{\partial^2 F^{(n,l)}}{\partial w_i \partial w_j}\,\partial_i\delta(x-x_1)\,\partial_j\delta(x-x_2) \tag{4.17}$$

up to irrelevant terms. The differentiation gives

$$\begin{aligned}\partial^2 F^{(n,l)}/\partial w_i\partial w_j &= 2s(w^2)^{s-2}(\lambda w)^l[\delta_{ij}w^2 + 2(s-1)w_iw_j]\\ &\quad + l(l-1)(w^2)^s(\lambda w)^{l-2}\lambda_i\lambda_j\\ &\quad + 2ls(w^2)^{s-1}(\lambda w)^{l-1}(w_i\lambda_j + w_j\lambda_i),\end{aligned} \tag{4.18}$$

where $w^2 = w_kw_k$ and $(\lambda w) = \lambda_k w_k$. Two more factors w_pw_r are attached to the bottom of the diagram, the derivatives coming from the vertices $\theta'(v\partial)\theta$. The UV divergence is logarithmic, and one can set all the external frequency and momentum equal to zero; then the core of the diagram takes on the form

$$\int\frac{d\omega}{2\pi}\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,k_ik_j\,D_{pr}(\omega,\mathbf{k})\,\frac{1}{\omega^2 + w^2\nu^2k^4}. \tag{4.19}$$

Here the first factor comes from the derivatives in (4.17), D_{pr} from (3.18) is the velocity correlation function (2.14), and the last factor comes from the two propagators $\langle\theta'\theta\rangle_0$. The substitutions $Z\to 1$, $c\to 0$ are made; cf. the discussion in Sec. III C.

Integrations over the frequency are easily performed using (4.7); then all the resulting integrals over $\mathbf{k}$ are reduced to the scalar integral (3.21) using the relations (3.20). Combining all the factors, contracting the tensor indices and expressing the result in $n = l + 2s$ and l gives

$$\Gamma_n(x;\theta) = F^{(n,l)}(x)\left\{1 - \frac{\hat g}{2yd(d+2)}\left(\frac{\mu}{m}\right)^y \times\left[\frac{Q_1}{2w(1+w)} + \alpha\frac{Q_2}{2wu(u+w)}\right]\right\}, \tag{4.20}$$

where

$$\begin{aligned}Q_1 &= -n(n+d)(d-1) + (d+1)l(l+d-2),\\ Q_2 &= -n(3n+d-4) + l(l+d-2),\end{aligned} \tag{4.21}$$

and $\hat g$ is defined in (2.26). Then the renormalization constant $Z_{(n,l)}$ in the MS scheme reads

$$Z_{(n,l)} = 1 - \frac{\hat g}{2yd(d+2)}\left\{\frac{Q_1}{2w(1+w)} + \alpha\frac{Q_1}{2wu(u+w)}\right\}; \tag{4.22}$$

see the explanation in Sec. IV A below Eq. (4.8). Then for the corresponding anomalous dimension Eq. (2.37) gives

$$\gamma_{(n,l)} = \frac{\hat g}{2d(d+2)}\left\{\frac{Q_1}{2w(1+w)} + \alpha\frac{Q_1}{2wu(u+w)}\right\}, \tag{4.23}$$

with the higher-order corrections in $\hat g$.

Finally, for the scaling dimension, associated with the operators (4.14), the general expression (2.47) gives

$$\Delta_{(n,l)} = n + n\Delta_\theta + \gamma^*_{(n,l)} = ny/6 + \gamma^*_{(n,l)}. \tag{4.24}$$

Substituting the fixed-point values (2.41) and $w_* = 1$ into (4.23), one finally obtains

$$\Delta_{(n,l)} = ny/6 + \frac{y\{Q_1 + \alpha Q_1\}}{6(d-1)(d+2)}, \tag{4.25}$$

with the higher-order corrections in y.

In particular, for the scalar operator with $l = 0$ one obtains

$$\Delta_{(n,0)} = \frac{-yn\{(n-2)(d-1) + \alpha(3n+d-4)\}}{6(d-1)(d+2)}. \tag{4.26}$$

Again, we meet an infinite family of dangerous operators with the spectrum of dimensions not bounded from below. For a fixed n, the dimension (4.25) increases with the rank l, so that for the maximum possible rank $l = n$ one always has $\Delta_{(n,n)} > 0$. This hierarchy, which is conveniently expressed by the inequality $\partial_l\Delta_{(n,l)} > 0$, becomes more strongly pronounced when α grows: $\partial_l\partial_\alpha\Delta_{(n,l)} > 0$. All these properties will be important in the OPE analysis of Sec. V.

C. More tensor operators

We also need to know the critical dimensions of the lth rank irreducible tensor operators, built only of two fields θ and l spatial derivatives. An example is provided by the operator

$$F_{i_1,\ldots,i_l}(x) = \theta(x)\partial_{i_1}\cdots\partial_{i_l}\theta(x) + \cdots. \tag{4.27}$$

As earlier in (4.14), the ellipsis stands for the subtractions with Kronecker's δ symbols that make the operator irreducible. Of course, for any given $l > 1$, there are several such operators with different placement of the derivatives: In the special case (4.27), all the derivatives act on the same field. However, all the other such operators differ from (4.27) by a total derivative, which is easily seen from the relation

$$F(x)\partial G(x) = -G(x)\partial F(x) + \partial(F(x)G(x)). \tag{4.28}$$

Thus, the set of independent lth rank operators can be chosen as (4.27) and the operators having the forms of derivatives, for example, for $l = 2$, as $\theta\partial_i\partial_j\theta + \cdots$ and $\partial_i\partial_j(\theta\theta) + \cdots$. In the calculation of their critical dimensions, it is sufficient to consider the SO(d) covariant model without the noise (3.3); see the discussion in the preceding section. Then the operators with different ranks do not mix in renormalization. The analysis of renormalization also shows that the operator (4.27) can mix only with its own "family" of derivatives: The operators with additional derivatives (like ∂_t or ∂^2) or with the fields θ', ϕ, ϕ', v' have too high canonical dimensions d_F, the appearance of v is forbidden by Galilean symmetry, and extra θ's are forbidden by the linearity of the model.

The same relation (4.28) also shows that for odd l, the operator (4.27) itself reduces to a derivative (more precisely, to a linear combination of derivatives). In the following, we are interested only in the operators not reducible to derivatives, and thus, from now on, we consider only even values of l. Then (4.27) is nontrivial and it cannot admix to the derivatives from its family, although they can admix to (4.27). Thus, the corresponding renormalization matrix Z_F appears block triangular, and so is the matrix Δ_F. The eigenvalue, associated with the nontrivial operator (4.27), coincides with the corresponding diagonal element of Δ_F. We conclude that in the calculation of the critical dimension, associated with the operator (4.27), the latter can be treated as if it were multiplicatively renormalizable.

Like in the preceding section, it is convenient to contract the operator (4.27) with an arbitrary constant vector $\boldsymbol{\lambda}=\{\lambda_i\}$. The resulting scalar operator has the form

$$F_l = \theta(\lambda_i\partial_i)^l\theta + \cdots, \tag{4.29}$$

where the terms, denoted by the ellipsis, necessarily involve the factors of λ^2. In order to find the corresponding renormalization constant Z_l, it is sufficient to keep only the principal monomial, explicitly shown in (4.16), and to retain in the result for the counterterm only terms of the same form. Then the relevant part of the vertex factor (4.4) is

$$V(x;x_1,x_2) = \delta(x-x_1)(\lambda_i\partial_i)^l\delta(x-x_2) + \{x_1 \leftrightarrow x_2\}. \tag{4.30}$$

The one-loop approximation for the functional (4.2) for the operator (4.29) has the same form (4.3). Let us choose the external momentum $\mathbf{p}$ to flow into the diagram through the left lower vertex and to flow out through the right lower one. The external momentum flowing through operator's vertex and all the external frequencies are set equal to zero: This is sufficient to find the needed counterterm. Furthermore, we put $w = u = 1$ in the propagators from the very beginning, because we are eventually interested in the value of the anomalous dimension at the fixed point $w_* = u_* = 1$.

Let us begin with the tracer case. Then the core of the diagram in (4.3) takes on the form

$$p_i p_j \int \frac{d\omega}{2\pi}\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\, 2i^l(\boldsymbol{\lambda}\cdot\mathbf{q})^l\,\frac{g\mu^y\nu^3k^{4-d-y}}{\omega^2+\nu^2k^4} \times\{P^{\perp}_{ij}(\mathbf{k})+\alpha P^{\parallel}_{ij}(\mathbf{k})\}\frac{1}{\omega^2+\nu^2q^4}. \tag{4.31}$$

Here the factor p_ip_j comes from the vertices (3.9), the factor $2i^l(\boldsymbol{\lambda}\cdot\mathbf{q})^{\mathbf{l}}$ comes from the vertex (4.30) for even l [for the odd l the two terms in (4.30) would cancel each other and instead of factor 2 one would get 0], the factors depending on $\mathbf{k}$ represent the velocity correlation function from (2.14) with the proper substitutions, including $c = 0$ and $w = u = 1$. The last factor comes from the propagators $\langle\theta'\theta\rangle_0$. The momentum $\mathbf{k}$ flows through the velocity propagator, so that $\mathbf{q} = \mathbf{k} + \mathbf{p}$.

In the resulting expression we retain only terms of the form $(\boldsymbol{\lambda}\cdot\mathbf{p})^{\mathbf{l}}$ and drop all the other terms, containing λ^2 or p^2. Thus, we can replace

$$p_ip_j\{P^{\perp}_{ij}+\alpha P^{\parallel}_{ij}\} \to (\alpha-1)(\mathbf{p}\cdot\mathbf{k})^2/k^2.$$

The integration over ω in (4.31) is easily performed using (4.7) and gives

$$g\mu^y(\alpha-1)i^l\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,(\mathbf{p}\cdot\mathbf{k})^2(\boldsymbol{\lambda}\cdot\mathbf{q})^l\,\frac{k^{-d-y}}{q^2(k^2+q^2)}. \tag{4.32}$$

Now we expand all the denominators in the integrand of (4.31) in $\mathbf{p}$ (dropping all the terms with p^2),

$$\frac{1}{q^2} \simeq \frac{1}{k^2+2(\mathbf{p}\cdot\mathbf{k})} = \frac{1}{k^2}\sum_{s=0}^{\infty}\frac{(-2)^s(\mathbf{p}\cdot\mathbf{k})^s}{k^{2s}},$$
$$\frac{1}{k^2+q^2} \simeq \frac{1}{2(k^2+\mathbf{p}\cdot\mathbf{k})} = \frac{1}{2k^2}\sum_{m=0}^{\infty}\frac{(-1)^m(\mathbf{p}\cdot\mathbf{k})^m}{k^{2m}}, \tag{4.33}$$

and expand the numerator using Newton's binomial formula:

$$(\boldsymbol{\lambda}\cdot\mathbf{q})^l = \sum_{n=0}^{l}C^n_l(\boldsymbol{\lambda}\cdot\mathbf{k})^n(\boldsymbol{\lambda}\cdot\mathbf{p})^{l-n}. \tag{4.34}$$

In the resulting threefold series over n,m,s,

$$\sum_{n=0}^{l}C^n_l(\boldsymbol{\lambda}\cdot\mathbf{p})^{l-n}\sum_{m,s=0}^{\infty}\frac{(-1)^m(-2)^s(\mathbf{p}\cdot\mathbf{k})^{m+s+2}(\boldsymbol{\lambda}\cdot\mathbf{k})^n}{k^{2(s+m)}},$$

we only need to collect the terms proportional to $(\boldsymbol{\lambda}\cdot\mathbf{p})^{\mathbf{l}}$, which leads to the restriction $n = s + m + 2$ and hence to the finite double sum

$$\sum_{s,m=0}^{s+m+2\leqslant l}(-1)^m(-2)^sC^{s+m+2}_l \times\frac{(\boldsymbol{\lambda}\cdot\mathbf{p})^{l-m-s-2}(\mathbf{p}\cdot\mathbf{k})^{m+s+2}(\boldsymbol{\lambda}\cdot\mathbf{k})^{s+m+2}}{k^{2(s+m)}}. \tag{4.35}$$

Substituting it to the (4.32) gives rise to the integrals

$$J_{i_1,\ldots,i_{2n}}(m) = \int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,k^{-d-y}\,\frac{k_{i_1},\ldots,k_{i_{2n}}}{k^{2n}}, \tag{4.36}$$

with $n = s + m + 2 \geqslant 2$. They are easily found using the isotropy considerations, cf. (3.20),

$$J_{i_1,\ldots,i_{2n}}(m) = \frac{\delta_{i_1i_2},\ldots,\delta_{i_{2n-1}i_{2n}} + \text{all permutations}}{d(d+2)\cdots(d+2n-2)}\,J(m), \tag{4.37}$$

with $J(m)$ from (3.21). The sum over all possible permutations of $2n$ tensor indices in the numerator of (4.37) involves $(2n-1)!! = (2n)!/2^nn!$ terms, but we have to keep only the terms that give rise to the structure $(\boldsymbol{\lambda}\cdot\mathbf{p})^{\mathbf{n}}$ after the contraction with the vectors $\boldsymbol{\lambda}$ and $\mathbf{p}$ in (4.35). It is easy to grasp that there are only $n!$ such permutations.

Collecting all the factors gives for the core (4.31) of the diagram in (4.3) the expression

$$i^l(\boldsymbol{\lambda}\cdot\mathbf{p})^l(\alpha-1)\hat{g}\left(\frac{\mu}{m}\right)^y\frac{1}{2y}\,\mathcal{S}_l(d), \tag{4.38}$$

where $\hat{g}$ is defined in (2.26) and

$$\mathcal{S}_l(d) = \sum_{s,m=0}^{s+m+2\leqslant l}\frac{(-1)^{s+m}2^sC^{s+m+2}_l(s+m+2)!}{d(d+2)\cdots[d+2(s+m)+2]}. \tag{4.39}$$

For $l = 0$, the sums (4.35) and (4.39) contain no terms, so that $\mathcal{S}_0(d) = 0$.

For the functional (4.2) we then obtain ($ip_i \to \partial_i$)

$$\Gamma_2(x) = F_l(x)\left\{1 + (\alpha - 1)\frac{\hat{g}}{4y}\left(\frac{\mu}{m}\right)^y \mathcal{S}_l(d)\right\}, \tag{4.40}$$

with the operator F_l from (4.29); note the additional factor $1/2$ from the symmetry coefficient in (4.2). Then for the renormalization constant from the relation $F_l = Z_l F_l^R$ in the MS scheme we obtain

$$Z_l = 1 + (\alpha - 1)\frac{\hat{g}}{4y}\,\mathcal{S}_l(d), \tag{4.41}$$

and the corresponding anomalous dimension is

$$\gamma_l(g) = -(\alpha - 1)\frac{\hat{g}}{4}\,\mathcal{S}_l(d). \tag{4.42}$$

The sum $\mathcal{S}_l(d)$ in (4.39) can be reduced to a simpler onefold sum for general l. Let us pass from s and m to the new summation variables $k = s + m$ and m and substitute the explicit expression for the binomial coefficient $C_l^{k+2} = l!/(k+2)!(l-k-2)!$. This gives

$$\mathcal{S}_l(d) = l! \sum_{k=0}^{k+2\leqslant l}\left\{\sum_{m=0}^{k}\frac{1}{2^m}\right\} \times \frac{(-2)^k}{(l-k-2)!\,d(d+2)\cdots(d+2k+2)}. \tag{4.43}$$

Now the internal summation over m is readily performed to give $2 - 2^{-k}$, so that, after changing the summation variable $k \to k+2$, we obtain

$$\mathcal{S}_l(d) = 2\mathcal{N}_l(d) - \mathcal{M}_l(d), \tag{4.44}$$

where

$$\mathcal{N}_l(d) = l! \sum_{k=2}^{l}\frac{(-2)^{k-2}}{(l-k)!\,d(d+2)\cdots(d+2k-2)},$$
$$\mathcal{M}_l(d) = l! \sum_{k=2}^{l}\frac{(-1)^{k-2}}{(l-k)!\,d(d+2)\cdots(d+2k-2)}. \tag{4.45}$$

The first sum can be calculated explicitly for any l (cf. [30]),

$$\mathcal{N}_l(d) = \frac{4l(l-1)}{d(d+2l-2)}, \tag{4.46}$$

while the second can be easily calculated for any given l.

For the critical dimension, associated with the operator (4.27), from the relation (2.47) we finally obtain

$$\Delta_l = l + 2\Delta_\theta + \gamma_l^* = l - 2 + y/3 + \gamma_l^*, \tag{4.47}$$

where from (4.42) and (2.41) we find

$$\gamma_l^* = \gamma_l(g^*) = -\frac{yd(\alpha-1)}{3(d-1)}\,\mathcal{S}_l(d), \tag{4.48}$$

with the higher-order corrections in y.

For $l = 0$, expressions (4.42) and (4.47) agree with the exact result (4.13) for the operator θ^2 [we recall that $\mathcal{S}_0(d) = 0$], while for $l = 2$ they agree with the results (4.23)–(4.25) with $n = l = 2$.

Now let us turn to the density case. Then the factor $p_i p_j$ in (4.31) should be replaced with $q_i q_j$ (and, of course, moved into the integrand). It is convenient to write

$$q_i q_j\{P_{ij}^{\perp}(\mathbf{k}) + \alpha P_{ij}^{\parallel}(\mathbf{k})\} = p_i p_j\{P_{ij}^{\perp}(\mathbf{k}) + \alpha P_{ij}^{\parallel}(\mathbf{k})\} + \alpha(q^2 - p^2). \tag{4.49}$$

The first term gives the old expression (4.31), and the last one is proportional to p^2 and can be dropped. Thus, we only need to calculate the contribution of the term αq^2 to the analog of expression (4.31). Then the analog of (4.32) takes on the form

$$g\mu^y\alpha\, i^l \int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,(\boldsymbol{\lambda}\cdot\mathbf{q})^l\,\frac{k^{2-d-y}}{(k^2+q^2)}. \tag{4.50}$$

Applying the expansions (4.33) and (4.34) leads to the double sum

$$\sum_{n=0}^{l} C_l^n(\boldsymbol{\lambda}\cdot\mathbf{p})^{l-n}\sum_{m=0}^{\infty}\frac{(-1)^m(\mathbf{p}\cdot\mathbf{k})^m(\boldsymbol{\lambda}\cdot\mathbf{k})^n}{k^{2m}}.$$

We have to retain only the terms proportional to $(\boldsymbol{\lambda}\mathbf{p})^l$, which leads to the restriction $n = m$ and hence to the finite sum:

$$\sum_{m=0}^{l}(-1)^m C_l^m\,\frac{(\boldsymbol{\lambda}\cdot\mathbf{p})^{l-m}(\mathbf{p}\cdot\mathbf{k})^m(\boldsymbol{\lambda}\cdot\mathbf{k})^m}{k^{2m}}. \tag{4.51}$$

Substituting it into (4.50) gives rise to the integrals (4.36) with all $n \geqslant 0$. In the sum (4.37) over all possible permutations we have to keep only $n! = m!$ terms that give rise to the structure $(\boldsymbol{\lambda}\mathbf{p})^n$ after the contraction with the vectors $\boldsymbol{\lambda}$ and $\mathbf{p}$ in (4.51). To avoid possible confusion, we write the terms with $m = 0$ and $m = 1$ separately and for $m \geqslant 2$ apply the formula (4.37). Then collecting all terms gives the result for (4.50)

$$i^l(\boldsymbol{\lambda}\cdot\mathbf{p})^m\,\frac{g\mu^y\alpha}{2}\left\{1 - \frac{l}{d} + \mathcal{M}_l(d)\right\} J(m), \tag{4.52}$$

with $J(m)$ from (3.21) and the sum $\mathcal{M}_l(d)$ from (4.45).

Proceeding as before for the tracer case, we arrive at the following expression for the renormalization constant Z_l in the MS scheme:

$$Z_l = 1 + (\alpha - 1)\frac{\hat{g}}{4y}\mathcal{S}_l(d) + \alpha\frac{\hat{g}}{4y}\left\{1 - \frac{l}{d} + \mathcal{M}_l(d)\right\}. \tag{4.53}$$

Here the contribution with $\mathcal{S}_l(d)$ comes from the first term in (4.49) and the last term with curly brackets comes from (4.53). Then for the anomalous dimension, using the expressions (4.44)–(4.46), we obtain

$$\gamma_l(g) = -\alpha\frac{\hat{g}}{4}\left(1 - \frac{l}{d}\right) + (1-\alpha)\frac{\hat{g}}{4}\mathcal{N}_l(d) - \frac{\hat{g}}{4}\mathcal{M}_l(d), \tag{4.54}$$

with higher-order corrections in g.

In the expression (4.47) for the critical dimension one has

$$\gamma_l^* = -\alpha\frac{y(l-d)}{3(d-1)} + (1-\alpha)\frac{8l(l-1)y}{3(d-1)(d+2l-2)} - \frac{dy}{3(d-1)}\mathcal{M}_l(d), \tag{4.55}$$

with higher-order corrections in y. For $l=0$ this result is in agreement with the expression (4.12) for the operator θ^2 in the density case.

V. OPERATOR PRODUCT EXPANSION AND THE ANOMALOUS SCALING

A. The case of a density field

Consider the equal-time pair correlation function of two UV finite quantities $F_{1,2}(x)$ with definite critical dimensions, for example, those of the primary fields or renormalized local composite operators. We restrict ourselves with equal-time correlators, because they are usually Galilean invariant and do not bear strong dependence on the IR scale, caused by the so-called sweeping effects. From the (canonical) dimensionality considerations it follows that

$$\langle F_1(t,\mathbf{x}_1)F_2(t,\mathbf{x}_2)\rangle = \nu^{d_F^\omega}\mu^{d_F}\eta(\mu r,mr,c/(\mu\nu)), \tag{5.1}$$

where d_F^ω and d_F are the canonical dimensions of the correlation function, given by simple sums of the corresponding dimensions of the operators, $r=|\mathbf{x}_2-\mathbf{x}_1|$, and $\eta(\cdots)$ is a function of completely dimensionless variables. We have written the right-hand side in terms of renormalized parameters, when the reference mass substitutes the typical UV momentum scale Λ. The behavior of the function η in the IR range, that is, for $\mu r\gg 1$, is determined by the IR attractive fixed point of the RG equation. Solving the RG equation in a standard way, one derives the following asymptotic expression:

$$\langle F_1(t,\mathbf{x}_1)F_2(t,\mathbf{x}_2)\rangle \simeq \nu^{d_F^\omega}\mu^{d_F}(\mu r)^{-\Delta_F}\zeta(mr,c(r)). \tag{5.2}$$

Here Δ_F is the critical dimension of the correlation function, given by simple sum of the dimensions of the operators. The RG equation does not determine the form of the scaling function ζ; it only determines the form of its arguments. They are canonically and critically dimensionless: In particular,

$$c(r)=c(\mu r)^{\Delta_c}/(\mu\nu), \tag{5.3}$$

with Δ_c from (2.49) can be interpreted as the effective speed of sound; more detailed discussion of this point can be found in [47].

For the correlation functions of two operators of the type $\theta^n(x)$ the general expression (5.2) gives

$$\langle\theta^p(t,\mathbf{x}_1)\theta^k(t,\mathbf{x}_2)\rangle \simeq \mu^{-(p+k)}(\mu r)^{-\Delta_p-\Delta_k}\zeta_{pk}(mr,c(r)), \tag{5.4}$$

with the dimensions Δ_n from (4.12). In the following, we do not display the dependence on the UV parameters μ and ν and omit the indices of the scaling functions.

The inertial-convective range corresponds to the additional condition that $mr\ll 1$. The behavior of the functions ζ at $mr\to 0$ can be studied by means of the OPE [13,14]. In the case at hand, it has the form

$$F_1(t,\mathbf{x}_1)F_2(t,\mathbf{x}_2) \simeq \sum_F C_F(mr,c(r))\,F(t,\mathbf{x}), \tag{5.5}$$

where $\mathbf{x}_2-\mathbf{x}_1\to 0$ and $\mathbf{x}=(\mathbf{x}_1+\mathbf{x}_1)/2$ is fixed. The summation in (5.5) is taken, in general, over all possible renormalized local composite operators allowed by the symmetries of the model and of the left-hand side, C_F being numerical coefficient functions analytical in mr and $c(r)$. In our model, due to the linearity in the field θ, the number of such fields in the operators F cannot exceed their number on the left-hand side. This restriction, which our model shares with the Kraichnan's model and its relatives [12] will be very important in the following.

The correlation function (5.2) is obtained by averaging (5.5) with the weight $\exp\mathcal{S}_R$ with the renormalized action functional from (3.4). The mean values $\langle F(x)\rangle$ appear on the right-hand side. Without loss of generality, it can be assumed that the expansion in (5.5) is made in irreducible tensor operators. Then, if the model is SO(d) covariant [the correlation function of the scalar noise (3.3) depends only on $r=|\mathbf{r}|$], only scalar operators survive the averaging. It can also be assumed that the expansion is made in the operators with definite critical dimensions. Then their mean values, in the asymptotic region of small m, take on the forms

$$\langle F(x)\rangle \simeq m^{\Delta_F}\xi(c(1/m)), \tag{5.6}$$

with another set of scaling functions ξ and the argument $c(\cdots)$ from (5.3). Since the diagrams of the perturbation theory have finite limits both for $c\to\infty$ and $c\to 0$, we may assume that the functions ξ are restricted for all values of c and can be estimated by some constants. What is more, for y large enough, including the most realistic case $y\to 4$, the dimension Δ_c becomes negative; see expression (2.49). Thus, the argument $c(1/m)\sim cm^{-\Delta_c}$ becomes small for fixed c and $m\to 0$, and the function ξ can be replaced by its (finite) limit value $\xi(0)$. We finally conclude that, in the IR range,

$$\langle F(x)\rangle \sim m^{\Delta_F}. \tag{5.7}$$

Then combining expressions (5.2), (5.5), and (5.7) gives the desired asymptotic expression for the scaling functions,

$$\zeta(mr,c(r)) \simeq \sum_F A_F(mr,c(r))\,(mr)^{\Delta_F}, \tag{5.8}$$

where the summation runs over Galilean-invariant scalar operators, with the coefficient functions A_F analytical in their arguments.

Divergences for $mr\to 0$ (and hence the anomalous scaling) result from the contributions of the operators with *negative* critical dimensions, termed "dangerous" in [15]. Clearly, the leading contribution is determined by the operator with the lowest (minimal) dimension; the others determine the corrections. All the operators θ^n are dangerous, and the spectrum of their dimensions is not restricted from below (there is no "most dangerous" operator); see expression (4.12). Fortunately, for *a given* correlation function, only a finite number of those operators can contribute to the OPE. For (5.4), these are the operators with $n\leqslant p+k$. Thus,

$$\zeta(mr,c(r)) \simeq \sum_{n=0}^{p+k} A_n(mr,c(r))\,(mr)^{\Delta_n}+\cdots, \tag{5.9}$$

with Δ_n from (4.12); the ellipsis stands for the "more distant" corrections, related to the operators with derivatives and other types of fields. The leading term of the small-mr behavior in (5.9) is given by the operator with the maximum possible $n=p+k$, so that the final expression has the form

$$\langle\theta^p(t,\mathbf{x}_1)\theta^k(t,\mathbf{x}_2)\rangle \simeq \mu^{-(p+k)}(\mu r)^{-\Delta_p-\Delta_k}(mr)^{\Delta_{p+k}}. \tag{5.10}$$

It is worth noting that the set of operators θ^n is "closed with respect to the fusion" in the sense that the leading term in the OPE for the pair correlator of two such operators is given by the operator from the same family with the summed exponent. This fact along with the inequality $\Delta_p + \Delta_k > \Delta_{p+k}$, which follows from the explicit expression (4.12), can be interpreted as the statement that the correlations of the scalar field in the density model reveal multifractal behavior; see [54].

B. The case of the tracer field

For the tracer model, the critical dimensions of the operators θ^n are linear in n: $\Delta[\theta^n] = n\Delta_\theta$; see Eq. (4.13). Then the dependence on the separation r in the asymptotic expressions (5.10) disappears: The leading terms of the inertial-range behavior are constants. More "vivid" quantities are the equal-time structure functions defined as

$$S_n(r) = \langle[\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}']^{2n}\rangle = (\nu\mu^2)^{-n}\eta(\mu r, mr, c/(\mu\nu)),$$
$$r = |\mathbf{x}' = \mathbf{x}|; \tag{5.11}$$

the second equality with dimensionless functions η follows from dimensionality considerations. Solving the RG equations gives the asymptotic expressions for $\mu r \gg 1$,

$$S_n(r) = (\nu\mu^2)^{-n}(\mu r)^{-2n\Delta_\theta}\zeta(mr, c(r)), \tag{5.12}$$

with $c(r)$ from (5.3) and some scaling functions ζ. It is important here that the pair correlation functions $\langle\theta^p\theta^k\rangle$ with $k + p = 2n$, appearing in the binomial decomposition of S_n, have similar asymptotic representations (5.4) with the same critical dimension $\Delta_k + \Delta_p = 2n\Delta_\theta$, and together they form the single asymptotic expression (5.12). The constant leading terms for those correlators, related to the contributions of the operator θ^n in the corresponding OPE, cancel each other in the structure function, and the latter acquires nontrivial dependence on r in the inertial range.

Indeed, both the functions (5.11) and the action (3.8) for the tracer (not for the density) are invariant with respect to the constant shift $\theta(x) \to \theta(x) + \text{const}$. Then the operators entering the corresponding OPE,

$$[\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}']^{2n} \simeq \sum_F C_F(mr, c(r))\, F(t,\mathbf{x}), \quad r \to 0,$$
$$\mathbf{x} = (\mathbf{x} + \mathbf{x}')/2, \tag{5.13}$$

must also be all invariant, so that they can involve the field θ only in the form of derivatives. Clearly, the leading term of the small-m behavior will be determined by the scalar operator with maximum possible number of the fields θ (namely, $2n$ for the given S_n) and the minimum possible number of spatial derivatives (namely, $2n$: one derivative for each θ). This is nothing other than the operator $F^{(2n,0)} = (\partial_i\theta\partial_i\theta)^n$ from (4.14). Thus, the desired leading-order expression for S_n in the inertial range is

$$S_n(r) \sim (\nu\mu^2)^{-n}(\mu r)^{-2n\Delta_\theta}(mr)^{\Delta_{(2n,0)}}, \tag{5.14}$$

with the dimension $\Delta_{(2n,0)}$ given in (4.25). The operators $F^{(2p,0)}$ with $p < n$ determine the main corrections to (5.14), the operators with extra derivatives and/or other types of fields correspond to more "distant" corrections (they all must be invariant with respect to the Galilean transformation and the shift of θ).

For the tracer, the "multifractal" behavior is demonstrated by the family of the operators $F^{(n,0)}$ rather than by the simple powers θ^n; see the end of the preceding section. Indeed, it is easy to grasp that the inertial-range behavior of the pair correlation function $\langle F^{(p,0)}F^{(k,0)}\rangle$ of two such operators is determined by the contribution to the OPE from their "elder brother" $F^{(n,0)}$ with $n = p + k$ and has the form (omitting the dependence on the UV parameters μ and ν)

$$\langle F^{(p,0)}(t,\mathbf{x})F^{(k,0)}(t,\mathbf{x}')\rangle \sim r^{-\Delta_{(p,0)}-\Delta_{(k,0)}+\Delta_{(n,0)}}. \tag{5.15}$$

The required inequality $\Delta_{(n,0)} < \Delta_{(p,0)} + \Delta_{(k,0)}$ [54] follows from the explicit one-loop expression (4.25). It remains to note that the operator $F^{(2,0)}$ can be interpreted as the local dissipation rate of fluctuations of our scalar field.

C. Effects of the large-scale anisotropy

Now consider the effects of the anisotropy, introduced into the system at large scales $\sim L$ through the correlation function of the random noise (3.3). As an illustration, consider first the case of uniaxial anisotropy: Assume that the function $C(\mathbf{r}/L)$ in (3.3) depends also on a constant unit vector $\mathbf{n} = \{n_i\}$ that determines a certain distinguished direction.

Then the irreducible tensor composite operators acquire nonzero mean values, with the tensor factors built of the vector $\mathbf{n}$. For example, the mean value of the operator (4.15) is proportional to the irreducible tensor $n_i n_j - \delta_{ij}/d$. In general, the mean value of any lth rank irreducible operator is proportional to the tensor $n_{i_1}, \ldots, n_{i_l} + \cdots$, where the ellipsis stands for the contributions with the Kronecker δ symbols that make it irreducible. Upon substitution into the OPE (5.13), their tensor indices are contracted with the corresponding indices of the coefficient functions $C_F(\mathbf{r})$. This gives rise to the (d-dimensional generalizations of the) Legendre polynomials $P_l(\cos\vartheta)$, where ϑ is the angle between the vectors $\mathbf{r}$ and $\mathbf{n}$.

Thus, the OPE expansion in irreducible composite operators provides the expansion in the irreducible representations of the SO(d) group. The main contribution to the "shell" with a given l is determined by the lth rank operator with the lowest critical dimension (of course, it should respect the symmetries of the model and of the left-hand side). Clearly, for the structure function S_n and $l \leqslant 2n$ the needed operator is $F^{(2n,l)}_{i_1,\ldots,i_l}$ from (4.14). For $l > 2n$ we need the operators that contain more derivatives than fields.

The expansion that takes into account only the leading term in each shell has the form (again, we omit ν and μ)

$$S_n = r^{-2n\Delta_\theta}\sum_{l=0}^{2n} A_l(r)\, P_l(\cos\vartheta)\,(mr)^{\Delta_{(2n,l)}} + \cdots, \tag{5.16}$$

with the dimension $\Delta_{(2n,l)}$ from (4.24); the ellipsis stands for the contributions with $l > 2n$. For the general large-scale anisotropy, all the spherical harmonics Y_{ls} will appear in the expansion, with the exponents depending only on l.

From the explicit leading-order expressions (4.25) it follows that the dimensions (4.24), for a fixed n, monotonically increase with l,

$$\Delta_{n,l} > \Delta_{n,p} \quad \text{if} \quad l > p, \tag{5.17}$$

or, in the differential form, $\partial \Delta_{n,l}/\partial l > 0$. Similar inequalities were derived earlier in various models of passively advected vector [35] and scalar [19] fields. This fact has a clear physical interpretation: In the presence of the large-scale anisotropy, anisotropic contributions in the inertial range exhibit a hierarchy, related to the "degree of anisotropy" l. The leading contribution is given by the isotropic shell ($l = 0$); the corresponding anomalous exponent is the same as for the purely isotropic case. The contributions with $l > 1$ give only corrections which become relatively weaker as $mr \to 0$, the faster the higher the degree of anisotropy l is. This effect gives quantitative support for Kolmogorov's hypothesis of the local isotropy restoration and appears rather robust, being observed for the real fluid turbulence [55].

The hierarchy (5.17) becomes more strongly pronounced as the degree of compressibility α increases, which can be expressed by the inequality $\partial^2\Delta_{n,l}/\partial l\partial\alpha > 0$. Thus, the anisotropic corrections become further from one another and from the isotropic term, in contrast to the situation observed earlier for passive scalar [31,32] and vector [36] fields, advected by Kraichnan's ensemble. The same inequality holds for the "frozen" regime in the Gaussian model with finite correlation time, the fact overlooked in [31].

For $l > 2n$, the leading contributions to the lth shell are determined by the operators that involve more derivatives than fields. The calculation of their dimensions is a difficult task because of the mixing of such operators in renormalization. The hierarchy relations remain valid due to the contributions of the canonical dimensions to the general expression (2.47): Clearly, their critical dimensions have the forms $l - 2n + O(y)$.

Fortunately, for the pair correlation functions, the full analog of the expression (5.16) can be presented, with all the shells included. Indeed, it is clear that the leading term of the lth shell now is determined by the single operator (4.27) with two fields and l tensor indices: It is unique up to derivatives, which have vanishing mean values and do not contribute to the quantities of interest. Thus, the desired asymptotic expression has the form

$$\langle\theta(t,\mathbf{x})\theta(t,\mathbf{x}')\rangle = r^{-2\Delta_\theta}\sum_{l=0}^{\infty} A_l(r)\, P_l(\cos\vartheta)\,(mr)^{\Delta_l}, \tag{5.18}$$

with the dimensions Δ_l from (4.47) and (4.48) for the tracer and (4.47) and (4.55) for the density case. The hierarchy of anisotropic contributions, similar to (5.17), holds, at least for small y, due to the contribution of the canonical dimensions to (4.47): $\Delta_l = l - 2 + O(y)$. Thus, the leading term in (5.18) is given by the scalar operator θ^2. When one passes to the structure function S_2 for the tracer that term is subtracted, and the leading role is inherited by the scalar operator $F^{(2,0)}$ from (4.14) in agreement with (5.16). The hierarchy is getting weaker as the compressibility parameter α grows: $\partial^2\Delta_{n,l}/\partial l\partial\alpha < 0$, as follows from the analysis of the explicit one-loop expressions (4.47), (4.48), and (4.55). Here our results agree with those for the Kraichnan model: Anisotropic corrections become closer to each other and to the isotropic term; cf. [30].

VI. DISCUSSION AND CONCLUSION

We have studied two models of passive scalar advection: the case of the density of a conserved quantity and the case of a tracer, described by the advection-diffusion equations (3.1) and (3.2), respectively, and subject to a random large-scale forcing (3.3). The advecting velocity field is described by the Navier-Stokes equations for a compressible fluid (2.7) and (2.8) with an external stirring force with the correlation function $\propto k^{4-d-y}$; see (2.10) and (2.11).

The full stochastic problems can be formulated as field theoretic models with the action functionals specified in (2.12), (3.5), and (3.8). Those models appear multiplicatively renormalizable, so that the corresponding RG equations can be derived in a standard fashion. They have the only IR attractive fixed point in the physical range of the model parameters, and the correlation functions reveal scaling behavior in the IR region (inertial and energy-containing ranges).

Their inertial-range behavior was studied by means of the OPE; existence of anomalous scaling (singular power-like dependence on the integral scale L) was established. The corresponding anomalous exponents were identified with the scaling (critical) dimensions of certain composite fields (composite operators): powers of the scalar field for the density and powers of its spatial gradients for the tracer, so that they can be systematically calculated as series in the exponent y. The practical calculations were performed in the leading order (one-loop approximation) and are presented in (4.12) and (4.25). The results (2.48) and (3.27) for primary fields and (4.13) for the operators θ^n for the tracer are given by this approximation exactly.

Thus, we removed two important restrictions of the previous treatments of the passive compressible problem: absence of time correlations and Gaussianity of the advecting velocity field. We stress that in contrast to previous studies that combined compressibility with finite correlation time [31,32], the present model is manifestly Galilean covariant, and this fact holds in all orders of the perturbation theory.

In a few respects, however, the results obtained here are very similar to those obtained earlier for the compressible version of Kraichnan's rapid-change model [28–30] and the Gaussian model with finite correlation time [31,32]. First of all, the mechanism of the origin of anomalous scaling is essentially the same: The anomalous exponents are identified with the dimensions of *individual* composite operators.

Second, those dimensions are insensitive to the specific choice of the random force (3.3), because the propagator $\langle\theta\theta\rangle_0$ does not enter into the relevant Green's functions. In particular, this means that the anomalous exponents remain intact if the artificial noise is replaced by an imposed linear gradient, a more realistic formulation of the problem. The force maintains the steady state and thus provides nonvanishing mean values for the composite operators, but it does not affect their dimensions.

For the rapid-change case, this fact is naturally interpreted within the zero-mode approach, where the equal-time correlation functions satisfy certain differential equations, and the anomalous exponents are related to the solutions of their *homogeneous* analogs, where the forcing terms are discarded; see [9,11]. On the contrary, the amplitudes are found by matching of these inertial-range zero-mode solutions with the large-scale solutions of the full inhomogeneous equations, which are nontrivial only in the presence of the forcing terms.

From the field theoretic viewpoint, the zero-mode approach is a realization of a more general idea of self-consistent ("bootstrap") equations, which involve skeleton diagrams with exact ("dressed") propagators and discarded bare terms; see e.g., Sec. 4.35 in [14]. Owing to special features of rapid-change models (linearity in the passive field and time decorrelation of the velocity), such equations there are exactly given by one-loop approximations and in the coordinate space take form of rather simple differential equations, written in an explicit closed form. Furthermore, in contrast to the case of finite correlation time, closed equations can be obtained for the *equal-time* correlation functions, which are Galilean invariant and, therefore, not affected by the so-called "sweeping effects" that obscure relevant physical interactions and lead to strong IR divergences. Thus, the problem of IR divergences in self-consistency equations, known for the NS equation since the seminal paper [56], in the rapid-change models is absent.

From a more physical point of view, zero modes can be interpreted as statistical conservation laws in the dynamics of particle clusters [57]. The close resemblance in the RG + OPE pictures of the origin of anomalous scaling for the present model and its rapid-change predecessors suggests that for the former, the concept of zero modes (and thus that of statistical conservation laws) is also applicable here. This observation is rather encouraging because in our model no simple differential equations can be derived for the equal-time correlation functions due to the fact that the advecting velocity has a finite correlation time. Owing to the linearity in the advected field, closed equations can be derived for its correlation functions, but only for different-time ones, and they would involve infinite diagrammatic series.

In this connection it remains to note that in the RG + OPE approach the IR divergences, caused by the sweeping effects, are related to the composite operators v^n (powers of the velocity field), which become dangerous for $y > 3$ [15]. However, they give no contribution to the OPEs for Galilean-invariant quantities like the equal-time correlation functions of the scalar field. For the incompressible case, these issues are discussed in Refs. [15–17,20,49] in detail; nothing essential changes for the compressible case. The relation between the RG approach and statistical conservation laws is also discussed in [58].

Although the anomalous exponents are independent of the specific choice of the noise, they do depend on the exponent y, the dimension of space d, and the parameter α that measures the degree of compressibility. In this respect, our results are also similar to those obtained for simpler models. An important difference with Gaussian models appears when possible dependence on the time scales is studied. It was argued that the exponents can depend on more details of the velocity ensemble than only the exponents, namely, on the dimensionless ratio of the correlation times of the scalar and velocity fields; see, e.g., the discussion in [7]. Indeed, analytic results obtained for Gaussian models with a finite correlation time within the zero-mode technique [59] and the RG + OPE approach [19,31,32] show that such a dependence indeed takes place, at least for some of the possible scaling regimes.

In the present case, the exponents could depend, in principle, on the dimensionless parameters u_0, v_0, w_0, the ratios of various viscosity and diffusion coefficients. After the RG treatment, these parameters are replaced with the corresponding invariant variables, which exactly have the meaning of the ratios of the correlation times of the transverse and longitudinal components of the velocity field, the pressure, and the scalar field; for a detailed discussion of this issue, see [19]. The existence of the unique IR attractive fixed point shows that in the IR range these ratios tend to their fixed-point values u_*, v_*, w_* irrespective of the initial values u_0, etc. We conclude that the anomalous exponents are independent on the time scale; the dependence observed in previous treatments is an artifact of simplified Gaussian statistics.

Another essential difference between our results and those obtained for Kraichnan's rapid-change model is that in the latter the anomalous exponents have a finite limit when the parameter that measures degree of compressibility [analog of α from (2.11) in our model] goes to infinity, that is, for the purely potential velocity field. In our case all the nontrivial anomalous dimensions grow with α without bound. Formally, the difference is due to the fact that the coordinate of the fixed point (2.41) in our model is independent on α and the dependence on it appears only in the numerators of the expressions like (4.12) and (4.25). This fact also means that the one-loop contributions in the critical dimensions become large as α grows, and the one-loop approximation can hardly be trusted even for small y. One may think that the real RG expansion parameter then becomes $y\alpha$ rather than y. In this connection we also recall that the IR attractive fixed point in the one-loop approximation exists for all α but ceases to exist for $\alpha = \infty$ (purely potential forcing). These facts suggest that, beyond the one-loop approximation, the fixed point (2.41), in fact, disappears or loses its stability, and the corresponding scaling regime undergoes some qualitative changeover, the possibility supported by the phase transition to a purely chaotic state observed in [27] for a simplified model.

To investigate this issue, it is necessary to go beyond the leading one-loop approximation (of course, starting with the compressible Navier-Stokes equation itself) and to discuss the existence, stability, and the dependence on α of the fixed point at least at the two-loop level, which seems to be a difficult technical task. Another interesting generalization of our present investigation is to derive a more realistic expression for the random force correlator (2.11) in order to determine realistic values of α and to express it in terms of measurable quantities. Here the combination of the RG techniques and the energy balance equation seems promising; see [60] for the incompressible case. This work remains for the future and is partly in progress.

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